Math 241 Fall 05 Final Exam B

Instructions: Do four of five in Part One and six of the seven in Part Two. If you try more than the required number, only your best will be counted. You may use one standard size 8.5 by 11 inch “cheat sheet” and your calculator. Turn in your exam book and this question sheet but not your “cheat sheet”.

Part One (Ten points each: Do four)

1. Find the Fourier series for the function which is 2π periodic and whose value for 0 ≤ x < 2π is |x|

2. Use the Fourier sine transform to solve the following version of the heat equation:
   \[
   \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad t > 0, \quad x > 0 \text{ subject to } u(0,t) = 0, \quad u(x,0) = e^{-x}.
   \]
   Use the fact (easy to prove and similar to Laplace transform) that
   \[
   \mathcal{F} \left[ e^{-x} \right] = \frac{\alpha}{1 + \alpha^2}.
   \]

3. Solve the Sturm-Liouville problem \( y'' + \lambda y = 0 \) subject to \( y'(0) = 0 \) and \( y'(\pi) = 0 \) on the interval \([0, \pi]\). Be sure to deliver ALL eigenvalues and ALL eigenfunctions and to demonstrate that you have found ALL such

4. Solve the (boundaryless) wave equation \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \) \(-\infty < x < \infty, \quad t > 0 \text{ subject to} \ u(x,0) = f(x) \) (initial deflection) and \( \frac{\partial u}{\partial t} (x, 0) = 0 \) (initial speed zero). For best results use the full Fourier transform. Your answer will of course have to depend on \( f(x) \).

5. Solve Laplace’s equation \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \), \( 0 \leq x \leq a \), \( 0 \leq y \leq b \) subject to \( u(x,0) = 0 \), \( u(x,b) = 100 \), \( u(0,y) = 0 \), \( u(a,y) = 0 \)
Part Two (Ten points each; do six)

6. Let \( f(z) = z^2 + 1 \). Let \( F \) be the path parametrized by 
\[
z(t) = (t+1) + i(t^2 + 2t + 1) \\
0 \leq t \leq 1
\]
Compute, by any legitimate method of your choice, 
\[
\int_{F} f(z)dz
\]

7. For \( f(z) = \frac{1}{z^2(z-3)} \), compute the Laurent series valid for \( 0 < |z| < 3 \)
Use your result to compute \( \text{Residue}(f(z), z=0) \)

8. The function \( f(z) = u(x, y) + i v(x, y) \) is analytic. If \( u(x, y) = x^3 - 3xy^2 \) then
(i) Compute \( v(x, y) \)  (ii) compute \( f'(z) \)

9. Compute \( \oint_{C} \frac{ze^z}{(z-2)^2}dz \) where \( C \) is the circle \( |z| = 3 \)

10. Compute 
\[
\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2 + 1)^2 \left( x^2 + 2x + 2 \right)}
\]

11. For \( f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)} \) find the residue of \( f(z) \) at the singular point \( z = -1 \).

12. Compute 
\[
\int_{0}^{\pi} \frac{d\theta}{13 - 5\sin\theta}
\]