Final, Math 241, Fall 2009

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You may use one sheet of 8 x 11” paper on which you write any information you like. No calculator. Good luck. **Show all work**, even on multiple choice questions.

(1) Compute the principal value of the integral

\[ \int_{0}^{\infty} \frac{\sin x}{x(x^2 + 1)} \, dx. \]

(a) 0  
(b) \( \frac{1}{2}(2 - e^{-1}) \)  
(c) \( \frac{\pi}{2x} \)  
(d) \( \frac{\pi}{2}(1 - e^{-1}) \)  
(e) \( \frac{\pi}{2}(2 - e^{-1}) \)
(2) Evaluate \( \int_C \frac{\sin(2z)}{(6z-\pi)^3} \, dz \), where \( C \) is the ellipse given by
\[ x^2 + 4y^2 = 4 \]
oriented counter-clockwise.
(a) 0
(b) 1/2
(c) \( \pi i \)
(d) \(-\sqrt{3}\)
(e) \(-2\pi i \sqrt{3}\)
(3) Evaluate the integral of $f(z) = z \cos(z^2)$ along the contour $C$ that begins at 0, moves along the real axis to 1, moves counterclockwise around the circle of radius 1 until it reaches $-1$, then moves down along a vertical path to $-1 - i$. (Hint: there is a shortcut.)

(a) $0$
(b) $\frac{i}{2}(e^{-2} - e^{2})$
(c) $\frac{1}{2}(1 + i)(e^{2} - e^{-2})$
(d) $\frac{i}{4}(e^{2} - e^{-2})$
(e) $\frac{1}{2}(e^{2} - e^{-2})$
(4) Compute a Laurent expansion of the function \( f(z) = \frac{1}{(z-2i)(z+i)} \)
valid on the annulus given by \( 1 < |z| < 2 \).
(5) (a) Compute all possible values of $i^{\pi/2}$.
(b) Compute all possible solutions of the equation $\cos(z) = 2$. 
(6) Compute the eigenvalues and eigenfunctions of the Sturm Liouville problem

\[ x^2 y'' + xy' + 25\lambda y = 0, \text{ subject to } y'(1) = 0 \text{ and } y(e) = 0. \]
(7) Evaluate the Cauchy-Principal value of the integral

\[ \int_{-\infty}^{\infty} \frac{3x^2}{(x^2 + 2x + 2)(x^2 + 1)^2} \, dx \]
(8) For each of the following functions determine all the singularities and classify them as removable, pole (and of what order) or essential.
(a) \( \frac{\cos(z)}{z^2} \)
(b) \( \frac{z}{\sin(z)} \)
(c) \( e^{1/z}/z \)
(9) What is the radius of convergence of the Taylor series centered at $2 + i$ of the function $\frac{\cos(z)}{z(z-\pi)}$?
(10) Suppose \( u(r, \theta) \) satisfies Laplace’s equations \( \Delta u = 0 \) on the unit disc \( r \leq 1, \theta \in [0, 2\pi] \) with \( u(1, \theta) = f(\theta) \). Calculate \( u(r, \theta) \).