No books, papers, calculators or electronic device may be used, other than a hand-written note sheet at most $5'' \times 7''$ in size. No questions will be answered or clarifications offered, during this exam.

This examination consists of ten multiple-choice questions and two short-answer questions. The multiple-choice questions are worth eight (8) points each, with no partial credit. The correct and most appropriate answer to a multiple-choice question will be, in each case, just one of the five choices (A), (B), (C), (D) and (E). Answer all multiple-choice questions on the answer sheet, which is page 13 of this exam. Only the answers on the answer sheet will be considered for grading.

The short-answer questions are worth ten (10) points each. You must show all your work and fill in your answers at the underlined space; credit for these questions are based mostly on your short answers. Partial credit will be given only when a substantial part of a problem has been worked out. Merely displaying some formulas is not a sufficient ground for receiving partial credit.

For this posted version, the answers appear at the end of the exam.
1. Let $S$ be the set of all complex numbers $w$ such that
\[ e^w (= \exp(w)) = 1 + i \quad \text{and} \quad |e^{w^2}| (= \exp(w^2)) > 1. \]
How many elements does the set $S$ have? (In other words, how many complex numbers $w$ are there satisfying both $e^w (= \exp(w)) = 1 + i$ and $|e^{w^2}| > 1$?)

A. 0 B. 1 C. 2 D. 3 E. None of the above.

2. Let $f(x)$ be a periodic function on the real line with period $2\pi$ such that
\[ f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ \pi - x, & \text{if } 0 < x < \pi \end{cases} \]
Write the Fourier series expansion of $f(x)$ as
\[ f(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx). \]
Then the infinite series
\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \]
A. converges to 0. B. converges to $2\pi$. C. converges to $\frac{\pi^2}{6}$. D. converges to $\frac{\pi}{2}$. E. diverges.

3. Consider the following five complex-valued functions in the complex variable $z = x + iy$:
\[ f(z) = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \quad g(z) = e^z \]
\[ h(z) = x - iy^2 \quad j(z) = \frac{z^2}{\cos(\frac{\pi}{4})} \]
\[ k(z) = z^{100} \]
Which ones of the above functions are holomorphic (or, complex analytic) at $z = 5$?
A. $f(z)$, $g(z)$ and $k(z)$ only.
B. $f(z)$, $g(z)$ and $h(z)$ only.
C. $f(z)$, $g(z)$, $h(z)$ and $j(z)$ only.
D. $g(z)$ and $k(z)$ only.
E. $f(z)$, $g(z)$, $h(z)$ and $k(z)$ only.
4. Denote by $I$ the complex line integral

$$I = \int_C z^2 \, dz,$$

where $C$ is the segment of the curve $y^2 = 4 - x$ going from $(0, -2)$ to $(0, 2)$.

What is the absolute value $|I|$ of this line integral?

A. $|I| = \frac{16}{3}$  
B. $|I| = 0$  
C. $|I| = \frac{8}{3}$  
D. $|I| = 2\pi$  
E. None of the above.

5. Suppose that $u(x, t)$ is a function defined for $0 \leq x \leq 1$, $t \geq 0$ such that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for all} \quad 0 \leq x \leq 1, \quad \text{all} \quad t \geq 0$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \text{for all} \quad t \geq 0$$

$$u(x, 0) = \cos(\pi x) \quad \text{for all} \quad 0 \leq x \leq 1$$

$$\frac{\partial u}{\partial t}(x, 0) = \cos(3\pi x) \quad \text{for all} \quad 0 \leq x \leq 1$$

What is the value of $u\left(\frac{1}{3}, \frac{1}{2}\right)$?

A. $\frac{1}{2}$  
B. $-\frac{1}{3\pi}$  
C. 0  
D. $\frac{\sqrt{2}}{4}$  
E. None of the above

6. Which of the following complex line integral is not equal to 0?

A. $\oint_{|z|=1} \frac{dz}{z^2+9}$  
B. $\oint_{|z|=1} \frac{dz}{z-1}$  
C. $\oint_{|z|=1} \frac{dz}{4z^2-1}$

D. $\oint_{|z|=1} \frac{e^z-1}{z^3} \, dz$  
E. None of the above

7. Suppose that $f(t)$ is a function defined for $t \geq 0$ such that its Laplace transform $\mathcal{L}\{f(t)\}(s)$ is equal to $\frac{e^{-3s}}{(s-1)^2}$. What is the value of $f(5)$?

A. $5 \, e^3$  
B. $2 \, e^2$  
C. $\frac{25}{2} \, e^2$  
D. $\frac{25}{2} \, e^5$  
E. None of the above
8. Consider the Sturm-Liouville equation for a function $y(x)$ defined for $0 \leq x \leq \pi$:

$$y'' + \lambda y = 0$$

with boundary conditions

$$y'(0) = 0, \quad y'(\pi) = 0.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the set of all eigenvalues of the above equation, and let $y_n(x)$ be the eigenfunction for the eigenvalue $\lambda_n$ such that $y(0) = 1$, $n \geq 1$. Which one of the following statements is true?

A. $\int_0^\pi y_n^2(x) \, dx = 0$ for all $n \geq 1$.
B. $\int_0^\pi x \, y_m(x) \, y_n(x) \, dx = 0$ for all $m \neq n$.
C. $\lim_{n \to \infty} y_n(x) = 0$.
D. $\sum_{n=0}^\infty y_n^2(x)$ converges for all $x \in [0, \pi]$.
E. None of the above

9. Let $u(x, t)$ be a function defined for $0 \leq x \leq \pi$, $t \geq 0$ such that

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial u}{\partial x}$$

and

$$u(0, t) = u(\pi, t) = 0 \quad \text{for all } t \geq 0,$$

$$u(x, 0) = e^{-\frac{x^2}{2}} \sin(x).$$

What is the value of $u\left(\frac{\pi}{2}, 1\right)$?

A. $e^{-2-\frac{\pi}{4}}$  
B. $e^{-1-\frac{\pi}{4}}$  
C. $e^{-1} + e^{-\frac{\pi}{4}}$  
D. $e^{-\frac{\pi}{4}}$  
E. None of the above
10. Suppose that $u(x, y, z)$ is a function on $\mathbb{R}^3$ which satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and

$$u(x, y, z) = 5z^3 \quad \text{if} \quad x^2 + y^2 + z^2 = 1.$$ 

What is the value of $u(1, 1, 1)$?

Notice that the values of $u(x, y, z)$ at the boundary sphere $\{x^2 + y^2 + z^2 = 1\}$ depends only on the angle $\phi$ with the positive $z$-axis, so that the solution $u(x, y, z)$ is symmetric under arbitrary rotation about the $z$-axis. The formula for the Laplacian in spherical coordinates is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}$$

Some Legendre polynomials are: $P_0(w) = 1$, $P_1(w) = w$, $P_2(w) = \frac{3w^2 - 1}{2}$, $P_3(w) = \frac{5w^3 - 3w}{2}$.

A. 5 B. 0 C. -1 D. 3 E. None of the above

11. The complex valued function $f(z) = \frac{e^z}{\cos z}$ has a Laurent series expansion centered at $z = 0$, for $|z| < 1/100$:

$$f(z) = \frac{e^z}{\cos z} = \sum_{n=-\infty}^{\infty} a_n z^n \quad |z| < 1/100$$

Find the coefficients $a_0, a_1, a_2, a_3$.

Your Answer:

$a_0 =$

$a_1 =$

$a_2 =$

$a_3 =$
12. Suppose that \( u(x, y) \) is a function defined for \( x^2 + y^2 \leq 1 \) such that

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

and

\[
u(x, y) = 4x^2 + x \quad \text{if} \quad x^2 + y^2 = 1.\]

Find a closed formula for \( u(x, y) \), and use it to evaluate \( u(\frac{1}{2}, \frac{1}{2}) \). (Recall that in polar coordinates, we have

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r^2} \left[ \left( r \frac{\partial}{\partial r} \right)^2 + \frac{\partial^2}{\partial \theta^2} \right].
\]

You might want to use the method of separation of variables.)

Your Answer:

\[
u(x, y) = \frac{1}{2} \left( x^2 - y^2 \right)
\]

\[
u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}
\]

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**ANSWERS**

1. A. (Since the real part of \( \log(\sqrt{2}) \) is less than \( \frac{\pi}{4} \).)
2. D
3. A.
4. A.
5. E. \( u(x, t) = \cos(\pi x) \cos(\pi t) + \frac{1}{3} \cos(3\pi x) \sin(3\pi t); u(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} \pi \)
6. D.
7. B. \( f(5) = 8 \; e^2; \; f(t) = \frac{1}{2} u(t - 3) (t - 3)^2 e^{t-3} \)
8. E.

9. E. Separation of variables gives the general solution: \( u(x, t) = \sum_n e^{-4n^2+1}t \; e^{-\frac{\pi}{4}} \sin(nx) \). Using the initial conditions we get \( u(x, t) = e^{-5t} e^{-\frac{\pi}{4}} \sin(x) \). Thus \( u(\frac{\pi}{2}, 1) = e^{-5-\frac{\pi}{4}} \).
10. C. \( u(x, y, z) = 2r^3 P_3(\cos \phi) + 3 \; r \; P_1(\cos \phi) = -3 \; x^2 \; z - 3 \; y^2 \; z + 2 \; z^3 + 3z, u(1, 1, 1) = -1 \)
11. f(z) = 1 + z + z^2 + \frac{z^3}{3} + \cdots.
12. C. \( u(\frac{1}{2}, \frac{1}{2}) = \frac{5}{2}; u(x, y) = 2x^2 - 2y^2 + x + 2 \)