Math 241
Final examination

Instructions. Answer the following problems carefully and completely. Show all your work. Do not use a calculator. You may use both sides of one $8\frac{1}{2} \times 11$ sheet of paper for handwritten notes you wrote yourself. Please turn in your sheet of notes with your exam. There are 100 points possible. Good luck!

Name ________________________
Instructors’s name ________________________
TA’s name and time ________________________

1. (2) __________________________
2. (14) __________________________
3. (6) __________________________
4. (2) __________________________
5. (3) __________________________
6. (8) __________________________
7. (8) __________________________
8. (5) __________________________
9. (5) __________________________
10. (10) __________________________
11. (11) __________________________
12. (6) __________________________
13. (6) __________________________
14. (14) __________________________
Total (100) __________________________

1
Here are some integrals you can use:

\[ \int_0^\infty xe^{-x} \sin(cx) \, dx = \frac{2c}{(1 + c^2)^2} \]
\[ \int_0^\infty xe^{-x} \cos(cx) \, dx = \frac{1 - c^2}{(1 + c^2)^2} \]

1. Write whether the following statement is true or false. (You do not need to show any work.) The product of an odd function \( f \) with an odd function \( g \) is an odd function.

   \[ \text{FALSE} \]
2. Use a Fourier transform, a sine transform, or a cosine transform to find the displacement $u(x, t)$, for $x > 0$ and $t > 0$, of a semi-infinite string if

$$u(0, t) = 0, \quad u(x, 0) = xe^{-x}, \quad \text{and} \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0.$$ 

You may assume the constant $a^2$ of the wave equation is equal to 1. Your final answer may contain an integral.

Let $U(x, t) = \int_0^\infty u(x, t) \sin(\alpha x) \, dx$ be the sine transform of $u(x, t)$.

$$\int_0^\infty \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = -\alpha^2 U(x, t) + \alpha u(x, 0, t)$$

$$= -\alpha^2 U(x, t)$$

$$\int_0^\infty \left\{ \frac{\partial^2 u}{\partial t^2} \right\} = \frac{\partial^2 U}{\partial t^2}.$$ 

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ transforms to

$$-\alpha^2 U(x, t) = \frac{\partial^2 U}{\partial t^2}.$$ 

$$\Rightarrow U(x, t) = A \cos(\alpha t) + B \sin(\alpha t)$$

where $A$ and $B$ are functions of $x$ alone.

$$0 = \int_0^\infty 0 \, dx = \int_0^\infty \left. \frac{\partial u}{\partial t} \right|_{t=0} \, dx = \left. \frac{\partial U}{\partial t} \right|_{t=0}.$$
\[ \Rightarrow B = 0 \quad \Rightarrow \quad \mathcal{U}(\alpha, t) = A \cos(\alpha t). \]

\[ A = \mathcal{U}(\alpha, 0) = \mathcal{F}_s \left\{ u(x, 0) \right\} = \mathcal{F}_s \left\{ x e^{-x^2} \right\} = \frac{2\alpha}{(1+\alpha^2)^2}. \]

\[ \Rightarrow \quad \mathcal{U}(\alpha, t) = \frac{2\alpha}{(1+\alpha^2)^2} \cos(\alpha t). \]

\[ \Rightarrow \quad \mathcal{U}(x, t) = \mathcal{F}_s \left\{ \mathcal{U}(\alpha, t) \right\} = \frac{2}{\pi} \int_0^\infty \frac{2\alpha}{(1+\alpha^2)^2} \cos(\alpha t) \sin(\alpha x) \, d\alpha. \]
3. Find *any two* independent solutions $u(x, y)$ to the following PDE:

$$\frac{\partial^2 u}{\partial x \partial y} = u$$

Neither of your solutions can be the zero function.

$$e^{x+y} \quad \text{is one solution.}$$

$$e^{2x+y/2} \quad \text{is another solution.}$$

$$\frac{e^{x+y}}{e^{2x+y/2}} = e^{-x+y/2} \quad \text{is not a constant \text{ \textit{fct.}}.}$$

$\Rightarrow$ they are independent.

4. Find $a$ and $b$ real numbers such that

$$\frac{10 - 5i}{6 + 2i} = a + ib.$$
5. Let

\[ z_1 = 2 \cos(\pi/8) + 2i \sin(\pi/8) \]
\[ z_2 = 4 \cos(3\pi/8) + 4i \sin(3\pi/8) \]

Find \( a \) and \( b \) real numbers such that

\[ \frac{z_1}{z_2} = a + ib. \]

\[ z_1 = 2e^{\pi i/8} \quad z_2 = 4e^{3\pi i/8} \]

\[ \frac{z_1}{z_2} = \frac{2e^{\pi i/8}}{4e^{3\pi i/8}} = \frac{1}{2} e^{-\frac{2\pi i}{8}} = \frac{1}{2} e^{-\frac{\pi i}{4}} = \frac{1}{2} (\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}) \]

\[ a = \frac{1}{2\sqrt{2}} \quad b = -\frac{1}{2\sqrt{2}} \]
6. Show the complex function \( f(z) = \bar{z} \) is not analytic at \( z = 0 \).

I will show the limit \( \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\bar{z}}{z} \) does not exist.

First let \( z = x \) for \( x \) real.

\[
\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} \frac{x}{x} = 1.
\]

Now let \( z = iy \) for \( y \) real.

\[
\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{i y}{y} = \lim_{y \to 0} \frac{-iy}{y} = -1.
\]

This shows the limit does not exist.
7. Find all points \( z \) in \( \mathbb{C} \) satisfying the equation

\[
\sin z = 2.
\]

Write the solutions in the form \( a + ib \) for \( a \) and \( b \) real numbers.

\[
2 = \sin z = \frac{4}{2i} (e^{iz} - e^{-iz})
\]

\[
4i e^{iz} = (e^{iz})^2 - 1
\]

\[
(e^{iz})^2 - 4i e^{iz} - 1 = 0.
\]

\[
\frac{e^{iz} = 4i + (-16 + 4)^{1/2}}{2} = 2i \pm \frac{1}{2} i \sqrt{12}
\]

\[
= 2i \pm i \sqrt{3} = i (2 \pm \sqrt{3}).
\]

Let \( z = x + iy \). Then

\[
e^{iz} = e^{-y} e^{ix} = i (2 \pm \sqrt{3})
\]

\[
2 + \sqrt{3} > 0 \Rightarrow \arg(i (2 + \sqrt{3})) = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2} + 2 \pi n.
\]

\[
|e^{iz}| = |e^{-y}| = e^{-y} = |i (2 + \sqrt{3})| = 2 + \sqrt{3}. \Rightarrow y = \log(2 + \sqrt{3})
\]

So one set of solns is \( \frac{\pi}{2} + 2 \pi n - i \log(2 + \sqrt{3}) \) for \( n \) any integer.

For \( i(2+\sqrt{3}) \)

\[
\]

Similarly \( 2 - \sqrt{3} > 0 \Rightarrow \) the other set of solns

\[
\]

is \( \frac{\pi}{2} + 2 \pi n - i \log(2 - \sqrt{3}) \) for \( n \) any integer.
8. Compute the contour integral

\[ \oint_C \frac{z}{z^2 - \pi^2} \, dz \]

where \( C \) is the circle \( |z| = 3 \).

The fact that \( \frac{z}{z^2 - \pi^2} \) is not analytic at \( \pm \pi \).

Neither of these points lie inside \( C \).

\[ \therefore \text{ by Cauchy's thm} \]

\[ \oint_C \frac{z}{z^2 - \pi^2} \, dz = 0. \]
9. Determine the pole(s) of $5 - 6/z^2$. Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

The fact $f(z) = 5 - 6/z$ is already written in the form of a Laurent series centered at 0, where it has a pole of order 2 with residue 0. $f(z)$ has no other poles.

10. Determine the pole(s) of

$$\frac{1}{1 - e^z}$$

Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

Let $f(z) = \frac{1}{1 - e^z}$. $f(z)$ is $2\pi i$-periodic, i.e. $f(z + 2\pi i) = f(z)$. $f(z)$ is not analytic at $z = 0$. Let's examine this singularity first.
Consider \( g(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \ldots \)

By the ratio test this power series converges on all of \( \mathbb{C} \).
\[ \Rightarrow g(z) \text{ is analytic everywhere. } g(0) = 1. \]
\[ \Rightarrow \frac{d}{dz} \left( \frac{z}{1-e^z} \right) = \frac{d}{dz} \left( \frac{-1}{g(z)} \right) = -1 \cdot (-1) \cdot (g(z))^{-2} \cdot g'(z) \]
\[ \Rightarrow \left. \frac{d}{dz} \left( \frac{z}{1-e^z} \right) \right|_{z=0} = g'(0) = \frac{1}{2}. \]
\[ \Rightarrow \frac{z}{1-e^z} \text{ is analytic at 0. } \Rightarrow f(z) \text{ has a simple pole at 0. The residue at 0 is} \]
\[ \lim_{z \to 0} \left( \frac{z}{1-e^z} \right) = - \lim_{z \to 0} g(z) = -1. \]

\( \Rightarrow \) By the \( 2\pi i \)-periodicity of \( f(z) \), the other poles are at \( 2\pi in \) for \( n \) any integer, and these poles are all simple with residue \(-1\).
11. Compute the integral

\[ \int_0^\pi \frac{1}{5 + 4 \cos \theta} \, d\theta \]

\[ \cos \theta \text{ is even so} \]

\[ \int_0^\pi \frac{d\theta}{5 + 4 \cos \theta} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{5 + 4 \cos \theta} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{5 + 2e^{i\theta} + 2e^{-i\theta}} \, d\theta = \frac{1}{2} \int_{i e^{i\theta}}^{i e^{-i\theta}} \frac{ie^{i\theta} \, d\theta}{5 + 2e^{i\theta} + 2e^{-i\theta}} \]

Let \( C(\theta) = e^{i\theta} \) for \(-\pi \leq \theta \leq \pi\).

\[ = \frac{1}{2i} \oint_C \frac{1}{z} \, d\frac{z}{5 + 2z + \frac{3}{2}z} = \frac{1}{2i} \oint_C \frac{dz}{2z^2 + 5z + 2} = \frac{1}{2i} \oint_C \frac{dz}{(z+1)(z+2)} = \frac{1}{2i} \oint_C \frac{dz}{5 + \frac{1}{4}} \]

\[ = \frac{5 - \frac{x}{4} - \frac{2z}{4}}{\frac{1}{4}} = \frac{5 - \frac{1}{4}}{2} \]

\[ = \frac{4}{4i} \cdot 2\pi i \cdot \text{Res} \left( \frac{1}{(z+1)(z+2)} \right) \]

only the simple pole \(-\frac{3}{2}\) lies inside \( C \)

\[ = \frac{\pi}{2} \cdot \frac{1}{(-\frac{3}{2} + 2)} = \frac{\pi}{3} \]
12. Let $C$ be the curve in the complex plane parametrized by $C(t) = \cos(t) + i \sin(t)$, for $0 \leq t \leq \pi$. (Note the $\pi$!) Compute the value of the contour integral

$$\oint_C \frac{dz}{z^2}$$

$C(t) = \cos t + i\sin t = e^{it}$, $0 \leq t \leq \pi$

$C'(t) = ie^{it}$

$$\oint_C \frac{dz}{z^2} = \int_0^\pi \frac{1}{e^{it}} \cdot i e^{it} \, dt = i \int_0^\pi \frac{1}{e^{it}} \, dt$$

$$= \frac{i}{-i} \cdot [e^{-it}]_0^\pi = -1 \cdot (-1 - 1) = 2.$$
13. Consider the function

\[ f(x) = \begin{cases} 
0 & \text{for } 0 \leq x \leq 1 \\
1 & \text{for } 1 < x \leq 2 
\end{cases} \]

defined on the interval \([0, 2]\). Let

\[ \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{2} \right) \]

be a sine series for \( f(x) \). Using the same values for \( B_n \), for all \( x \) in the real line define a function

\[ g(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{2} \right). \]

Find \( g(-5/2) \) and \( g(-5) \).

\( g(x) \) is a \( 4\pi \)-periodic fct.

\[ \Rightarrow g(-\frac{5}{2}) = g\left(-\frac{5}{2} + \frac{\pi}{2}\right) = g\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 1. \]

\[ g(-5) = g(-1) = -g(1) = -f(1) = -\frac{1}{2}(0 + 1) \]

\[ g \text{ is odd} \]

\[ = -\frac{1}{2}. \]
14. Solve the Laplace equation \( u_{xx} + u_{yy} = 0 \) for a function \( u(x, y) \) with \( 0 \leq x \leq 2, 0 \leq y \leq 1 \) and boundary conditions:

\[
\begin{align*}
    u(0, y) &= 0, \\
    \frac{\partial u}{\partial x}(2, y) &= 0, \\
    u(x, 0) &= 0, \\
    u(x, 1) &= 3\sin\left(\frac{\pi x}{4}\right) - 2\sin\left(\frac{5\pi x}{4}\right).
\end{align*}
\]

\[
u(x, 1) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) - \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{2}\right).
\]

Such a solution is only possible if \( \lambda = \frac{\pi}{4} + \frac{\pi}{2} n \) for \( n = 0, 1, 2, 3, \ldots \)

\[
u(x, 1) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) - \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{2}\right).
\]

\[
u(x, 1) = \begin{cases} 
A \quad \text{for } x = 0 \\
B \quad \text{for } x = 2
\end{cases}
\]

So \( \lambda = 0 \) is not an eigenvalue.
$a < 0$

**Case 1:** $F(x) = A \cosh(ax) + B \sinh(ax)$ for $a = \sqrt{1}$.

$F(0) = 0 \Rightarrow A = 0$.

$F'(a) = B \alpha \cosh(2ax) = 0 \Rightarrow B = 0$.

$\Rightarrow$ no eigenvalues $\neq 0$.

So the eigenvalues are $\lambda_n = \left(\frac{\pi}{4} + \frac{\pi n}{2}\right)^2$ for $n = 0, 1, 2, 3, \ldots$

with eigenfunc $F_n(x) = B_n \sinh(\sqrt{\lambda_n} \cdot x)$.

$\Rightarrow G_n''(y) = \lambda_n G_n(y) \quad G_n(0) = 0$

$\Rightarrow G_n(y) = C_n \sinh(\sqrt{\lambda_n} \cdot y)$.

$\Rightarrow u_n(x, y) = A_n \sinh\left(\frac{\pi}{4} + \frac{\pi n}{2}\right) \sin(\frac{\pi}{4} + \frac{\pi n}{2}) x$ for $n = 0, 1, 2, \ldots$

$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$.

$u(x, 1) = 3 \sin(\frac{\pi x}{4}) - 2 \sin(\frac{5\pi x}{4}) = \sum_{n=0}^{\infty} A_n \sinh\left(\frac{\pi}{4} + \frac{\pi n}{2}\right) \sin(\frac{\pi}{4} + \frac{\pi n}{2}) x$

$\Rightarrow$ only the $n = 0$ and $n = 2$ coeff. are nonzero.

$A_0 = \frac{3}{\sinh(\frac{\pi}{4})}$, $A_2 = \frac{-2}{\sinh(\frac{5\pi}{4})}$

$u(x, y) = \frac{3}{\sinh(\frac{\pi}{4})} \sinh\left(\frac{\pi x}{4}\right) \sin(\frac{\pi x}{4}) - \frac{2}{\sinh(\frac{5\pi}{4})} \sinh\left(\frac{5\pi x}{4}\right) \sin(\frac{5\pi x}{4})$. 

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