Name: ____ PennID: __________________

Math 241
Final Exam
December 17, 2015

Instructions:

Turn off and put away your cell phone.
Please write your Name and PennID on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No calculators or any other electronic devices are allowed during this exam.
You may only consult 1 page (letter size) with notes on both sides. No other consultation material is allowed.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write all your solutions clearly and legibly; no credit will be given for illegible solutions.
Please try to be as organized as possible. If any question is not clear, ask for clarification.

#	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	10	
7	15	
8	10	
9	10	
Total	100	

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Table 1: Boundary value problems for $\phi''(x) = -\lambda\,\phi(x)$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\phi'(0) = 0$ $\phi'(L) = 0$	$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi'(L)$
Eigenvalues	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$ $+ \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$	$A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$	$a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,\mathrm{d}x$ $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$ $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$

Table 2: Orthogonality relations for sines and cosines

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

Problem 1 (10 pts): Solve the heat equation $u_t = 5\,u_{xx}$, where $0 < x < 1$, with boundary conditions and initial condition given by

$$u_x(0,t) = u_x(1,t) = 0 \quad \text{and} \quad u(x,0) = 4 + 2\cos(3\pi x).$$

Problem 2 (15 pts): Compute the Fourier series of the function $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = 1 - |x|$. For what values of $-1 \le x \le 1$ does this series converge to $f(x)$?

Problem 3 (10 pts): Consider the boundary value problem

$$
\begin{aligned}
& x^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + x\frac{\mathrm{d}\phi}{\mathrm{d}x} + \lambda\phi = 0, \quad 1 \le x \le e \\
& \phi(1) = \phi(e) = 0.
\end{aligned}
$$

a) Find all eigenvalues and eigenfunctions;

b) With respect to what weight function are the above eigenfunctions orthogonal?

Problem 4 (10 pts): Consider the boundary value problem

$$\phi'' + (3 - 3x)\phi' + \lambda\phi = 0, \quad \phi(0) = \phi(2) = 0.$$

a) Rewrite the above equation in Sturm-Liouville form;

b) Verify that $\phi(x) = x(x-1)(x-2)$ is an eigenfunction for this problem, and compute its eigenvalue λ;

c) Prove that the eigenvalue λ obtained in b) is the *second* eigenvalue λ_2 of this Sturm-Liouville problem.

Problem 5 (10 pts): Let D be a disk of radius R and consider the eigenvalue problem

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } D, \\ \langle \nabla\phi, \vec{n} \rangle = \phi & \text{on } \partial D, \end{cases}$$

where ϕ is bounded at the origin. For what values of R is $\lambda = 0$ is an eigenvalue?

Problem 6 (10 pts): Suppose u satisfies the heat equation $u_t = \Delta u$, inside a bounded region $B \subset \mathbb{R}^3$, subject to the following initial condition and boundary condition:

$$u(x, y, z, 0) = f(x, y, z) \text{ in } B,$$
$$\langle \nabla u, \vec{n} \rangle = g \text{ on } \partial B, \text{ for all } t \geq 0.$$

Let $E(t)$ be the total heat energy at time t, that is,

$$E(t) = \iiint_B u(x, y, z, t)\, dx\, dy\, dz$$

What is the total heat energy at time $t = 100$?

Problem 7 (15 pts): Let D be the disk of radius 1 and consider the Poisson equation

$$\begin{cases} \Delta u = r^2 & \text{in } D \\ u(1,\theta) = \cos(2\theta) & \text{on } \partial D. \end{cases}$$

a) Find a radial solution $v(r,\theta) = v(r)$ of the equation $\Delta v = r^2$ satisfying $v(1) = 0$ and $|v(0)| < +\infty$;

b) Find a solution $w(r,\theta)$ of the associated homogeneous problem

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w(1,\theta) = \cos(2\theta) & \text{on } \partial D. \end{cases}$$

c) Use v and w to write the solution $u(r,\theta)$ to the original nonhomogeneous problem.

Problem 8 (10 pts): Use Fourier transforms to find an explicit formula for the solution $u(x, t)$ of the PDE $u_t + 7u_x = 0$, where $-\infty < x < \infty$, $t \geq 0$, and $u(x, 0) = f(x)$.

Problem 9 (10 pts): The vertical displacement $u(r,\theta,t)$ of a vibrating circular membrane of radius $r_0 = 1$, fixed at the boundary, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad 0 \le r \le 1, \, -\pi \le \theta \le \pi, t \ge 0,$$
$$u(1,\theta,t) = 0.$$

Let $J_0(z)$ be the 0th Bessel function of first kind, and denote its zeroes by $z_{01} < z_{02} < z_{03} < \cdots < z_{0m} < \ldots$. Find a formula for the solution $u(r,\theta,t)$ subject to the initial conditions

$$u(r,\theta,0) = 0, \quad u_t(r,\theta,0) = \sum_{m=1}^{\infty} \frac{1}{m^2} J_0(z_{0m} r).$$

You may assume without explanations that separating variables $u(r,\theta,t) = R(r)V(\theta)T(t)$ for the above PDE gives the following ODEs:

$$T'' + \lambda T = 0, \quad V'' + \mu V = 0, \quad r(rR')' + (\lambda r^2 - \mu)R = 0, \quad \lambda, \mu \ge 0.$$
