Math 241 Final Exam
Spring 2014

Name:

Instructor: Hynd Weber

Please clear your desk and put away all notes, books, electronic devices, etc. No outside material is allowed during the exam. Make sure to clearly indicate your responses; box your final answers if necessary.

My signature below certifies that I will comply with the University of Pennsylvania’s Code of Academic Integrity in completing this examination.

Your signature

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A Partial Table of Integrals

\[ \int_0^x u \cos nu \, du = \frac{\cos nx + nx \sin nx - 1}{n^2} \quad \text{for any real } n \neq 0 \]

\[ \int_0^x u \sin nu \, du = \frac{\sin nx - nx \cos nx}{n^2} \quad \text{for any real } n \neq 0 \]

\[ \int_0^x e^{mu} \cos nu \, du = \frac{e^{mx}(m \cos nx + n \sin nx) - m}{m^2 + n^2} \quad \text{for any real } n, m \]

\[ \int_0^x e^{mu} \sin nu \, du = \frac{e^{mx}(-n \cos nx + m \sin nx) + n}{m^2 + n^2} \quad \text{for any real } n, m \]

\[ \int_0^x \sin nu \cos mu \, du = \frac{m \sin nx \sin mx + n \cos nx \cos mx - n}{m^2 - n^2} \quad \text{for any real numbers } m \neq n \]

\[ \int_0^x \cos nu \cos mu \, du = \frac{m \cos nx \sin mx - n \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n \]

\[ \int_0^x \sin nu \sin mu \, du = \frac{n \cos nx \sin mx - m \sin nx \cos mx}{m^2 - n^2} \quad \text{for any real numbers } m \neq n \]

Facts About Bessel Functions

- Bessel’s equation: \( r^2 f''(r) + rf'(r) + (\alpha^2 r^2 - m^2) f(r) = 0 \) for each integer \( m \geq 0 \). The only solutions which are bounded at \( r = 0 \) are \( f(r) = c J_m(\sqrt{\alpha r}) \) for a constant \( c \).

- Orthogonality relation: Writing \( z_{mn} \) as the \( n \)th zero of \( J_m(z) \),

\[ \int_0^1 r J_n(z_{mn}) J_n(z_{mk}) \, dr = 0, \quad n \neq k \]

for each \( m = 0, 1, \ldots \)

Fourier Transform

\[ \mathcal{F}[u](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x) e^{i\omega x} \, dx, \quad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega) e^{-i\omega x} \, d\omega \]

Table of Fourier Transform Pairs \( (\alpha, \beta > 0) \)

<table>
<thead>
<tr>
<th>( u(x) = \mathcal{F}^{-1}[U] )</th>
<th>( U(\omega) = \mathcal{F}[u] )</th>
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<td>( e^{-\alpha x^2} )</td>
<td>( \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}} )</td>
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<td>} )</td>
<td>( \frac{2\alpha}{2\pi x^2 + \alpha^2} )</td>
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Laplacian in Polar Coordinates

\[ \nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \]
1. Consider the wave equation for a vibrating string

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L \]

subject to the boundary conditions

\[ u(0, t) = u(L, t) = 0. \]

Which of the following statements are correct? Note that these statements are graded +1 for each correct answer, −1 for each incorrect answer, 0 for no answer.

(a) \( u \) represents the horizontal displacement of the string. \quad Y \quad N

(b) Newton’s second law was used to derive this equation. \quad Y \quad N

(c) \( c \) has units of speed. \quad Y \quad N

(d) The model is valid for large displacements in the string. \quad Y \quad N

(e) The endpoints of the string are fixed. \quad Y \quad N

(f) Increasing the length of the string, while keeping the density and tension fixed, will increase the frequency at which the string vibrates. \quad Y \quad N
2. (a) Write the forward time, centered spatial finite difference scheme for the heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \]

Here \( 0 < x < L \) and \( 0 < t < T \). Use the notation \( u^{(m)}_{j} \) for the approximation of the values \( u(j\Delta x, m\Delta t) \) for the true solution for \( j = 0, 1, \ldots, N \), and \( m = 0, 1, \ldots, M \).

(b) When is the approximation method from part (a) numerically stable?
3. Set

\[ f(x) = \begin{cases} 
  x - 1, & -1 \leq x \leq 0 \\
  x + 1, & 0 < x \leq 1 
\end{cases}. \]

(a) Compute the coefficients in the Fourier series of \( f(x) \)

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)). \]
(b) Plot the Fourier series of $f(x)$ on the interval $[-3, 3]$. 
4. (a) Find the harmonic function $u(r, \theta)$ on the disk $r^2 \leq 1$ that satisfies the boundary condition

$$u(1, \theta) = 1 + \sin(2\theta).$$

(b) Explain why this solution is always less than or equal to 2.
5. Solve the heat equation

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \]

subject to the initial condition

\[ u(x, 0) = e^x. \]
6. Find \( u(x,t) \) that satisfies the nonhomogeneous heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-x^2t} \sin(3\pi x) \quad \text{for } 0 < x < 1, \ t > 0
\]

with boundary conditions \( u(0,t) = u(1,t) = 0 \) and initial condition \( u(x,0) = 2 \).
7. Consider a vibrating circular membrane of radius 1 that has no displacement on the boundary. The associated PDE is the wave equation
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.
\]
(a) After separating time, we are lead to the eigenvalue problem
\[
\nabla^2 \phi + \lambda \phi = 0, \quad \phi(1, \theta) = 0.
\]
Separate variables \( \phi(r, \theta) = f(r)g(\theta) \) and derive ODE for \( f(r) \) and \( g(\theta) \).
(b) Find the eigenvalues $\lambda_{mn}$ and the corresponding eigenfunctions $\phi_{mn}$. You may assume all eigenvalues are positive.
(c) Find a general solution $u(r,t)$ when the membrane is initially circularly symmetric

$$u(r,0) = \alpha(r), \quad \frac{\partial u}{\partial t}(r,0) = 0.$$
8. (Note: Respond if HYND is your instructor) Consider the energy

$$E(t) = \frac{1}{2} \int_0^1 \left( \frac{\partial u}{\partial t}(x,t) \right)^2 + \left( \frac{\partial u}{\partial x}(x,t) \right)^2 \, dx.$$ 

associated with a solution $u(x,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, \ t \geq 0.$$ 

Show that if the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(1,t)$$

are satisfied for all $t > 0$, then

$$E(t) = E(0), \quad t \geq 0.$$
8. (Note: Respond if Weber is your instructor) Consider the second order differential equation on the domain \([1, 2]:(\text{Note: Respond if Weber is your instructor})\)

\[
x^2 \frac{d^2 f}{dx^2} + 4x \frac{df}{dx} + (\lambda - x^2) f = 0, \quad f(1) = 0, \quad f(2) = 0.
\]

(0.1)

This is almost, but not quite, a Bessel-type differential equation.

a) Put the equation into Sturm-Liouville form. What are \(p(x), q(x),\) and \(\sigma(x)\)?

b) According to the Sturm-Liouville theory, the eigenvalues \(\{\lambda_n\}_{n=1}^\infty\) come as a discrete list, and to each eigenvalue \(\lambda_n\) corresponds an eigenfunction \(\varphi_n(x)\). The eigenfunctions satisfy certain orthogonality relations. For the differential equation (0.1), write out this relation in terms of the appropriate integral or integrals.
c) If \( \{ \lambda, \varphi(x) \} \) constitute an eigenvalue-eigenfunction pair for this Sturm-Liouville equation, show that necessarily \( \lambda > 0 \). Give a specific reason why \( \lambda = 0 \) is not actually an eigenvalue. (Hint: Remember the Rayleigh quotient.)