Math 360: Advanced Calculus

These are lecturer's notes for MATH 360: Advanced Calculus, taught in Spring 2013 at the University of Pennsylvania.

I make no pretence that these notes are complete. Emails pointing out errata or egregious gaps are greatly appreciated.

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Why we need real numbers

This is a course about real numbers.

What is a real number?

This is not nearly as simple a question as it seems. The answer was only discovered in the 1800s!

Def'n (old) The natural numbers are the set

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (Note: some omit 0 from $\mathbb{N}$. always check!)

The integers are the natural numbers and their negatives:

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \mathbb{N} \cup -\mathbb{N}$, with the agreement that $0 = -0$.

Kronecker: "Die ganzen Zahlen hat der liebe Gott gemacht; alles anderes ist Menschenwerk."

"Beloved God made the natural numbers; everything else is man's work."

As with many famous math. quotes, this may be apocryphal.

$\mathbb{N}$ is closed under addition and multiplication; $\mathbb{Z}$ is closed under addition and multiplication and has additive inverses

Def'n The rational numbers are

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m,n \in \mathbb{Z},\ n \neq 0,\ m \,\&\, n \text{ have no common divisors} \right\}$$

$\mathbb{Q}$ is closed under $+, -, \cdot, /$.

Prop'n Every linear equation with rational coefficients has a rational solution.

Question: Is there $q \in \mathbb{Q}$ with $q^2 = 2$?

(Heretical) Answer: No!

Proof Suppose so, i.e. there is $q = \frac{m}{n} \in \mathbb{Q}$ so that

$$\left(\frac{m}{n}\right)^2 = 2$$

Since $\frac{m}{n} \in \mathbb{Q}$, at most one of m, n is even.

$$m^2 = 2n^2$$

So m^2 is even. The only way a square can be even is if the thing it's a square of is even. So

$$m = 2k \quad \text{for some } k.$$

Then $2n^2 = m^2 = 4k^2$

$$n^2 = 2k^2$$

So n^2 is even. Hence n is even. But this contradicts the choice of m & n. ∎

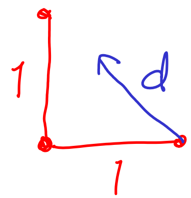
(This is an example of a reductio ad absurdum proof.)

reduction to absurdity

"proof by contradiction"

This poses two major problems:

1. algebraic: Can't solve equations, even simple ones!
2. geometric: If we have a square lot with side length 1 earth measure and we want to walk across it...



Pythagoras says: Can walk any distance d, so long as $d^2 \leq 1^2 + 1^2 = 2$.

The set of distances we can walk is therefore

$$S = \{ q \mid q^2 \leq 2 \} \subset \mathbb{Q}.$$

Compare this set to

$$R = \{ q \mid q^2 \leq 1 \} \subset \mathbb{Q}$$

which contains a least upper bound, that is,

"upper bound" (A) If r is in R, then $1 \geq r$.

"least" (B) Any other number b which satisfies property (A) is larger than 1.

"contains" (C) 1 is in R.

The set S does not have this property, even though $\{q \mid q^2 \le 2\}$ and $\{q \mid q^2 \le 1\}$ are formally almost identical!

Ordered Fields

Def'n A set is a collection of objects ("elements") for which the question "Is object x an element?" has an unambiguous yes-or-no answer.

(Homework: Google "Russell's paradox.")

Notation $x \in A$ "x is an element of the set A"

$x \notin A$ "x is not an element of the set A"

Def'n A set B is a subset of a set A if every element of B is also an element of A.

$B \subset A$ "B is a subset of A"

We call two sets equal if they are mutually subsets

$$A = B \iff A \subset B \text{ and } B \subset A$$

Notation every element of B is also an element of A

universal quantifier $\rightarrow$ $\forall x \in B, \quad x \in A$

for all x elements of B, x is an element of A

$$x \in B \Rightarrow x \in A$$

x is an element of B implies x is an element of A

Def'n An ordered set is a pair $(S, \leq)$ of a set S and an ordering $\leq$, such that:

1. For any $x, y \in S$, either $x \leq y$ or $y \leq x$ (or both!)
2. For any $x \in S$, $x \leq x$
3. If $x \leq y$ and $y \leq x$, then $x = y$
4. If $x \leq y$ and $y \leq z$, then $x \leq z$.

E.g. alphabetisation

E.g. $\mathbb{Q}$ with the standard "less than or equal to"

Def'n A field is a set F together with operations $+$ and $\cdot$ such that:

addition axioms:

- $\forall x, y \in F,\ x + y = y + x$
- $\forall x, y, z \in F,\ (x+y)+z = x+(y+z)$
- $\exists 0 \in F \ni \forall x \in F,\ x + 0 = x$ (existential quantifier)
- $\forall x \in F,\ \exists (-x) \in F \ni x + (-x) = 0$

multiplication axioms:

- $\forall x, y \in F,\ x \cdot y = y \cdot x$
- $\forall x, y, z \in F,\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $\exists 1 \in F \ni \forall x \in F,\ 1 \cdot x = x$
- $\forall x \in F \ni x \neq 0,\ \exists x^{-1} \in F \ni x \cdot x^{-1} = 1$ } sad orphan multiplication axiom
- $x \cdot (y + z) = (x \cdot y) + (y \cdot z)$ distributive axiom
- $1 \neq 0$ nondegeneracy axiom

E.g. $\mathbb{Q}$ with ordinary operations.

Def'n An ordered field is an ordered set with field operations, subject to the compatibility requirements:

- $\forall x, y, z,\ x \leq y \Rightarrow x + z \leq y + z$
- $\forall x, y,\ 0 \leq x$ and $0 \leq y \Rightarrow 0 \leq xy$

We call elements $x \in F$ which satisfy $0 \leq x$ nonnegative.

Notation $x < y$ means $x \leq y$ and $x \neq y$

Observe Inside any ordered field $(F, \leq, +, \cdot)$, there is 1, $1+1$, $1+1+1$, etc. So F contains a copy of $\mathbb{N}$. And a copy of $\mathbb{Z}$. And a copy of $\mathbb{Q}$.

"$\mathbb{Q}$ is the smallest ordered field."

Now we're ready to prove some facts about ordered fields.

Proposition In an ordered field, $-0=0$

Proof $0+0=0$, so 0 is an additive inverse of itself.

Proposition In an ordered field $x\cdot 0=0$ for any x.

Proof
$$
\begin{aligned}
x\cdot 0 &= x\cdot(0+0) = x\cdot(0-0)\\
&= x\cdot 0-(x\cdot 0)\\
&= x\cdot 0 + -(x\cdot 0) = 0 \quad \blacksquare
\end{aligned}
$$

Proposition In an ordered field, $0<1$.

Proof Suppose otherwise, i.e. that $1\le 0$

By assumption, $0\neq 1$, so we have $1<0$.

Then $-1+1<-1+0$

$$0<-1$$

Then $(-1)\cdot(-1)$ is the product of positive numbers hence positive.

By a homework problem, $(-1)\cdot(-1)=1$.

Thus $0<1$. But we assumed $1<0$. $\rightarrow\leftarrow$ $\blacksquare$

Bounds

Def'n In an ordered set $(S,\le)$, we say $b\in S$ is an upper bound for a subset $A\subset S$ if, for every $a\in A$, we have $a\le b$.

Note that b need not be an element of A, only of S.

b is a lower bound for A if, for every $a\in A$, $b\le a$.

Fact Every subset of $\mathbb{N}$ has a lower bound.

$\forall A\subset\mathbb{N}\ \exists b\in\mathbb{N}$ s.t. $\forall a\in A\quad b\le a$.

In fact we can use the same lower bound for all the subsets namely $0\in\mathbb{N}$.

$\exists b\in\mathbb{N}$ s.t. $\forall A\subset\mathbb{N}\quad \forall a\in A\quad b\le a$

Order of quantifiers matters!

Fact
- Not every subset of $\mathbb{N}$ has an upper bound.
  (E.g.: $\mathbb{N}$ itself
  - even natural numbers)
- Not every subset of $\mathbb{Z}$ has an upper bound

- Not every subset of $\mathbb{Z}$ has a lower bound.

Defn An upper bound b for a subset A of an ordered set is called *least* if it is $\le$ any other upper bound for A.

b is a least upper bound if $\forall a \in A,\ a \le b$ and $\forall b' \ni \forall a \in A\ a \le b',\ b \le b'$

A least upper bound for A is called the *supremum* of A

Lemma The least upper bound of a set A is unique (if there is one)

Proof If b_1 and b_2 are LUBs, then $b_1 \le b_2$ and $b_2 \le b_1$. ∎

So we write $b = \sup A$ unambiguously.

Fact Every nonempty subset of $\mathbb{Z}$ which has an upper bound has a least upper bound.

Proof. If $A \subset \mathbb{Z}$ has an upper bound b', do the following:
- If $b' \in A$, then $b' = \sup A$
- If $b' \notin A$, try $b'-1$.

The process terminates after at most $b'-a$ steps, where a is any element of A.

Defn We say an ordered set S has the *least upper bound property* if any nonempty subset $A \subset S$ which has an upper bound has a least upper bound.

E.g. $\mathbb{N}$ and $\mathbb{Z}$ have the LUB property.
(Note this does not mean $\mathbb{N}$ or $\mathbb{Z}$ has a least upper bound!)

E.g. $\mathbb{Q}$ does not have the LUB property!

Proof Let $A = \{q \mid q^2 \le 2\} \subset \mathbb{Q}$.
Consider $B = \{q \mid q^2 \ge 2\}$.
Given $q \in \mathbb{Q}$ define p by:

$$p = \frac{2q+2}{q+2} =$$

Then

$$p^2 - 2 = \frac{(2q+2)^2}{(q+2)^2} - 2 = \frac{(2q+2)^2 - 2(q+2)^2}{(q+2)^2} = \frac{4q^2+8q+4-(2q^2+8q+8)}{(q+2)^2} = \frac{2(q^2-2)}{(q+2)^2}$$

So $p^2 < 2 \iff q^2 < 2$
$p^2 > 2 \iff q^2 > 2$

On the other hand,

$$p = \frac{2q+2}{q+2} = q - \frac{q^2-2}{q+2}$$

So $p > q \Leftrightarrow q^2 < 2$

$q > p \Leftrightarrow q^2 > 2$.

Thus no q with $q^2 < 2$ can be an upper bound for A.

So any upper bound b for A has $b^2 \geq 2$.

Elements of B are all upper bounds for A. (Homework)

So to be least, an upper bound b must be smaller than every elt of B, hence have $b^2 \leq 2$.

So the only LUB for A is one with $b^2 = 2$

But There is no rational number b with $b^2 = 2$.

So A has an upper bound but no least upper bound

Therefore $\mathbb{Q}$ lacks the LUB property.

Theorem There is an ordered field with the LUB property.

Proof: Complicated and not very enlightening.

One version is in Appendix of Rudin's chapter 1.

Theorem There is only one ordered field with the LUB property.

That is, if I construct two such fields, there is a way to identify them. We'll see the proof much later.

Def'n The real numbers are the unique ordered field with the LUB property. We write $\mathbb{R}$.

Def'n The extended real numbers are $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ with the ordering $-\infty \leq x \leq \infty$ for all $x \in \mathbb{R}$.

Prop'n If $A \subset \mathbb{R}$ is not bounded above, then $\sup A = \infty$.

Proof. "A is not bounded above": $\forall b \in \mathbb{R}, \exists a \in A$ s.t. $b < a$.

So if $\sup A \geq a$ for all $a \in A$, we must have $\sup A \geq b$ for all $b \in \mathbb{R}$. ∎

Prop'n $\sup \emptyset = -\infty$

Proof Any real number is an upper bound for $\emptyset$. So $\sup \emptyset$ must be smaller than every real number. ∎

Sequences in an Ordered Field

Def'n A **sequence** (in an ordered field) is an assignment, to each $n \in \mathbb{N}$, of an element x_n.

E.g. $x_n = n^2$ formula

$x_0 = 1$, $x_{k+1} = x_k + 1$ recurrence relation

Def'n The **absolute value** of an element x of an ordered field is $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Def'n A sequence $(x_n)_{n \in \mathbb{N}}$ **converges to** x if: for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N$ guarantees $|x_n - x| < \varepsilon$

Notation $\lim_{n \to \infty} x_n = x$ or $x_n \to x$

Here ε, each x_n, and x are all elements of the ordered field.

Def'n A sequence $(x_n)_{n \in \mathbb{N}}$ is **monotone** (increasing) if for each $n \in \mathbb{N}$, $x_n \leq x_{n+1}$

Def'n An ordered field has the **monotone sequence property** if every monotone sequence which is bounded above converges.

Theorem An ordered field with the least upper bound property has the monotone sequence property.

Proof.

To show MS holds let $(x_n)_{n \in \mathbb{N}}$ be a monotone sequence which is bounded above.

Then the set $\{x_n\}_{n \in \mathbb{N}}$ is a set which is bounded above, so by LUB $\exists$ $b = \sup\{x_n\}_{n \in \mathbb{N}}$

Claim $x_n \to b$

Proof To show $x_n \to b$, we must ensure, for any given $\varepsilon > 0$, that $|x_n - b| < \varepsilon$. So for our given $\varepsilon > 0$...

Notice $x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots \leq b$

So $b - x_n \geq 0$ and $b - x_n \geq b - x_{n+1}$.

Thus if a particular n has $b - x_n < \varepsilon$, so do all $k \geq n$.

So if $x_n \not\to b$, it must be the case that, for all n,

$b - x_n = |b - x_n| \geq \varepsilon$ for our given ε.

Then $b \geq \varepsilon + x_n$

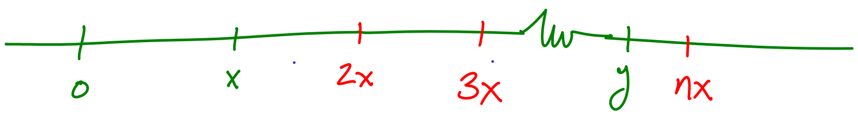
$b - \varepsilon \geq x_n$

So $b - \varepsilon$ is an upper bound for $\{x_n\}$. But b was the least upper bound, and $b - \varepsilon < b$. $\rightarrow\leftarrow$

(MS $\Rightarrow$ LUB)

To show LUB, suppose A is a set with some upper bound M. We want to find $\sup A$.

Defn An ordered field has the Archimedean property if for any x, y with $0 < x < y$, there is $n \in \mathbb{N}$ so that $y < nx$.



Theorem An ordered field with the MS property has the Archimedean property.

Proof Given $0 < x < y$, consider the sequence $(nx)_{n \in \mathbb{N}}$. This sequence is monotone.

If the Archimedean property failed, $y \geq nx$ for all n.

So $(nx)_{n \in \mathbb{N}}$ is bounded above, hence by MSP

$nx \to L$ for some limit L. Then for any $\varepsilon > 0$, we could find n so that, for any $k \in \mathbb{N}$,

$L - nx < \varepsilon$ and $L - (n+k)x < \varepsilon$

Then $kx = (n+k)x - nx \leq L - nx < \varepsilon$

Choosing $\varepsilon = x$ and $k = 2$ gives a contradiction.

Cor. In an ordered field with the MS property, for any $0 < x$, there is $n \in \mathbb{N}$ with $0 < \frac{1}{n} < x$.

Proof Case: $x \geq 1$. Then any n will do.

Case: $x < 1$. Apply Archimedean property to $0 < x < 1$ to get n with $nx > 1$.

Then $x > \frac{1}{n}$.

Proposition In an Archimedean field, $\frac{1}{n} \to 0$.

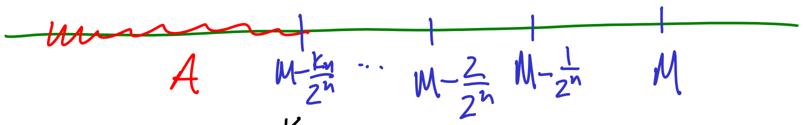
Short proof, just the definitions.

Theorem An ordered field with the MS property has the LUB prop.

Proof. Given A a subset of the field, with some upper bound M, define the sequence b_n as follows:

for each $n \in \mathbb{N}$, consider $M - \frac{k}{2^n}$ for $k \in \mathbb{N}$.

Define k_n to be the first k for which $M - \frac{k}{2^n}$ is not an upper bound for A.



Define $b_n = M - \frac{k_n}{2^n}$.

Note that by our choice of k_n, $b_n + \frac{1}{2^n}$ is an upper bound for A.

Claim $b_n \leq b_{n+1}$ (i.e. $(b_n)_{n \in \mathbb{N}}$ is monotone.)

Proof. $M - \frac{2k_n}{2^{n+1}} = M - \frac{k_n}{2^n}$

So $k_{n+1} \leq 2k_n$

So $\underbrace{M - \frac{k_{n+1}}{2^{n+1}}}_{= b_{n+1}} \geq M - \frac{2k_n}{2^{n+1}} = \underbrace{M - \frac{k_n}{2^n}}_{= b_n}$ ■

$(b_n)_{n \in \mathbb{N}}$ is bounded above by M, so by MSP, $b_n \to b$.

Claim $b = \sup A$.

Proof Notice each b_n is chosen to be smaller than any upper bound of A.

Therefore, given any upper bound b', the condition

$$b_n \leq b' \text{ gives } b = \lim b_n \leq b'.$$

So b is least.

To see that b is an upper bound, consider $c_n = b_n + \frac{1}{2^n}$.

Then $\lim c_n = \lim b_n + \frac{1}{2^n} = \lim b_n + \lim \frac{1}{2^n}$

$$= b + 0 = b.$$

Each $a \in A$ has $a \leq c_n$, so $a \leq b$.

Thus b is an upper bound for A. ■

We've just shown LUB $\Leftrightarrow$ MSP.

Def'n An ordered field with either LUB or MSP is called complete.

The problem with $\mathbb{Q}$ is that $\mathbb{Q}$ is not complete.
Can you think of a sequence in $\mathbb{Q}$ which is monotone and bounded, but does not converge?

$(1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots)$

Theorem (Useful characterisation of supremum)
Given $A \subset \mathbb{R}$ a nonempty set which is bounded above.
Then $\alpha = \sup A$ iff

- α is an upper bound for A
- $\forall \varepsilon > 0 \ \exists a_\varepsilon \in A$ with $\alpha - \varepsilon < a_\varepsilon \leq \alpha$

Proof Assume $\alpha = \sup A$. Then α is an upper bound for A.
Suppose the theorem fails. Then $\exists \varepsilon > 0$ s.t. $\forall a \in A$, $a \leq \alpha - \varepsilon$. So $\alpha - \varepsilon$ is an upper bound, and $\alpha - \varepsilon < \alpha$. $\rightarrow\leftarrow$

Now assume α satisfies the two conditions.
If b is some upper bound for A, then for all $a \in A$, $b \geq a$. In particular, for each $\varepsilon > 0$, we have $\alpha - \varepsilon < a_\varepsilon \leq b$. Taking $\varepsilon = \frac{1}{n}$, we have $b > \alpha - \frac{1}{n}$,
so $\lim b \geq \lim \alpha - \lim \frac{1}{n}$
$b \geq \alpha$. So α is least. ∎

Proposition (Limit laws)
Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences with limits x and y respectively. Then:

1) $\lim (x_n + y_n) = x + y$
2) $\lim (x_n y_n) = xy$
3) If no $y_n = 0$ and $y \neq 0$, then $\lim y_n^{-1} = y^{-1}$
4) $\lim \sqrt[p]{x_n} = \sqrt[p]{x}$

Proof: Should have seen first two already.

1) Given $\varepsilon > 0$, consider $|(x_n + y_n) - (x + y)|$
$$= |x_n - x + y_n - y| \underset{\text{homework}}{\leq} |x_n - x| + |y_n - y|.$$
Since $x_n \to x$ and $y_n \to y$,
there is K s.t. $n \geq K \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$.
Similarly, there is M s.t. $n \geq M \Rightarrow |y_n - y| < \frac{\varepsilon}{2}$.
So take $N = \max\{M, K\}$. Then $n \geq N$ guarantees
$$|x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

2) Similar to 1)
3) Given $\varepsilon > 0$,
We want to control $|y_n^{-1} - y^{-1}| = |y y^{-1} y_n^{-1} - y_n y_n^{-1} y^{-1}|$

$$= |(y - y_n) y^{-1} y_n^{-1}| = |y - y_n|\,|y^{-1}|\,|y_n^{-1}|$$

Since $y_n \to y$, there is K so that $n \geq K \Rightarrow |y_n - y| < \frac{1}{2}|y|$

$$|y| - |y_n| < |y_n - y| < \tfrac{1}{2}|y|$$

$$-|y_n| < -\tfrac{1}{2}|y|$$

$$|y_n| > \tfrac{1}{2}|y|.$$

Also, there is M so that $n \geq M \Rightarrow |y_n - y| < \frac{1}{2}|y|^2\varepsilon$.

Taking $N = \max\{M, K\}$, if $n \geq N$, we have

$$|y_n^{-1} - y^{-1}| = |y - y_n|\,|y^{-1}|\,|y_n^{-1}|$$

$$< \tfrac{1}{2}|y|^2\varepsilon\,|y^{-1}|\,2\,|y| = \varepsilon$$

4) You do!

Henceforth we may assume that limits and algebraic operations commute.

Def'n A subsequence of $(x_n)_{n\in\mathbb{N}}$ is a choice, for each $k \in \mathbb{N}$, of index $n_k \in \mathbb{N}$, so that $n_k < n_{k+1}$. (That is, a subsequence is an increasing function from $\mathbb{N}$ to $\mathbb{N}$.)

E.g. $x_n = n^2$, $K_n = 2K$

Then we have a sequence

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	
x_n:	0	1	4	9	16	25	36	...
x_{n_k}:	0 ($k=0$, $n_k=0$)		4 ($k=1$, $n_k=2$)		16 ($k=2$, $n_k=4$)		36 ($k=3$, $n_k=6$)	...

Note that a subsequence is a sequence in its own right.

Def'n $x \in \mathbb{R}$ is a cluster point of the sequence (x_n) if for any $\varepsilon > 0$, there are infinitely many $n \in \mathbb{N}$ with $|x_n - x| < \varepsilon$.

E.g. x_n: $0 \quad 1 \quad \frac{1}{2} \quad 1 \quad \frac{1}{4} \quad \frac{1}{5} \quad 1 \quad \frac{1}{7} \quad \frac{1}{8} \quad 1 \quad \frac{1}{10} \quad \frac{1}{11} \quad 1 \quad \ldots$

Cluster points are 0 and 1.

Proposition Given a sequence $(x_n)_{n\in\mathbb{N}}$ and $x\in\mathbb{R}$,

i) x is a cluster point of $(x_n)_{n\in\mathbb{N}}$ iff for any $\varepsilon>0$ and all $N\in\mathbb{N}$, there is $n>N$ with $|x_n-x|<\varepsilon$

ii) x is a cluster point of $(x_n)_{n\in\mathbb{N}}$ iff there is a subsequence of x_n which converges to x.

iii) $x_n\to x$ iff <u>every</u> subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

iv) $x_n\to x$ iff $(x_n)_{n\in\mathbb{N}}$ is bounded and x is the only cluster point of $(x_n)_{n\in\mathbb{N}}$.

Proof i) is just the definition of "infinite".

ii) The subsequence is given by picking indices:

Let $n_0=0$. For $k\geq1$, choosing $\varepsilon=\frac{1}{k}$ and $N=n_{k-1}$, there is n with $n>n_{k-1}$

$|x_n-x|<\frac{1}{k}$.

Use $n_k=n$.

Claim: $(x_{n_k})_{k\in\mathbb{N}}$ converges to x.

iv) of the above says to see if a limit exists, we need to investigate its cluster points.

Defn If $(x_n)_{n\in\mathbb{N}}$ is bounded above, define its limit superior of $(x_n)_{n\in\mathbb{N}}$ as the supremum of the set of its cluster points.

If $(x_n)_{n\in\mathbb{N}}$ is bounded below its limit inferior is the infimum of its cluster points

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, write $\limsup x_n=\overline{\lim}\, x_n=\infty$

If $(x_n)_{n\in\mathbb{N}}$ is not bounded below

$$\liminf x_n=\underline{\lim}\, x_n=-\infty$$

E.g. $x_n: 1,2,3,\frac{1}{4},5,6,\frac{1}{7},8,9,\frac{1}{10},11,\dots$

$\underline{\lim}\, x_n=\inf\{0\}=0$

$\overline{\lim}\, x_n=\infty$

Proposition $\lim x_n=x$ iff $\underline{\lim}\, x_n=\overline{\lim}\, x_n$

Theorem Every sequence in $\mathbb{R}$ has a monotone (either increasing or decreasing) subsequence.

Proof Given a sequence $(x_n)_{n\in\mathbb{N}}$, we say a term x_k has an open horizon if $x_k > x_n$ for all $n \geq k$.

Case. There are infinitely many terms with open horizons. Then they must be arranged

$$x_{n_0} > x_{n_1} > x_{n_2} > \cdots$$

with $n_0 < n_1 < n_2 < \cdots$.

Case There are finitely many terms with open horizon. Then there is some M so that $n \geq M$ implies x_n does not have open horizon.

So for each n $\exists\, m > n$ with $x_m \geq x_n$.

Take $n_0 = M$, $n_1 = m$, etc. ▨

Note The same sequence could have monotone increasing and monotone decreasing subsequences.

Bolzano-Weierstrass Theorem (Our first real Theorem)

If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, ie. $x_n \in [-B,B]$ for all $n\in\mathbb{N}$, then (x_n) has a convergent subsequence.

Proof By previous theorem, there is some monotone subsequence $(x_{n_k})_{k\in\mathbb{N}}$. This subsequence is bounded, hence converges by MSP. ▨

Cauchy Sequences

Recall the definition of convergence:

$$x_n \to x \text{ if } \forall \varepsilon>0\ \exists N \ni n>N \Rightarrow |x_n - x| < \varepsilon$$

This seems to require knowledge of the limit x. What if we don't know the limit (e.g. as in HW 1) but we want to investigate convergence? If the sequence is monotone & bounded we use MSP. Otherwise, what to do?

Defn A sequence $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* if for all $\varepsilon>0$ there is $N\in\mathbb{N}$ so that $m,n\geq N$ guarantees

$$|x_m - x_n| < \varepsilon.$$

Theorem Convergent sequences are Cauchy.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges to x. Given $\varepsilon>0$, choose N so that $n\geq N$ guarantees $|x_n - x| < \frac{\varepsilon}{2}$.

Then if $n,m\geq N$, we have:

$$|x_m - x_n| = |x_m - x + x - x_n|$$
$$\leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So this N works, and $(x_n)_{n\in\mathbb{N}}$ is Cauchy. $\boxtimes$

Theorem (Homework) Cauchy sequences converge.

Theorem The Cauchy sequences in $\mathbb{R}$ are exactly the convergent sequences.

Proposition Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence with $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$. Then $(x_n)_{n\in\mathbb{N}}$ converges.

Proof. We'll show $(x_n)_{n\in\mathbb{N}}$ is Cauchy. For $m>n$,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n|$$
$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$\leq \frac{1}{2}|x_{m-1} - x_{m-2}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$\leq \frac{1}{4}|x_{m-2} - x_{m-3}| + \frac{1}{2}|x_{m-2} - x_{m-3}| + \cdots + |x_{n+1} - x_n|$$
$$\leq \left(\frac{1}{2^{m-n}} + \frac{1}{2^{m-n-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2}\right)|x_n - x_{n-1}|$$

$$\leqslant \left(\sum_{k=1}^{m-n} \frac{1}{2^k}\right) \frac{1}{2^{n-1}} |x_1 - x_0|$$

Claim $\sum_{k=1}^{m-n} \frac{1}{2^k} < 1$

Claim $\frac{1}{2^{n-1}} \to 0$.

So given $\varepsilon > 0$, there is N so that $n \geqslant N$ guarantees $\frac{1}{2^{n-1}} < \frac{\varepsilon}{|x_1 - x_0|}$.

Then $m, n \geqslant N$ guarantees $|x_m - x_n| < \varepsilon$.

So $(x_n)_{n\in\mathbb{N}}$ is Cauchy, hence converges. ∎

Note: This proof works for any coefficient $A < 1$

Proposition A sequence with $|x_{n+1} - x_n| \leqslant A|x_n - x_{n-1}|$, for some $A<1$ converges.

The only difference is to check that $\sum_{k=1}^{m-n} A^k$ is bounded independent of m and n and that $A^{n-1} \to 0$.

Important Example

Consider the sequence $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Then $x_{n+1} - x_n = \frac{1}{n+1}$, which goes to 0 as $n \to \infty$.

Nevertheless, $(x_n)_{n\in\mathbb{N}}$ does not converge, and isn't Cauchy:

$$x_m - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}$$

(To see that $(x_n)_{n\in\mathbb{N}}$ does not converge, consider

$$x_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{2^k}$$
$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{16} + \dots$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$$
$$= 1 + \frac{1}{2}k$$

Clearly the sequence $(1 + \frac{1}{2}k)_{k\in\mathbb{N}}$ is not bounded, hence the subsequence $(x_{2^k})_{k\in\mathbb{N}}$ is not bounded, hence $(x_n)_{n\in\mathbb{N}}$ is not bounded, hence $(x_n)_{n\in\mathbb{N}}$ diverges.)

The Complex Field

Algebraists are interested in $\mathbb{C}$ because it is "algebraically closed."

Def'n A complex number is a formal sum $z = a + ib$ where $a, b \in \mathbb{R}$.

Addition and multiplication of complex numbers are defined by requiring the distributive law hold, and that $i^2 = -1$ and $0i = 0$.

Proposition $\mathbb{C}$ so equipped is a field.

Proof

- $(a+ib) + ((-a) + i(-b))$
$$= a - a + i(b-b) = 0 + 0i$$
- $(a+ib)\cdot\left(\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right)$
$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} + i\left(\frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2}\right) = 1$$

$\blacksquare$

Def'n We call a the real part and b the imaginary part of $z = a+ib$.

Def'n The conjugate of $z = a+ib$ is
$$\bar{z} = a - ib.$$

$$z\bar{z} = (a+ib)(a-ib) = a^2+b^2$$ ← This is a real number! It's ≥ 0.

Def'n The modulus of $z = a+ib$ is
$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}.$$

Proposition $|z| = 0$ iff $z = 0 \in \mathbb{C}$.

Proof. Suppose $a^2+b^2 = 0$.

Then $a^2+b^2 \geq a^2 \geq 0$ gives $a^2 = 0$.

Sim $b^2 = 0$. So $a = b = 0$, and

$z = a+ib = 0 + i\cdot 0 = 0$.

Suppose $a = b = 0$. Then $|z| = \sqrt{0} = 0$.

Proposition

- $\mathrm{Re}\, z \leq |z|$
- $\mathrm{Im}\, z \leq |z|$
- $|zw| = |z||w|$

These are inequalities of real numbers!

Proof.

- If $z = a+ib$, $(\mathrm{Re}\, z)^2 = a^2 \leq a^2+b^2 = |z|^2$
- Sim. for $\mathrm{Im}\, z = b$
- $|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}\, w\bar{w}$
$$= |z|^2|w|^2$$

Proposition $|z+w| \leq |z|+|w|$ Triangle inequality for $\mathbb{C}$

Proof $|z+w|^2 = (z+w)(\overline{z}+\overline{w})$

$$= z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w}$$
$$= |z|^2 + |w|^2 + z\overline{w} + \overline{z\overline{w}}$$
$$= |z|^2 + |w|^2 + 2\mathrm{Re}(z\overline{w})$$
$$\leq |z|^2 + |w|^2 + 2|z\overline{w}|$$
$$= |z|^2 + |w|^2 + 2|z|\,|w|$$
$$= (|z|+|w|)^2 \quad \blacksquare$$

Defn A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ converges to z if for all $\varepsilon>0$ there is some N so that $n \geq N$ guarantees $|z_n - z| < \varepsilon$

Notice this is formally identical to the definition of convergence for real sequences, but the symbols have new meanings:

n	still $\in \mathbb{N}$	$\lvert\cdot\rvert$	complex modulus
ε	still $\in (0,\infty) \subset \mathbb{R}$	$-$	complex subtraction
		z_n, z	complex numbers

Q. Does $\mathbb{C}$ have the sorts of holes that $\mathbb{Q}$ does?

To answer this question, we'll try to formulate it more clearly. Since $\mathbb{C}$ doesn't have an ordering compatible with its field structure (That's a homework problem) we can't talk about LUB or MSP.

Defn A complex sequence is Cauchy if for any $\varepsilon>0$ there is N such that $m,n \geq N$ guarantees $|z_m - z_n| < \varepsilon$.

Theorem $\mathbb{C}$ is Cauchy-complete; that is, the Cauchy sequences in $\mathbb{C}$ are precisely the convergent sequences

The proof uses the following:

Proposition $(a_n + ib_n)_{n\in\mathbb{N}}$ converges to $a+ib$ iff $a_n \to a$ and $b_n \to b$.

Proof $(\Rightarrow)$ $|a_n + ib_n - (a+ib)|^2 = (a_n-a)^2 + (b_n-b)^2$

$$\geq (a_n-a)^2$$

So if $a_n + ib_n \to a + ib$, for our given $\varepsilon > 0$, there is N so that $n \geq N$ guarantees

$$|a_n - a| \leq |a_n + ib_n - (a+ib)| < \varepsilon.$$

Thus $a_n \to a$. Similarly for $b_n \to b$.

($\Leftarrow$) If $a_n \to a$ and $b_n \to b$, there are N_1, N_2 so that
$$n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{\sqrt{2}}$$
$$n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{\sqrt{2}}$$

So choosing $N = N_1 + N_2$, $n \geq N$ guarantees

$$|a_n + ib_n - (a+ib)|^2 = |a_n - a|^2 + |b_n - b|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$$

So $|a_n + ib_n - (a+ib)| < \varepsilon$. ∎

Proposition $(a_n + ib_n)_{n\in\mathbb{N}}$ is Cauchy iff $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy.

(Basically the same proof.)

Euclidean Spaces (And other algebraic objects we'll do analysis on...)

Defn Euclidean n-space is

$$\mathbb{R}^n = \{(x^1, \dots, x^n) \mid x^i \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$$

$\mathbb{R}^n$ has operations:

$$x = (x^1, \dots, x^n) \qquad y = (y^1, \dots, y^n) \qquad \lambda \in \mathbb{R}$$

$$x + y = (x^1 + y^1, \dots, x^n + y^n)$$

$$\lambda x = (\lambda x^1, \dots, \lambda x^n)$$

Proposition
- $x + y = y + x$
- $\lambda(x+y) = \lambda x + \lambda y$ and $(\lambda + \mu)x = \lambda x + \mu x$
- $(\lambda\mu)x = \lambda(\mu x)$
- $(x+y)+z = x + (y+z)$
- $1 \cdot x = x$
- $0 = (0, \dots, 0)$ has $0 + x = x$ for all $x \in \mathbb{R}^n$
- $(-1) \cdot x + x = 0$

That is, $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$.

Def'n Define the scaler product or inner product on $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i$$

Proposition. For any $x, y, z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, we have:

- $\langle x,y\rangle = \langle y,x\rangle$
- $\langle x,x\rangle \geq 0$, with equality exactly when $x=0$.
- $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$
- $\langle \lambda x, y\rangle = \lambda\langle x,y\rangle = \langle x, \lambda y\rangle$

Def'n A $\mathbb{R}$-vector space along with an inner product $\langle\cdot,\cdot\rangle$ satisfying the above proposition is called an inner product space.

Proposition In an inner product space, set $\|x\| = \sqrt{\langle x,x\rangle}$. Then $\|\cdot\|$ satisfies:

- $\|x\| \geq 0$, equality iff $x=0$
- $\|x+y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda|\,\|x\|$

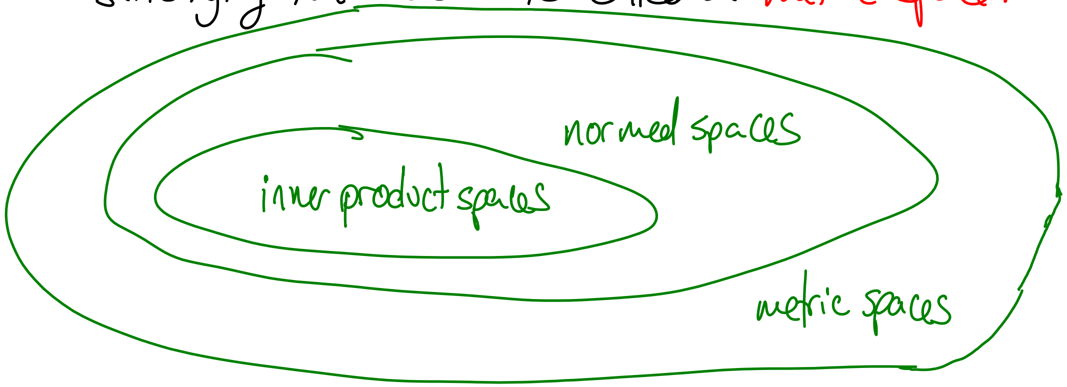
Def'n A vector space along with a norm $\|\cdot\|$ satisfying the above conditions is called a normed space.

Def'n In a normed space, define the metric (or distance function) by $d(x,y) = \|x-y\|$

Proposition The metric in a normed space satisfies:

- $d(x,y) \geq 0$, equality only if $x=y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$ triangle inequality!

Def'n A set M along with a function $d : M \times M \to \mathbb{R}$ satisfying the above is called a metric space.



Why work in such generality?

E.g. Consider the space $\mathcal{C}[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ (forget for the moment that we haven't defined what a continuous function is...)

For any $f, g \in \mathcal{C}[0,1]$, define:

$$(f,g)_{L^2} = \int_0^1 f(x)\, g(x)\, dx.$$

(Again, we haven't defined what $\int dx$ means, but bear with me...)

Define $(f+g)(x) = f(x) + g(x)$

$$(\lambda f)(x) = \lambda\, (f(x)).$$

Then $\mathcal{C}[0,1]$ is a $\mathbb{R}$-vector space. (Again, there's some work to be done here...)

and with $(\cdot,\cdot)_{L^2}$, it's an inner product space.

- $(f,g)_{L^2} = \int_0^1 f(x)\, g(x)\, dx = \int_0^1 g(x) f(x)\, dx = (g,f)_{L^2}$
- $(f,f)_{L^2} = \int_0^1 f(x)\, f(x)\, dx = \int_0^1 (f(x))^2\, dx \geq 0$ (Work to be done here, too...)
- If $(f,f)_{L^2} = \int_0^1 (f(x))^2 dx = 0$, then $f \equiv 0$
- $(f+g, h)_{L^2} = \int_0^1 (f(x)+g(x))\, h(x)\, dx$

$$= \int_0^1 (f(x)h(x) + g(x)h(x))\, dx$$

$$= \int_0^1 f(x)\, h(x)\, dx + \int_0^1 g(x)\, h(x)\, dx$$

$$= (f,h)_{L^2} + (g,h)_{L^2}$$

- $(\lambda f, g) = \int_0^1 \lambda f(x) g(x)\, dx = \lambda \int_0^1 f(x) g(x)\, dx = \lambda (f,g)_{L^2}$

The norm induced by $(\cdot,\cdot)_{L^2}$ is

$$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \left(\int_0^1 (f(x))^2 dx \right)^{1/2}$$

Metric Spaces

Def'n A **metric space** (M,d) is a set M together with a **distance function** or **metric**

$$d : M \times M \to \mathbb{R}$$

which satisfies the conditions for all $x,y,z \in M$

1) $d(x,y) \geq 0$.
2) $d(x,y) = 0$ if and only if $x = y$
3) $d(x,y) = d(y,x)$
4) $d(x,z) \leq d(x,y) + d(y,z)$

Eg.
- $\mathbb{R}$ with the metric $d(x,y) = |x-y|$
- $\mathbb{C}$ with the metric $d(z,w) = |z-w|$
- **Any set** with the **discrete metric**

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Try to draw a picture of the discrete metric...

$M = \{x\}$ $\quad$ $M = \{x,y\}$ $\quad$ $M = \{x,y,z\}$ $\quad$ $M = \{x,y,z,w\}$ (in $\mathbb{R}^3$)

We can't really do any better than drawing the discrete metric on 4 points. But the discrete metric is a metric, even on uncountable sets, since

1) $d_0(x,y)$ is either 0 or 1, both of which are nonnegative
2) If $x \neq y$, then $d_0(x,y) = 1 \neq 0$
3) If $x \neq y$, then $d_0(x,y) = 1 = d_0(y,x)$
   If $x = y$, then $d_0(x,y) = 0 = d_0(x,y)$
4) If $x = z$, then $d_0(x,z) = 0 \leq d_0(x,y) + d_0(y,z)$
   If $x \neq z$, then if $y = z$,
   $d_0(x,z) = 1 \quad d_0(x,y) = 1 \quad d_0(y,z) = 0$
   if $y \neq z$, then all distances are 1, so
   $1 \leq 1 + 1 = 2$ ✓

Another weird e.g.

Given **any** metric space (M,d), we can tweak the metric to the **induced bounded metric**

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Notice that since $d(x,y) < 1 + d(x,y)$

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1$$

(Exercise: Prove (M,ρ) is a metric space.)

The takeaway lesson is: metric spaces can be kinda weird.

Recall A sequence $x_n \to x$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that $n \geq N \Rightarrow |x_n - x| < \varepsilon$.

We could rewrite this as:

$x_n \to x$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that $n \geq N \Rightarrow x_n \in (x-\varepsilon, x+\varepsilon)$

It turns out ~~almost~~ all of our discussion of sequences in $\mathbb{R}$ can be rephrased in terms of little intervals $(x-\varepsilon, x+\varepsilon)$

Def'n In a metric space (M,d), the open ball of radius $R \geq 0$ about the point $x \in M$ is

$$D(x,R) = \{y \in M \mid d(x,y) < R\}$$

(Rudin calls these "neighborhoods" and reserves the term "ball" for the standard metric on $\mathbb{R}^n$.)

Prop'n The open balls in $\mathbb{R}$ are precisely the open intervals.

Proof The open ball $D(x,R)$ in $\mathbb{R}$ is

$$\{y \in \mathbb{R} \mid d(x,y) < R\} = \{y \in \mathbb{R} \mid |x-y| < R\}$$

$$= \{y \in \mathbb{R} \mid -R < x-y < R\} = \{y \in \mathbb{R} \mid -x-R < -y < R-x\}$$

$$= \{y \in \mathbb{R} \mid x+R > y > x-R\} = (x-R, x+R)$$

On the other hand, any open interval

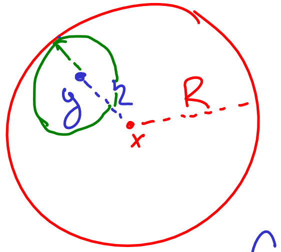
$$(a,b) = \left(\tfrac{1}{2}(a+b) - \tfrac{1}{2}(b-a), \tfrac{1}{2}(a+b) + \tfrac{1}{2}(b-a)\right).$$

$$= D\left(\tfrac{1}{2}(a+b), \tfrac{1}{2}(b-a)\right)$$

Def'n A subset of a metric space $A \subset (M,d)$ is open if for every $x \in A$, there is some $\varepsilon > 0$ such that $D(x,\varepsilon) \subset A$.

Proposition Open balls are open.

Proof Given an open ball $D(x,R)$, we need to show that for any $y \in D(x,R)$, there is $\varepsilon > 0$ with $D(y,\varepsilon) \subset D(x,R)$.



If $y = x$, take $R = \varepsilon$.

If $y \neq x$, note that $\eta = d(x,y) < R$.

So $R - \eta > 0$. Set $\varepsilon = R - \eta$.

Claim $D(y,\varepsilon) \subset D(x,R)$

Proof To show $D(y,\varepsilon) \subset D(x,R)$, suppose $z \in D(y,\varepsilon)$. Then

$$d(y,z) < \varepsilon = R - \eta.$$

So
$$d(x,z) \le d(x,y) + d(y,z) = \eta + d(y,z)$$
$$< \eta + R - \eta = R.$$

So $z \in D(x,R)$. ▨

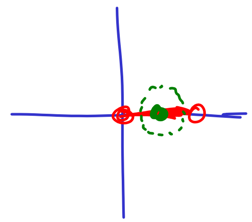
Eg. The open ball in $\mathbb{R}^2$ with the standard metric is an ordinary disc (hence "D")

Q. Is $(0,1)$ open?

A. It depends!

$(0,1) \subset \mathbb{R}$ is the open ball of radius $\frac{1}{2}$ centred at $\frac{1}{2}$. (hence open in $(\mathbb{R}, |\cdot - \cdot|)$)

But of course we could also view $(0,1)$ as a subset of $\mathbb{R}^2$ with the standard metric:



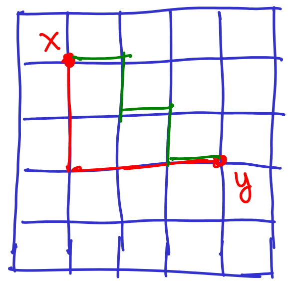
Any open ball about a point $(x,0)$ contains points with $x^2 > 0$ and points with $x^2 < 0$. So $(0,1)$ is not open as a subset of $\mathbb{R}^2$ with the standard metric.

So remember: Sets are not open; only subsets can be!

One way to understand what a metric space "looks like" is to look at its balls. (As you are asked to do in this week's homework.)

E.g. The taxicab metric on $\mathbb{R}^n$ is defined by:

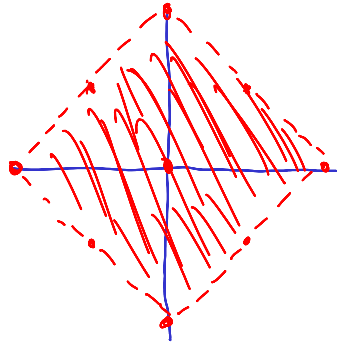
$$d_{taxi}(x,y) = |x^1 - y^1| + \cdots + |x^n - y^n|$$



Many equally-short routes from x to y. Lots of ways to get equality in the triangle inequality.

Balls in the taxicab metric look like:

(Points in the city accessible by a taxi ride of less than R blocks)



Open & Closed Sets

Def'n $A \subset (M,d)$ is open if for all $x \in A$, there is $\varepsilon > 0$ so that $D(x,\varepsilon) \subset A$.

Def'n Given two sets A, B,

their union is $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$

their intersection is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Given a collection of sets A_λ, define

the union $\bigcup_\lambda A_\lambda = \{x \mid \text{there is some } \lambda \text{ with } x \in A_\lambda\}$

the intersection $\bigcap_\lambda A_\lambda = \{x \mid \text{for all } \lambda,\ x \in A_\lambda\}$

$\exists \longleftrightarrow \cup \longleftrightarrow$ "or"

$\forall \longleftrightarrow \cap \longleftrightarrow$ "and"

Proposition Let A_λ, $\lambda \in \Lambda$ be a collection of open subsets of a metric space. Then

1) $\bigcup_\lambda A_\lambda$ is open

2) If there are finitely many λ, $\bigcap_\lambda A_\lambda$ is open.

Proof 1) To show $\bigcup_\lambda A_\lambda$ is open, consider an arbitrary $x \in \bigcup_\lambda A_\lambda$. Then there is some λ with $x \in A_\lambda$.

Since A_λ is open, there is $\varepsilon > 0$ with $D(x,\varepsilon) \subset A_\lambda$.

Then $D(x,\varepsilon) \subset A_\lambda \subset \bigcup_\lambda A_\lambda$ (why?) so this $\varepsilon > 0$ works for $\bigcup_\lambda A_\lambda$, too.

2) To show $A_1 \cap A_2 \cap \cdots \cap A_k$ is open, consider an arbitrary $x \in A_1 \cap A_2 \cap \cdots \cap A_k$. Since each A_i is open and $x \in A_i$, there is $\varepsilon_i > 0$ so that

$$D(x, \varepsilon_i) \subset A_i.$$

Set $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$. Then

$$D(x,\varepsilon) \subset D(x, \varepsilon_i) \subset A_i$$

So (why?) $D(x,\varepsilon) \subset A_1 \cap A_2 \cap \cdots \cap A_k$

Def'n Given two subsets $A, B \subset M$, define the set subtraction operation $A \setminus B = \{x \in A \mid x \notin B\}$.

Note that $A \setminus B$ and $B \setminus A$ are in general very different sets! (Exercise! when is it the case that $A \setminus B = B \setminus A$?)

Def'n The complement of $A \subset M$ is $A^c = M \setminus A$.

Def'n A subset of a metric space is closed if its complement is open. (This is not Rudin's definition but we'll see they're equivalent.)

Proposition (De Morgan's laws)

1) $(A \cup B)^c = A^c \cap B^c$
2) $(A \cap B)^c = A^c \cup B^c$

Proof This is really just the fact that not $\forall \rightsquigarrow \exists$ not and not $\exists \rightsquigarrow \forall$ not

1) Show $(A \cup B)^c \subset A^c \cap B^c$ by showing $(A \cup B)^c \subset A^c$ and $(A \cup B)^c \subset B^c$.

Since the statement is symmetric in A and B, we'll just show $(A \cup B)^c \subset A^c$.

Suppose $x \in (A \cup B)^c$. 2 cases: $x \in A$. Then $x \in A \cup B$, so $x \notin (A \cup B)^c$.

$x \notin A$. Then $x \in A^c$.

Now show $A^c \cap B^c \subset (A\cup B)^c$. If $x \in A^c \cap B^c$, then $x \notin A$ and $x \notin B$. 2 cases: $x \in A\cup B$. Then either $x\in A$ or $x \in B$.

$x \notin A \cup B$. Then done.

2) is similar.

Proposition In any metric space (M,d), ϕ and M are open.

Proof Show $M \subset M$ is open. Let $x \in M$, and choose $\varepsilon = 1$.

Then $D(x,1) = \{y \in M \mid d(x,y) < 1\} \subset M$.

To see that $\phi \subset M$ is open, recall that A open means:

for any $x \in A$, there is $\varepsilon > 0$ with $D(x,\varepsilon) \subset A$.

But this is true of the empty set since there aren't any such $x \in \phi$!

Corollary In any metric space (M,d), ϕ and M are closed.

Proof To see that ϕ is closed, show ϕ^c is open.

But $\phi^c = M$, which by the proposition is open.

To see that M is closed, show M^c is open.

But $M^c = \phi$, which by the proposition is open.

Defn A subset which is both open and closed is called clopen. (Yes, really.)

Eg: Consider $(\mathbb{Q}_+, d_{std})$ (Q: Is any subset of a metric space also a metric space?)

The set $A = \{q \in \mathbb{Q}_+ \mid q^2 < 2\}$ is open since for any $q \in A$, there is some $p \in A$ with $p > q$.

Since in an ordered field we can square both sides of an inequality, the rational interval

$$(0,p) = \{r \in \mathbb{Q} \mid 0 < r < p\} \subset A.$$

If we set $\varepsilon = \min\{q, p-q\} > 0$, then $D(q,\varepsilon) \subset (0,p) \subset A$.

Therefore A is open.

On the other hand, $A^c = \{q \in \mathbb{Q}_+ \mid q^2 \geq 2\}$

$= \{q \in \mathbb{Q}_+ \mid q^2 > 2\}$ (since no rational has $q^2 = 2$)

A similar argument shows A^c is open.

So A is closed.

Luckily for us, clopen sets are relatively rare in well-behaved metric spaces.

Def'n Given $A \subset (M,d)$, $x \in A$ is an interior point of A if there is some $\varepsilon > 0$ so that $D(x,\varepsilon) \subset A$.

Restatement of the definition of "open"

A is open if every point of A is an interior point of A.

Def'n The interior of $A \subset (M,d)$ is the set of all interior points of A, i.e. $\operatorname{int}(A) = \{x \in A \mid x \text{ is an interior point of } A\}$.

Notice that $\operatorname{int} A \subset A$ by definition.

Proposition Let $A \subset (M,d)$, $x \in A$, and suppose there is an open U with $x \in U$, $U \subset A$. Then $x \in \operatorname{int} A$.

Proof Since U is open and $x \in U$, there is $\varepsilon > 0$ with $D(x,\varepsilon) \subset U$. Since $U \subset A$, $D(x,\varepsilon) \subset A$. So x is interior to A. ∎

This sort of thing happens all the time — for many definitions that involve "there is $\varepsilon > 0$ so that $D(x,\varepsilon)\ldots$" we might as well have said "there is an open set U so that $x \in U$ and ...".

The reason to define "interior" is that not all subsets are open, and we'd like to somehow measure "how open" a given subset is. This is a general theme in the course (and more broadly...)

Proposition $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$

Proof ($\operatorname{int} A \cap B \subset \operatorname{int} A \cap \operatorname{int} B$)

If $x \in \operatorname{int} A \cap B$, then there is some open $U \subset A \cap B$ with $x \in U$. Since $A \cap B \subset A$, $U \subset A$. So $x \in \operatorname{int} A$. Sim. $x \in \operatorname{int} B$. So $x \in \operatorname{int} A \cap \operatorname{int} B$. ∎

($\operatorname{int} A \cap \operatorname{int} B \subset \operatorname{int} A \cap B$)

If $x \in \operatorname{int} A$, there is U_A open with $x \in U_A$, $U_A \subset A$.
If $x \in \operatorname{int} B$, there is U_B open with $x \in U_B$, $U_B \subset B$.
Then $U_A \cap U_B \subset A \cap B$ is open, and $x \in U_A \cap U_B$.
So $x \in \operatorname{int} A \cap B$.

E.g. • What is the interior of $\mathbb{R} \setminus \mathbb{Q} \subset (\mathbb{R}, d_{std})$?
Given $x \in \mathbb{R}$, for any $\varepsilon > 0$, there is $q \in \mathbb{Q}$ with $x - \varepsilon < q < x + \varepsilon$. So every interval in $\mathbb{R}$ contains

some rational numbers! Thus $\mathrm{int}(\mathbb{R}\setminus\mathbb{Q}) = \emptyset$.

- What is the interior of $\mathbb{Q}$?

  Given any $x \in \mathbb{R}$ and $\varepsilon > 0$, there is $r \notin \mathbb{Q}$ with $x - \varepsilon < r < x + \varepsilon$. So every interval in $\mathbb{R}$ contains some irrational number! Thus $\mathrm{int}(\mathbb{Q}) = \emptyset$

Just like we measure how open a set is with its interior points, we measure how closed a set is by:

Defn $x \in M$ is a limit point for $A \subset M$ if every open set $U \subset M$ which contains x also contains some $a \in A$ with $x \neq a$.

Note: x need not be an element of A to be a limit point for A.

E.g. Every $x \in \mathbb{R}$ is a limit point for $\mathbb{Q} \subset \mathbb{R}$.
Every $x \in \mathbb{R}$ is a limit point for $\mathbb{R}\setminus\mathbb{Q} \subset \mathbb{R}$.

Theorem Suppose $A \subset (M,d)$ and x is a limit point for A. Then for any $\varepsilon > 0$, $D(x,\varepsilon)$ contains infinitely many points of A.

Proof Suppose there were some $\varepsilon > 0$ with
$$D(x,\varepsilon) \cap A = \{a_1, \ldots, a_n\}$$
Letting $\eta = \frac{1}{2}\min\{d(x,a_i) \mid a_i \neq x\}$, we see that $D(x,\eta)$ is an open set containing x, but which does not contain any point of A, except possibly x itself. So x is not a limit point for A. ∎

Corollary Finite subsets have no limit points.

Theorem A subset $A \subset (M,d)$ is closed iff it contains all of its limit points.

Proof ($\Rightarrow$) Suppose A is closed. Then $M\setminus A$ is open. Let x be a limit point for A.

2 cases:
- $x \in A$ Then we're done.
- $x \notin A$. Since $M\setminus A$ is open, there is $\varepsilon > 0$ with $D(x,\varepsilon) \subset M\setminus A$. But this contradicts

the assumption that x is a limit point for A. So this case doesn't occur. ▨

($\Leftarrow$) To show A is closed, show $M\setminus A$ is open.
Let $x \in M\setminus A$. Since A contains all its limit points x can't be a limit point. So $\exists \varepsilon>0$ with $D(x,\varepsilon)\cap A$ containing no points of A, except possibly x. By assumption $x\notin A$, so $D(x,\varepsilon)\cap A=\phi$, i.e. $D(x,\varepsilon)\subset M\setminus A$.
Thus $M\setminus A$ is open. ▨

Def'n The closure of $A\subset(M,d)$ is $\mathrm{cl}(A)=A\cup\{x \mid x \text{ is a limit point for } A\}$

Prop'n
1) $A\subset \mathrm{cl}(A)$
2) $\mathrm{cl}(A)$ is closed
3) $A=\mathrm{cl}(A)$ iff A is closed.

Proof 1) is clear from defn.

2) Consider $M\setminus \mathrm{cl}A=\left\{x\in M \mid x\notin A \text{ and } x \text{ is not a limit point for } A\right\}$

Let $x\in M\setminus \mathrm{cl}A$.

Then $\exists\varepsilon>0$ so that $D(x,\varepsilon)\cap A=\phi$
(same argument as in the previous theorem.)

Claim $D(x,\varepsilon)\subset M\setminus \mathrm{cl}A$

Proof We already have that no point of $D(x,\varepsilon)$ lies in A; what remains to be shown is that no point of $D(x,\varepsilon)$ is a limit point for A.
Let $y\in D(x,\varepsilon)$ and let $\eta>0$ be such that $D(y,\eta)\subset D(x,\varepsilon)$. No point of $D(y,\eta)$ lies in A, (not even y), so y is not a limit point for A. ▨

3) ($\Rightarrow$) Since $\mathrm{cl}A$ is closed, $A=\mathrm{cl}A \Rightarrow A$ is closed
($\Leftarrow$) Suppose A is closed. Then A contains its limit points so $A=A\cup\{\text{limit points for } A\}=\mathrm{cl}A$.

Theorem Given $A \subset (M,d)$, consider the collection $\mathcal{F}_A = \{ F \subset M \mid F \text{ is closed and } A \subset F \}$ of all closed subsets of M which contain A. $\mathrm{cl}A = \bigcap_{F \in \mathcal{F}_A} F$.

Proof $(\subset)$ We'll show that for any $F \in \mathcal{F}_A$, $\mathrm{cl}A \subset F$.

Since $\mathrm{cl}A = A \cup \{\text{limit points for } A\}$, and $A \subset F$ by assumption, just need to show each limit point for A is in F.

Since $A \subset F$, every limit point for A is a limit point for F.

Since F is closed, F contains all its limit points.

So each limit point for A lies in F.

$(\supset)$ Suppose $x \in M$ is neither an element of A nor a limit point for A. There is some $\varepsilon > 0$ so that $D(x,\varepsilon)$ contains **no** point of A. Note that $D(x,\varepsilon)^c$ is a closed set which contains A. But $x \notin D(x,\varepsilon)^c$.

So $x \notin \bigcap_{F \in \mathcal{F}_A} F$.

Thus $(\mathrm{cl}A)^c \subset \left(\bigcap_{F \in \mathcal{F}_A} F\right)^c$, so $\bigcap_{F \in \mathcal{F}_A} F \subset \mathrm{cl}A$ ■

Thus any closed set which contains A also contains $\mathrm{cl}A$. $\mathrm{cl}A$ is the minimal closed set containing A.

Def'n Given $A \subset B \subset (M,d)$, say A is **dense in B** if $B \subset \mathrm{cl}A$. Equivalently, A is dense in B if for every $\varepsilon > 0$ and every $b \in B$, $D(b,\varepsilon)$ contains some element of a.

E.g. The rationals are dense in $\mathbb{R}$.

Stone-Weierstrass Theorem: Polynomials are dense in $\mathcal{C}[a,b]$.

(We don't quite know what this means — yet.)

Boundary

Def'n The boundary of a subset $A \subset (M,d)$ is

$$\partial A = cl(A) \cap cl(M\setminus A)$$

Immediate consequences:

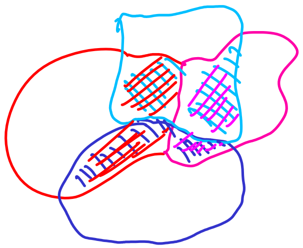
1) ∂A is closed
2) $\partial A = \partial(M\setminus A)$

Prop'n $x \in \partial A$ iff every open set U containing x also contains some point of A and some point not in A.

Proof.

$$\begin{aligned}\partial A &= cl(A)\cap cl(M\setminus A)\\ &= (A \cup \{\text{limit points for } A\}) \cap ((M\setminus A)\cup\{\text{limit points for } M\setminus A\})\\ &= (A\cap(M\setminus A)) \cup (A \cap \{\text{limit points for } M\setminus A\})\\ &\quad \cup (\{\text{limit points for } A\}\cap(M\setminus A)) \cup (\{\text{limit points for } A\}\cap\{\text{limit points for } M\setminus A\})\end{aligned}$$

(Intersection distributes over union, so we can FOIL.)



$(B\cup C)\cap(D\cup F)$ has four possible parts

$A\cap(M\setminus A)=\emptyset$, so there are three cases for $x\in\partial A$:

1) $x\in A$ and x is a limit point for $M\setminus A$.
   Then any open U which contains x contains a point of A (namely x) and contains (infinitely many) points of $M\setminus A$.
2) $x \in M\setminus A$ and x is a limit point for A
   Same as 1.
3) x is a limit point for $M\setminus A$ and for A (Actually this case can be done away with ...)
   Then every open set U containing x contains (infinitely many) points of A and of $M\setminus A$.

($\Leftarrow$) Two cases:

1) $x\in A$ Then $x\notin M\setminus A$. By hypothesis every open U contains some point of $M\setminus A$, which can't be x. So x is a limit point for $M\setminus A$.
2) $x\notin A$ Same as 1.

So any such $x \in (A\cap\{\text{limit points for } M\setminus A\}) \cup ((M\setminus A)\cap\{\text{limit points for } A\}) \subset \partial A$.

Eg. $\partial\mathbb{Q} = cl(\mathbb{Q}) \cap cl(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{R}\cap\mathbb{R} = \mathbb{R}$

Eg. Let's see what can be said about the closure and boundary of $D(x,R)$.

$$D(x,R) = \{y\in M \mid d(x,y) < R\}$$
$$D(x,R)^c = \{y\in M \mid d(x,y) \geq R\}$$

What is $cl(D(x,R))$?

Claim: $B(x,R) = \{y\in M \mid d(x,y) \leq R\}$ is closed.

Note: ∂A is ~~not~~ not just $cl(A)\setminus A$, e.g.

$A = \{0\} \subset \mathbb{R}$. $A = cl A = \{0\}$, $cl A\setminus A = \emptyset$

$\partial A = cl A \cap cl(\mathbb{R}\setminus\{0\}) = \{0\}\cap\mathbb{R} = \{0\}$.

Proposition $cl A\setminus A \subset \partial A$

Proof Let $x\in cl A\setminus A$. Then x is a limit point for A but $x\notin A$. Every open set containing x contains an element of $M\setminus A$ (namely x) and an element of A. ∎

Proposition $x\in cl(A)$ iff $\inf\{d(x,y) \mid y\in A\} = 0$

Proof. ($\Rightarrow$) Suppose $x\in cl A$. Two cases:

$x\in A$ Then $d(x,x)=0$, so $\inf\{d(x,y)\mid y\in A\}\leq 0$.

x is a limit point for A Then for each $\varepsilon>0$ there is some $y_\varepsilon\in A$ with $d(x,y_\varepsilon)<\varepsilon$. So $\inf\{d(x,y)\mid y\in A\}<\varepsilon$. Since ε was arbitrary, $\inf\{d(x,y)\mid y\in A\}\leq 0$. ∎

($\Leftarrow$) By the ε-characterisation of inf, for each $\varepsilon>0$ there is $y_\varepsilon\in A$ with $0\leq d(x,y_\varepsilon)<\varepsilon$.

If some ε has $y_\varepsilon = x$, then $x\in A\subset cl A$.

If no ε has $y_\varepsilon = x$, then y_ε is an element of $D(x,\varepsilon)\cap A$ not equal to x, so x is a limit point for A. ∎