Why we need real numbers

This is a course about real numbers.

What is a real number?

This is not nearly as simple a question as it seems.
The answer was only discovered in the 1800s!

Definition (old) The natural numbers are the set
\[ N = \{0, 1, 2, 3, \ldots \} \]
(Note: some omit 0 from N)

The integers are the natural numbers and their negatives:
\[ Z = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} = N \cup \{0\} \cup -N, \text{ with the }\]
agreement that 0 = -0.

Kronecker: "Die ganzen Zahlen hat der liebe Gott gemacht; alles andere ist Menschenwerk."

"Beloved God made the natural numbers; everything else is man's work."

\[ N \] is closed under addition and multiplication; \[ Z \] is closed under addition and multiplication and has additive inverses.
Definition. The rational numbers are
\[ \mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, \ n \neq 0, \ m, n \text{ have no common divisors} \right\} \]

\( \mathbb{Q} \) is closed under \(+, -, \cdot, /\).

Proof. Every linear equation with rational coefficients has a rational solution.

Question: Is there \( q \in \mathbb{Q} \) with \( q^2 = 2 \)?

(Historical) Answer: No!

Proof. Suppose so, i.e., there is \( q = \frac{m}{n} \in \mathbb{Q} \) so that
\[ \left( \frac{m}{n} \right)^2 = 2 \]

Since \( \frac{m}{n} \in \mathbb{Q} \), at most one of \( m, n \) is even.

\[ m^2 = 2n^2 \]

So \( m^2 \) is even. The only way a square can be even is if the thing it's a square of is even. So \( m = 2k \) for some \( k \).

Then \( 2n^2 = m^2 = 4k^2 \)  
\[ n^2 = 2k^2 \]

So \( n^2 \) is even. Hence \( n \) is even. But this contradicts the choice of \( m, n \). \( \square \)

(This is an example of a reduction ad absurdum proof.)

(reduction to absurdity)

"proof by contradiction"

This poses two major problems:

1. algebraic: Can't solve equations, even simple ones!
2. geometric: If we have a square lot with side length 1  
   earth measure and we want to walk across it...

Pythagoras says: Can walk any distance \( d \),  
So long as \( d^2 \leq 1^2 + 1^2 = 2 \).

The set of distances we can walk is therefore  
\( S = \{ q \mid q^2 \leq 2 \} \subset \mathbb{Q} \).

Compare this set to  
\( R = \{ q \mid q^2 \leq 13 \} \subset \mathbb{Q} \)

which contains a least upper bound, that is,  
"upper bound" \( 4 \) if \( c \) is in \( R \), then \( 1 < c \).
**Ordered Fields**

**Defs.** A set is a collection of objects ("elements") for which the question "Is object $x$ an element?" has an unambiguous yes-or-no answer.

( Homework: Google "Russell's paradox." )

**Notation.** $x \in A$ "$x$ is an element of the set $A$"

$x \not\in A$ "$x$ is not an element of the set $A$"

**Defs.** A set $B$ is a subset of a set $A$ if every element of $B$ is also an element of $A$.

$B \subseteq A$ "$B$ is a subset of $A$"

We call two sets equal if they are mutually subsets $A = B \iff A \subseteq B$ and $B \subseteq A$

**Notation.** every element of $B$ is also an element of $A$

\[ \forall x \in B, \ x \in A \]

for all $x$ elements of $B$, $x$ is an element of $A$

$x \in B \implies x \in A$

$x$ is an element of $B$ implies $x$ is an element of $A$
**Defn.** An ordered set is a pair \((S, \leq)\) of a set \(S\) and an ordering \(\leq\), such that:
1. For any \(x, y \in S\), either \(x \leq y\) or \(y \leq x\) (or both!)
2. For any \(x \in S\), \(x \leq x\)
3. If \(x \leq y\) and \(y \leq x\) then \(x = y\)
4. If \(x \leq y\) and \(y \leq z\), then \(x \leq z\).

**Ex.** Alphabetisation

**Ex.** \(\leq\) with the standard “less than or equal to”

**Defn.** A field is a set \(F\), together with operations \(+\) and \(\cdot\), such that:
- \(\forall x, y \in F, \; x + y = y + x\)
- \(\forall x, y, z \in F, \; (x + y) + z = x + (y + z)\)
- \(\exists 0 \in F \; \forall x \in F, \; x + 0 = x\)
- \(\forall x \in F, \; \exists (x) \in F. \; x + (x) = 0\)
- \(\forall x, y \in F, \; x \cdot y = y \cdot x\)
- \(\forall x, y, z \in F, \; (x \cdot y) \cdot z = x \cdot (y \cdot z)\)
- \(\exists 1 \in F. \; \forall x \in F, \; 1 \cdot x = x\).

**Addition axioms**

**Multiplication axioms**

- \(\forall x, y \in F, \; x \cdot 0 = 0\)
- \(\exists x, y \in F, \; x \cdot (y + z) = x \cdot y + x \cdot z\)
- \(\exists 1 \in F. \; \forall x \in F, \; 1 \cdot x = x\)

**Distributive axiom**

**Nondegeneracy axiom**

**Defn.** An ordered field is an ordered set with field operations, subject to the compatibility requirements:
- \(\forall x, y, z \in F, \; x \leq y \Rightarrow x + z \leq y + z\)
- \(\forall x, y \in F, \; 0 \leq x \text{ and } 0 \leq y \Rightarrow 0 \leq xy\)

We call elements \(x \in F\) which satisfy \(0 \leq x\) nonnegative.

**Notation.** \(x < y\) means \(x \leq y\) and \(x \neq y\)

**Observe.** Inside any ordered field \((F, \leq, +, \cdot)\), there is \(1, 1 + 1, 1 + 1 + 1\), etc. So \(F\) contains a copy of \(\mathbb{N}\). And a copy of \(\mathbb{Z}\). And a copy of \(\mathbb{Q}\).

"\(\mathbb{Q}\) is the smallest ordered field."
Now we're ready to prove some facts about ordered fields.

**Proposition** In an ordered field, \(-O = O\).

**Proof** \(O + O = O\), so \(O\) is an additive inverse of itself.

**Proposition** In an ordered field, \(x \cdot O = O\) for any \(x\).

**Proof**
\[
x \cdot O = x \cdot (O + O) = x \cdot (O - O) = x \cdot O - (x \cdot O) = x \cdot O + - (x \cdot O) = O
\]

**Proposition** In an ordered field, \(0 < 1\).

**Proof** Suppose otherwise, i.e. that \(1 \leq O\).

By assumption, \(O \neq 1\), so we have \(1 < O\).

Then \(-1 + 1 < -1 + 0\)

\[O < -1\]

Then \((-1) \cdot (-1)\) is the product of positive numbers, hence positive.

By a homework problem, \((-1) \cdot (-1) = 1\).

Thus \(0 < 1\). But we assumed \(1 < O\). \(\rightarrow \leftarrow\)

**Bounds**

**Def** In an ordered set \((S, \leq)\), we say \(b \in S\) is an upper bound for a subset \(A \subseteq S\) if, for every \(a \in A\), we have \(a \leq b\).

Note that \(b\) need not be an element of \(A\), only \(b\) is a lower bound for \(A\) if, for every \(a \in A\), \(b \leq a\).

**Fact** Every subset of \(\mathbb{N}\) has a lower bound.

\[\forall A \subseteq \mathbb{N} \exists b \in \mathbb{N} \cdot \forall a \in A \cdot b \leq a\]

In fact we can use the same lower bound for all the subsets, namely \(0 \in \mathbb{N}\).

\[\exists b \in \mathbb{N} \cdot \forall A \subseteq \mathbb{N} \cdot \forall a \in A \cdot b \leq a\]

Order of quantifiers matters!

**Fact** Not every subset of \(\mathbb{N}\) has an upper bound.

(Example: \(\mathbb{N}\) itself)

- Even natural numbers)

- Not every subset of \(\mathbb{Z}\) has an upper bound
Not every subset of \( \mathbb{Z} \) has a lower bound.

**Defn.** An upper bound \( b \) for a subset \( A \) of an ordered set is called a least upper bound if it is \( \leq \) any other upper bound for \( A \).

\[ b \text{ is a least upper bound if } \forall a \in A, a \leq b \text{ and } \forall b' \in A, a \leq b' \]

A least upper bound for \( A \) is called the supremum of \( A \).

**Lemma.** The least upper bound of a set \( A \) is unique.

**Proof.** If \( b_1 \) and \( b_2 \) are LUBs, then (if there is one)

\[ b_1 \leq b_2 \text{ and } b_2 \leq b_1. \]

So we write \( b = \text{sup } A \) unambiguously.

**Fact.** Every nonempty subset of \( \mathbb{Z} \) which has an upper bound has a least upper bound.

**Proof.** If \( A \subset \mathbb{Z} \) has an upper bound \( b' \), do the following:

- If \( b' \in A \), then \( b' = \text{sup } A \).
- If \( b' \notin A \), try \( b' \rightarrow 1 \).

The process terminates after at most \( b' - a \) steps, where \( a \) is any element of \( A \).

**Defn.** We say an ordered set \( S \) has the least upper bound property if any nonempty \( A \subset S \) which has an upper bound has a least upper bound.

**Exeg.** \( \mathbb{N} \) and \( \mathbb{Z} \) have the LUB property.

(Note this does not mean \( \mathbb{N} \) or \( \mathbb{Z} \) has a least upper bound!)

**Exeg.** \( \mathbb{Q} \) does not have the LUB property!

**Proof.** Let \( A = \{ q \mid q^2 \leq 2 \} \subset \mathbb{Q} \).

Consider \( B = \{ q \mid q^2 > 2 \} \).

Given \( q \in \mathbb{Q} \) define \( p \) by:

\[ p = \frac{2q + 2}{q + 2} \]

Then

\[ p^2 - 2 = \frac{(2q + 2)^2}{(q + 2)^2} - 2 \]

\[ = \frac{(2q + 2)^2 - 2(q + 2)}{(q + 2)^2} \]

\[ = \frac{4q^2 + 8q + 4 - 2q^2 + 8q + 8}{(q + 2)^2} \]

\[ = \frac{2(q^2 + 2)}{(q + 2)^2} \]

So

\[ p^2 < 2 \iff q^2 < 2 \]

\[ p^2 > 2 \iff q^2 > 2 \]
On the other hand,
\[ p = \frac{2q^2 + 2}{q+2} = q - \frac{q^2 - 2}{q+2} \]
So \( p > q \iff q^2 < 2 \)
\( q > p \iff q^2 > 2. \)
Thus no \( q \) with \( q^2 < 2 \) can be an upper bound for \( A \).
So any upper bound \( b \) for \( A \) has \( b^2 \geq 2. \)
Elements of \( \mathbb{B} \) are all upper bounds for \( A. \) (Homework)
So to be least, an upper bound \( b \) must be smaller than every elt of \( \mathbb{B} \), hence have \( b^2 \leq 2. \)
So the only LUB for \( A \) is one with \( b^2 = 2. \)

But there is no rational number \( b \) with \( b^2 = 2. \)
So \( A \) has an upper bound but no least upper bound.
Therefore \( \mathbb{Q} \) lacks the LUB property.

**Theorem.** There is an ordered field with the LUB property.

**Proof:** Complicated and not very enlightening.

One version is in Appendix of Rudin's chapter 1.

**Theorem.** There is only one ordered field with the LUB property.

That is, if I construct two such fields, there is a way to identify them. We'll see the proof much later.

**Def.** The real numbers are the unique ordered field with the LUB property. We write \( \mathbb{R} \).

**Def.** The extended real numbers are \( \mathbb{R} \cup \{ a \in \mathbb{R} : a \geq -\infty \} \)
with the ordering \( -\infty \leq x \leq \infty \) for all \( x \in \mathbb{R} \).

**Prop.** If \( A \subseteq \mathbb{R} \) is not bounded above, then \( \sup A = \infty. \)

**Proof.** "It is not bounded above": \( \forall b \in \mathbb{R}, \exists a \in A : a < b. \)
So if \( \sup A \geq a \) for all \( a \in A \), we must have \( \sup A \geq b \) for all \( b \in \mathbb{R} \).

**Prop.** \( \sup \emptyset = -\infty \)

**Proof.** Any real number is an upper bound for \( \emptyset. \) So \( \sup \emptyset \)
must be smaller than every real number.
**Sequences in an Ordered Field**

**Defn.** A sequence \((x_n)_{n \in \mathbb{N}}\) is an assignment, to each \(n \in \mathbb{N}\), of an element \(x_n\).

**Example:** \(x_n = n^2\) formula.

\[ x_0 = 1, \quad x_{k+1} = x_k + 1 \] recurrence relation.

**Defn.** The absolute value of an element \(x\) of an ordered field is:
\[ |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

**Defn.** A sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\) if:
for any \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that
\[ n \geq N \text{ guarantee } |x_n - x| < \varepsilon \]

**Notation:** \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\)

**Theorem.** An ordered field with the least upper bound property has the monotone sequence property.

**Proof.**
To show MS holds, let \((x_n)_{n \in \mathbb{N}}\) be a monotone sequence which is bounded above.

Then the set \(\{x_n \mid n \in \mathbb{N}\}\) is a set which is bounded above.

So by LUB \(\exists b = \sup \{x_n \mid n \in \mathbb{N}\}\)

Claim: \(x_n \to b\)

**Proof.** To show \(x_n \to b\), we must ensure, for any given \(\varepsilon > 0\), that \(|x_n - b| < \varepsilon\). So for our given \(\varepsilon > 0\)...

Notice \(x_0 \leq x_1 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots \leq b\)

So \(b - x_n \geq 0\) and \(b - x_n \geq b - x_{n+1}\).

This if a particular \(n\) has \(b - x_n < \varepsilon\), so do all \(k > n\).
So if \( x_n \neq b \), it must be the case that, for all \( n \),
\[ b - x_n = |b - x_n| \geq \varepsilon \] for our given \( \varepsilon \).
Then \( b \geq \varepsilon + x_n \)
\[ b - \varepsilon \geq x_n \]
So \( b - \varepsilon \) is an upper bound for \( \{x_n\} \). But \( b \) was the least upper bound, and \( b - \varepsilon < b \).

(\( \text{MS} \Rightarrow \text{LUB} \))
To show \( \text{LUB} \), suppose \( A \) is a set with some upper bound \( M \). We want to find sup \( A \).

Def. An ordered field has the Archimedean property if for any \( x, y \) with \( 0 < x < y \), there is \( n \in \mathbb{N} \) so that \( y < nx \).

\[ 0 \quad \frac{x}{1} \quad \frac{2x}{2} \quad \frac{3x}{3} \quad \ldots \quad \frac{nx}{n} \]

Theorem. An ordered field with the \( \text{MS} \) property has the Archimedean property.

Proof. Given \( 0 < x < y \), consider the sequence \( (nx)_{n \in \mathbb{N}} \). This sequence is monotone.

If the Archimedean property failed, \( y = nx \) for all \( n \).
So \( (nx)_{n \in \mathbb{N}} \) is bounded above, hence by \( \text{MS} \)
\[ nx \to L \] for some limit \( L \). Then for any \( \varepsilon > 0 \),
we could find \( n \) so that, for any \( k \in \mathbb{N} \),
\[ L - nx < \varepsilon \quad \text{and} \quad L - (n + k)x < \varepsilon \]
Then \( kx = (n + k)x - nx \leq L - nx < \varepsilon \)
Choosing \( \varepsilon = \frac{\varepsilon}{x} \) and \( k = 2 \) gives a contradiction.

Cor. In an ordered field with the \( \text{MS} \) property, for any \( 0 < x \), there is \( n \in \mathbb{N} \) with \( 0 < \frac{1}{n} < x \).

Proof. Case: \( x \geq 1 \). Then any \( n \) will do.
Case: \( x < 1 \). Apply Archimedean property to \( 0 < x < 1 \) to get \( n \) with \( nx > 1 \).
Then \( x > \frac{1}{n} \).

Proposition. In an Archimedean field, \( \frac{1}{n} \to 0 \).

Short proof, just the definitions.
**Theorem** An ordered field with the MS property has the LUB property.

**Proof.** Given $A$ as subset of the field, with some upper bound $M$, define the sequence $b_n$ as follows:

- For each $n \in \mathbb{N}$, consider $M - \frac{K_n}{2^n}$ for $K_n \in \mathbb{N}$.
- Define $K_n$ to be the first $K$ for which $M - \frac{K_n}{2^n}$ is not an upper bound for $A$.

Define $b_n = M - \frac{K_n}{2^n}$.

Note that by our choice of $K_n$, $b_n + \frac{1}{2^n}$ is an upper bound for $A$.

Claim $b_n < b_{n+1}$ (i.e. $(b_n)_{n \in \mathbb{N}}$ is monotone.)

**Proof.**

- $M - \frac{2K_n}{2^{n+1}} = M - \frac{K_n}{2^n}$
- So $K_n + 2^n < 2K_n$
- So $M - \frac{K_n + 2^n}{2^{n+1}} > M - \frac{2K_n}{2^{n+1}} = M - \frac{K_n}{2^n}$

$(b_n)_{n \in \mathbb{N}}$ is bounded above by $M$, so by MSP, $b_n \to b$.

Claim $b = \sup A$.

**Proof.** Notice each $b_n$ is chosen to be smaller than any upper bound of $A$.

Therefore, given any upper bound $b'$, the condition $b_n \leq b'$ gives $b = \lim b_n \leq b'$.

So $b$ is least.

To see that $b$ is an upper bound, consider $c_n = b_n + \frac{1}{2^n}$.

Then $\lim c_n = \lim b_n + \frac{1}{2^n} = \lim b_n + \lim \frac{1}{2^n} = b + 0 = b$.

Each $a \in A$ has $a \leq c_n$, so $a \leq b$.

Thus $b$ is an upper bound for $A$.

We've just shown LUB $\iff$ MSP.

**Def:** An ordered field with either LUB or MSP is called complete.
The problem with \( \mathbb{Q} \) is that it is not complete.
Can you think of a sequence in \( \mathbb{Q} \) which is monotonic and bounded, but does not converge?

\[
(1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots)
\]

**Theorem (Useful characterization of supremum)**

Given \( A \subseteq \mathbb{R} \) a nonempty set which is bounded above. Then \( \alpha = \sup A \) iff

- \( \alpha \) is an upper bound for \( A \)
- \( \forall \varepsilon > 0 \exists a \in A \) with \( \alpha - \varepsilon < a \leq \alpha \)

**Proof**

Assume \( \alpha = \sup A \). Then \( \alpha \) is an upper bound for \( A \).

Suppose the theorem fails. Then \( \exists \varepsilon > 0. \forall a \in A \),
\[ \alpha \leq \alpha - \varepsilon. \]
So \( \alpha - \varepsilon \) is an upper bound, and \( \alpha - \varepsilon < a \) for some \( a \in A \).

Now assume \( \alpha \) satisfies the two conditions.

If \( b \) is some upper bound for \( A \), then \( \forall a \in A \), \( b \geq a \). In particular, for each \( \varepsilon > 0 \), we have
\[ \alpha - \varepsilon < a \leq b. \]
Take \( \varepsilon = \frac{1}{n} \), we have \( b > \alpha - \frac{1}{n} \),
so \( \lim b \geq \lim a - \lim \frac{1}{n} \), so \( \alpha \) is least.

**Proposition (Limit laws)**

Suppose \( (x_n) \) and \( (y_n) \) are sequences with limits \( x \) and \( y \) respectively. Then:

1. \( \lim (x_n + y_n) = x + y \)
2. \( \lim (x_n y_n) = xy \)
3. If \( \forall n \geq 0 \) and \( y \neq 0 \), then \( \lim y_n^{-1} = y^{-1} \)
4. \( \lim \sqrt{x_n} = \sqrt{x} \)

**Proof**

Should have seen first two already.
1) Given \( \varepsilon > 0 \), consider \( |(x_n + y_n) - (x + y)| \)
\[ = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y|. \]
Since \( x_n \to x \) and \( y_n \to y \),
there is \( K \in \mathbb{N} \) such that \( n \geq K \Rightarrow |x_n - x| < \frac{\varepsilon}{2} \).
Similarly, there is \( M \in \mathbb{N} \) such that \( n \geq M \Rightarrow |y_n - y| < \frac{\varepsilon}{2} \).
So take \( N = \max\{M, K\} \). Then \( n \geq N \) guarantees
\[ |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

2) Similar to 1)
3) Given \( \varepsilon > 0 \),
We want to control \( |y_n^{-1} - y^{-1}| = |y_n^{-1} - y_n y^{-1}| \)

\[ |(y - y_n) y_n^{-1}| = |y - y_n| |y_n| |y_n^{-1}| \]

Since \( y_n \to y \), there is \( K \) so that \( n \geq K \Rightarrow |y_n - y| < \frac{1}{2} |y| \)
\[ |y| < \frac{1}{2} |y| \]
\[ |y_n| > \frac{1}{2} |y| \]

Also, there is \( M \) so that \( n \geq M \Rightarrow |y_n - y| < \frac{1}{2} |y|^2 \).

Taking \( N = \max \{ M, K^2 \} \), if \( n \geq N \) we have
\[ |y_n - y| = |y - y_n| |y_n| |y_n^{-1}| \]
\[ < \frac{1}{2} |y|^2 \cdot |y| \cdot 2 |y| = \frac{1}{4} |y|^3 \]

4) You do! \( \square \)

Henceforth we may assume that limits and algebraic operations commute.

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**Def:** A subsequence of \( (x_n)_{n \in \mathbb{N}} \) is a choice, for each \( \mathbb{N} \), of index \( n_k \in \mathbb{N} \), so that \( n_k < n_{k+1} \).
(That is a subsequence is an increasing function from \( \mathbb{N} \) to \( \mathbb{N} \).)

**Ex:** \( x_n = n^2 \), \( k_n = 2k \)

Then we have a sequence \( n=0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots \)
\[ x_n: 0, 1, 4, 9, 16, 25, 36, \ldots \]
\[ n_k: 0, 2, 4, 6, 8, 10, 12, 14, \ldots \]

**Ex:** \( x_n = n \)

Note that a subsequence is a sequence in its own right.

**Def:** \( x \in \mathbb{R} \) is a **cluster point** of the sequence \( (x_n) \) if for any \( \varepsilon > 0 \), there are infinitely many \( n \in \mathbb{N} \) with \( |x_n - x| < \varepsilon \).

**Ex:** \( x_n: 0, \frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{5}, 1\frac{1}{7}, 1\frac{1}{10}, 1\frac{1}{11}, \ldots \)
Cluster points are 0 and 1.
Proposition. Given a sequence \((x_n)_{n \in \mathbb{N}}\) and \(x \in \mathbb{R}\),

i) \(x\) is a cluster point of \((x_n)_{n \in \mathbb{N}}\) iff for any \(\varepsilon > 0\) and all \(N \in \mathbb{N}\), there is \(n > N\) with \(|x_n - x| < \varepsilon\).

ii) \(x\) is a cluster point of \((x_n)_{n \in \mathbb{N}}\) iff there is a subsequence of \(x_n\) which converges to \(x\).

iii) \(x \rightarrow x\) iff every subsequence of \((x_n)_{n \in \mathbb{N}}\) converges to \(x\).

iv) \(x \rightarrow x\) iff \((x_n)_{n \in \mathbb{N}}\) is bounded and \(x\) is the only cluster point of \((x_n)_{n \in \mathbb{N}}\).

Proof: i) is just the definition of "infinite".

ii) The subsequence is given by picking indices:

Let \(n_0 = 0\). For \(k \geq 1\), choose \(\varepsilon = \frac{1}{k}\) and \(N = n_{k-1}\), there is \(n > n_{k-1}\) with \(|x_n - x| < \frac{1}{k}\).

Use \(n_k = n\).

Claim: \((x_{n_k})_{n \in \mathbb{N}}\) converges to \(x\).

The above says to see if a limit exists we need to investigate its cluster points.

Def: If \((x_n)_{n \in \mathbb{N}}\) is bounded above, define its limit superior of \((x_n)_{n \in \mathbb{N}}\) as the supremum of the set of its cluster points.

If \((x_n)_{n \in \mathbb{N}}\) is bounded below its limit inferior is the infimum of its cluster points.

If the sequence \((x_n)_{n \in \mathbb{N}}\) is not bounded above, write \(\limsup x_n = \overline{\lim x_n} = \infty\).

If \((x_n)_{n \in \mathbb{N}}\) is not bounded below, write \(\liminf x_n = \underline{\lim x_n} = -\infty\).

E.g. \(x_n = 1, 2, 3, \frac{1}{2}, 5, 6, \frac{1}{3}, 8, 9, \frac{1}{4}, \ldots\)

\[\lim x_n = \inf \{0.3\} = 0\]

\[\overline{\lim x_n} = \infty\]

Proposition \(\lim x_n = x\) iff \(\liminf x_n = \overline{\lim x_n}\).
Theorem. Every sequence in \( \mathbb{R} \) has a monotone (either increasing or decreasing) subsequence.

Proof. Given a sequence \((x_n)_{n \in \mathbb{N}}\), we say a term \(x_k\) has an open horizon if \(x_k > x_n\) for all \(n > k\).

Case 1. There are infinitely many terms with open horizons. Then they must be arranged:
\[
x_{n_0} > x_{n_1} > x_{n_2} > \ldots
\]
with \(n_0 < n_1 < n_2 < \ldots\).

Case 2. There are finitely many terms with open horizons. Then there is some \(M\) so that \(n \geq M\) implies \(x_n\) does not have open horizon. So for each \(n \geq M\), \(x_{n+1} \geq x_n\). Take \(n_0 = M\), \(n_1 = m\), etc.

Note. The same sequence could have monotone increasing and monotone decreasing subsequences.

Bolzano-Weierstrass Theorem (Our first real Theorem)

If \((x_n)_{n \in \mathbb{N}}\) is a bounded sequence of real numbers, i.e. \(x_n \in [B, B]\) for all \(n \in \mathbb{N}\), then \((x_n)\) has a convergent subsequence.

Proof. By previous theorem, there is some monotone subsequence \((x_{n_k})_{k \in \mathbb{N}}\). This subsequence is bounded, hence converges by MCT. \(\Box\)
**Cauchy Sequences**

Recall the definition of convergence:

\[ x_n \to x \text{ if } \forall \varepsilon > 0 \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |x_n - x| < \varepsilon \]

This seems to require knowledge of the limit \( x \).

What if we don't know the limit (e.g., as in HW 1) but we want to investigate convergence?

If the sequence is monotone and bounded we use MCT. Otherwise, what to do?

**Definition** A sequence \((x_n)_{n \in \mathbb{N}}\) is Cauchy if for all \( \varepsilon > 0 \) there is \( N_0 \in \mathbb{N} \) so that \( n, m \geq N \) guarantees

\[ |x_n - x_m| < \varepsilon. \]

**Theorem** Convergent sequences are Cauchy.

**Proof** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence which converges to \( x \). Given \( \varepsilon > 0 \), choose \( N \) so that \( n \geq N \) guarantees

\[ |x_n - x| < \frac{\varepsilon}{2}. \]

Then if \( n, m \geq N \), we have:

\[ |x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

So this \( N \) works, and \((x_n)_{n \in \mathbb{N}}\) is Cauchy.

**Theorem** (Homework) Cauchy sequences converge.

**Theorem** The Cauchy sequences in \( \mathbb{R} \) are exactly the convergent sequences.

**Proposition** Suppose \((x_n)_{n \in \mathbb{N}}\) is a sequence with

\[ |x_{n+1} - x_n| \leq \frac{1}{2} |x_n - x_{n-1}|. \]

Then \((x_n)_{n \in \mathbb{N}}\) converges.

**Proof** We'll show \((x_n)_{n \in \mathbb{N}}\) is Cauchy. For \( m > n \),

\[ |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \ldots + x_{n+1} - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \leq \frac{1}{2} |x_m - x_{m-1}| + \frac{1}{2} |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \leq \frac{1}{2} |x_{m-1} - x_{m-2}| + \frac{1}{2} |x_{m-2} - x_{m-3}| + \ldots + |x_{n+1} - x_n| \leq \left( \frac{1}{2^{m-n-1}} + \frac{1}{2^{m-2}} + \ldots + \frac{1}{2} \right) |x_m - x_{n-1}|. \]
\[
\left( \sum_{k=1}^{m} \frac{1}{2^k} \right) \frac{1}{2^{n+1}} |x_n - x_0|< \varepsilon
\]

**Claim** \[
\sum_{k=1}^{m} \frac{1}{2^k} < 1
\]

**Claim** \[
\frac{1}{2^{n+1}} \to 0.
\]

So given \(\varepsilon > 0\), there is \(N\) so that \(n \geq N\) guarantees \[
\frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{|x_n - x_0|}.
\]

Then \(m, n \geq N\) guarantees \[
|x_m - x_n| < \varepsilon.
\]

So \((x_n)_{n \geq N}\) is Cauchy, hence converges. **Note:** This proof works for any coefficient \(A < 1\)

**Proposition** A sequence with \[
|x_{n+1} - x_n| \leq A |x_n - x_{n-1}|
\]

for some \(A < 1\) converges.

The only difference is to check that \[
\sum_{k=1}^{m} A^k
\]

is bounded independent of \(m\) and \(n\) and that \(A^{n+1} \to 0\).

**Important Example**

Consider the sequence \(x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\).

Then \(x_{n+1} - x_n = \frac{1}{n+1}\), which goes to 0 as \(n \to \infty\).

Nevertheless, \((x_n)_{n \geq N}\) does not converge, and isn't Cauchy:

\[
x_m - x_n = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}
\]

To see that \((x_n)_{n \geq N}\) does not converge, consider \[
x_{2k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{2k}
\]

\[
> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} + \cdots
\]

\[
= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}
\]

\[
= 1 + \frac{1}{2}k
\]

Clearly the sequence \((1 + \frac{1}{2}k)_{k \in \mathbb{N}}\) is not bounded, hence the subsequence \((x_{2k})_{k \in \mathbb{N}}\) is not bounded, hence \((x_n)_{n \in \mathbb{N}}\) diverges.
The Complex Field

Algebraists are interested in \( \mathbb{C} \) because it is "algebraically closed."

**Definition** A complex number is a formal sum

\[ z = a + ib \] when \( a, b \in \mathbb{R} \).

Addition and multiplication of complex numbers are defined by requiring the distributive law hold and that \( i^2 = -1 \) and \( 0i = 0 \).

**Proposition** \( \mathbb{C} \) so equipped is a field.

**Proof**

\[
(a + ib) + (c + id) = (a + c) + i(b + d) = a - a + i(b - b) = 0 + 0i
\]

\[
(a + ib) \cdot (\frac{a}{a^2 + b^2} + i\frac{b}{a^2 + b^2}) = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} + i\left(\frac{ab}{a^2 + b^2} - \frac{ab}{a^2 + b^2}\right) = 0
\]

**Definition** We call \( a \) the real part and \( b \) the imaginary part of \( z = a + ib \).

**Definition** The conjugate of \( z = a + ib \) is \( \overline{z} = a - ib \).

\[ z \overline{z} = (a + ib)(a - ib) = a^2 + b^2 \text{ This is a real number!} \]

\[ |z| = \sqrt{z \overline{z}} = \sqrt{a^2 + b^2} \]

**Proposition** \( |z| = 0 \) if \( z = 0 \in \mathbb{C} \).

**Proof** Suppose \( a^2 + b^2 = 0 \). Then \( a^2 + b^2 = a^2 = 0 \) gives \( a = 0 \).

\[ \text{Suppose } a = b = 0. \text{ So } a = b = 0 \text{, and } z = a + ib = 0 + i \cdot 0 = 0. \]

Suppose \( a = b = 0 \). Then \( |z| = \sqrt{0} = 0 \).

**Proposition**

\[ \Re z = \frac{a}{a^2 + b^2} \]

\[ \Im z = \frac{b}{a^2 + b^2} \text{ These are inequalities of real numbers!} \]

**Proof**

\[ |zw| = |z||w| \]

\[ |zw|^2 = z\overline{w} = zw \overline{w} = z\overline{w} \overline{w} = z\overline{w} \overline{w} = 1|z|^2 |w|^2 \]
**Theorem.** \( C \) is Cauchy-complete; that is, the Cauchy sequences in \( C \) are precisely the convergent sequences.

The proof uses the following:

**Proposition.** \((a_n + ib_n)\) converges to \(a + ib\) if \( a_n \to a \) and \( b_n \to b \)

**Proof.** \[ |a_n + ib_n - (a + ib)|^2 = (a_n - a)^2 + (b_n - b)^2 \]

\( \Rightarrow (a_n - a)^2 \)
So if \( a_n + i b_n \to a + i b \) for any given \( \varepsilon > 0 \), there is \( N \) so that \( n \geq N \) guarantees
\[
|a_n - a| + |b_n - b| < \varepsilon.
\]
Thus \( a_n \to a \). Similarly for \( b_n \to b \).

\((\Leftarrow)\) If \( a_n \to a \) and \( b_n \to b \), there are \( N_1, N_2 \)
so that \( n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2} \)
\( n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \)

So choosing \( N = N_1 + N_2 \), \( n \geq N \) guarantees
\[
|a_n + i b_n - (a + i b)| = |a_n - a|^2 + |b_n - b|^2
< \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2
\]

So \( |a_n + i b_n - (a + i b)| < \varepsilon \).

**Proposition.** \( (a_n + i b_n)_{n \in \mathbb{N}} \) is Cauchy iff \( (a_n)_{n \in \mathbb{N}} \)
and \( (b_n)_{n \in \mathbb{N}} \) are Cauchy.

(Basically the same proof.)

**Euclidean Spaces (And other algebraic objects well do analysis on...)**

**Definition.** Euclidean \( n \)-space is
\[
\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}
\]

\( \mathbb{R}^n \) has operations:
\[
\begin{align*}
x &= (x_1, \ldots, x_n) \quad y = (y_1, \ldots, y_n) \quad \lambda \in \mathbb{R} \\
x + y &= (x_1 + y_1, \ldots, x_n + y_n) \\
\lambda x &= (\lambda x_1, \ldots, \lambda x_n)
\end{align*}
\]

**Proposition.**
\[
\begin{align*}
x + y &= y + x \\
\lambda(x + y) &= \lambda x + \lambda y \quad \text{and} \quad (\lambda + \mu)x = \lambda x + \mu x \\
(\lambda \mu)x &= \lambda(\mu x) \\
(x + y) + z &= x + (y + z) \\
1 \cdot x &= x \\
0 = (0, \ldots, 0) \quad \text{has} \quad 0 + x = x \quad \forall x \in \mathbb{R}^n \\
(-1) \cdot x + x &= 0
\end{align*}
\]

That is, \( \mathbb{R}^n \) is a vector space over the field \( \mathbb{R} \).
**Definition.** Define the scalar product or inner product on $\mathbb{R}^n$ by 
\[ \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \]
\[ \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \]

**Proposition.** For any $x, y, z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, we have:
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$, with equality exactly when $x=0$.
- $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \langle x, \lambda y \rangle$

**Definition.** A $\mathbb{R}$-vector space along with an inner product $\langle \cdot, \cdot \rangle$ satisfying the above proposition is called an inner product space.

**Proposition.** In an inner product space, set $\|x\| = \sqrt{\langle x, x \rangle}$.
Then $\| \cdot \|$ satisfies:
- $\|x\| \geq 0$, equality if $x=0$
- $\|x+y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda| \|x\|$

**Definition.** A vector space along with a norm $\| \cdot \|$ satisfying the above conditions is called a normed space.

**Definition.** In a normed space, define the metric (or distance function) by $d(x, y) = \|x - y\|$

**Proposition.** The metric in a normed space satisfies:
- $d(x, y) \geq 0$, equality only if $x=y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

**Definition.** A set $M$ along with a function $d : M \times M \rightarrow \mathbb{R}$ satisfying the above is called a metric space.
Why work in such generality?

Consider the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ (forget for the moment that we haven't defined what a continuous function is...)

For any $f,g \in C[0,1]$, define:

\[(f,g)_{L^2} = \int_0^1 f(x)g(x)\,dx.\]

(Again, we haven't defined what $\int \,dx$ means, but bear with me...)

Define \[(f+g)(x) = f(x) + g(x)\]

\[(\lambda f)(x) = \lambda f(x).\]

Then \(C[0,1]\) is a $\mathbb{R}$-vector space, work to be done here...

...and with \((\cdot,\cdot)_{L^2}\), it's an inner product space.

\[(f,g)_{L^2} = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = (g,f)_{L^2}\]

\[(f,f)_{L^2} = \int_0^1 f(x)^2\,dx = \int_0^1 (f(x))^2\,dx \geq 0 \quad \text{(Work to be done here, too...)}

If \((f,f)_{L^2} = \int_0^1 |f(x)|^2\,dx = 0\), then \(f=0\)

\[(f+g,h)_{L^2} = \int_0^1 (f(x)+g(x))h(x)\,dx = \int_0^1 f(x)h(x)\,dx + \int_0^1 g(x)h(x)\,dx = (f,h)_{L^2} + (g,h)_{L^2}

\[(\lambda f,g)_{L^2} = \int_0^1 \lambda f(x)g(x)\,dx = \lambda \int_0^1 f(x)g(x)\,dx = \lambda (f,g)_{L^2}\]

The norm induced by \((\cdot,\cdot)_{L^2}\) is

\[\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}\]
Metric Spaces

**Defn**: A metric space \((M,d)\) is a set \(M\) together with a distance function or metric

\[
d: M \times M \to \mathbb{R}
\]

Which satisfies the conditions for all \(x, y, z \in M\)

1. \(d(x, y) \geq 0\)
2. \(d(x, y) = 0\) if and only if \(x = y\)
3. \(d(x, y) = d(y, x)\)
4. \(d(x, z) \leq d(x, y) + d(y, z)\)

**Ex**: \(\mathbb{R}\) with the metric \(d(x, y) = |x - y|\)

- \(\mathbb{C}\) with the metric \(d(z, w) = |z - w|\)
- Any set with the discrete metric

\[
d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}
\]

Try to draw a picture of the discrete metric...

- \(M = \{x, y, z\}\)
- \(\mathbb{R}^3\)

We can't really do any better than drawing the discrete metric on 4 points. But the discrete metric is a metric on uncountable sets, since

1. \(d_0(x, y)\) is either 0 or 1, both of which are nonnegative.
2. If \(x \neq y\), then \(d_0(x, y) = 1 \neq 0\)
3. If \(x \neq y\), then \(d_0(x, y) = 1 = d_0(y, x)\)
   - If \(x = y\), then \(d_0(x, y) = 0 = d_0(y, x)\)
4. If \(x = z\), then \(d_0(x, z) = 0 \leq d_0(x, y) + d_0(y, z)\)
   - If \(x \neq z\), then if \(y = z\),
     \[
d_0(x, z) = 1 \quad d_0(x, y) = 1 \
     \]
     \[
d_0(y, z) = 0
     \]
   - if \(y \neq z\), then all distances are 1, so \(1 + 1 = 2\)

Another weird **Ex**:

Given any metric space \((M, d)\), we can tweak the metric to the induced bounded metric

\[
\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}
\]
Notice that since \( d(x,y) < 1 + d(x,y) \)
\[
\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1
\]
(Exercise: Prove \((M,\rho)\) is a metric space.)

The takeaway lesson is: metric spaces can be kinda weird.

Recall: A sequence \( x_n \to x \) if for any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) so that \( n \geq N \implies |x_n - x| < \varepsilon \).

We could rewrite this as:
\[
x_n \to x \text{ if for any } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ so that }\]
\[
n \geq N \implies x_n \in (x-\varepsilon, x+\varepsilon)
\]

It turns out almost all of our discussion of sequences in \( \mathbb{R} \)

can be rephrased in terms of little intervals \((x-\varepsilon, x+\varepsilon)\).

**Defn.** In a metric space \((M,d)\), the open ball of radius \( R > 0 \) about the point \( x \in M \) is
\[
D(x,R) = \{ y \in M \mid d(x,y) < R \}
\]
(Poehl calls these "neighborhoods" and reserves the term "ball" for
the standard metric on \( \mathbb{R} \)).

**Proposition.** The open balls in \( \mathbb{R} \) are precisely the open intervals.

**Proof.** The open ball \( D(x,R) \) in \( \mathbb{R} \) is
\[
\{ y \in \mathbb{R} \mid d(x,y) < R \} = \{ y \in \mathbb{R} \mid |x-y| < R \}
\]
\[
= \{ y \in \mathbb{R} \mid -R < x-y < R \} = \{ y \in \mathbb{R} \mid -R < x-y \}
\]
\[
= \{ y \in \mathbb{R} \mid x+R > y > x-R \} = (x-R, x+R)
\]

On the other hand, any open interval
\[
(a,b) = (\frac{1}{2}(a+b) - \frac{1}{2}(a-b), \frac{1}{2}(a+b) + \frac{1}{2}(b-a))
\]
\[
= D(\frac{1}{2}(a+b), \frac{1}{2}(b-a))
\]

**Defn.** A subset of a metric space \( A \subset (M,d) \) is open
if for every \( x \in A \), there is some \( \varepsilon > 0 \) such that
\( D(x,\varepsilon) \subset A \).

**Proposition.** Open balls are open.

**Proof.** Given an open ball \( D(x,R) \), we need to show that for any \( y \in D(x,R) \), there is \( \varepsilon > 0 \) with
\( D(y,\varepsilon) \subset D(x,R) \).
If \( y = x \), take \( R = \varepsilon \).
If \( y \neq x \), note that \( \varepsilon = d(y, x) < R \).
So \( R - \varepsilon > 0 \). Set \( \varepsilon = R - \varepsilon \).

Claim: \( D(y, \varepsilon) \subseteq D(x, R) \)

Proof: To show \( D(y, \varepsilon) \subseteq D(x, R) \).
Suppose \( z \in D(y, \varepsilon) \). Then
\[
d(y, z) < \varepsilon = R - \varepsilon.
\]
So \( d(x, z) \leq d(x, y) + d(y, z) = \varepsilon + d(y, z) < \varepsilon + (R - \varepsilon) = R. \)
So \( z \in D(x, R) \).

Eqn: The open ball in \( \mathbb{R}^2 \) with the standard metric is an ordinary disc (hence "D")

Q: Is \((0, 1)\) open?
A: It depends!

\((0, 1) \subset \mathbb{R} \) is the open ball of radius \( \frac{1}{2} \) centered at \( \frac{1}{2} \).
(hence open in \((R, 1)\))

But of course we could also view \((0, 1)\) as a subset of \( \mathbb{R}^2 \) with the standard metric:

Any open ball about a point \((x, 0)\) contains points with \( x > 0 \) and points with \( x < 0 \). So \((0, 1)\) is not open as a subset of \( \mathbb{R}^2 \) with the standard metric.

So remember: sets are not open; only subsets can be!

One way to understand what a metric space "looks like" is to look at its balls. (As you are asked to do in this week's homework.)

Eqn: The taxicab metric on \( \mathbb{R}^n \) is defined by:
\[
d_{\text{taxi}}(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|
\]

Many equally-short routes from \( x \) to \( y \).
Lots of ways to get equality in the triangle inequality.
Balls in the taxi cab metric look like:
(Points in the city accessible by a taxi ride of less than $R$ blocks)

**Open and Closed Sets**

**Defn.** $A \subset (\mathbb{R}, d)$ is open if for all $x \in A$, there is $\varepsilon > 0$ so that $D(x, \varepsilon) \subset A$.

**Defn.** Given two sets $A, B$,
- their union is $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
- their intersection is $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

Given a collection of sets $A_\lambda$, define
- the union $\bigcup_\lambda A_\lambda = \{ x \mid \text{there is some } \lambda \text{ with } x \in A_\lambda \}$
- the intersection $\bigcap_\lambda A_\lambda = \{ x \mid \text{for all } \lambda, x \in A_\lambda \}$

$\exists \iff \lor \quad \forall \iff \land$

"or"  "and"

**Proposition** Let $A_\lambda, \lambda \in \Lambda$ be a collection of open subsets of a metric space. Then
1) $\bigcup_\lambda A_\lambda$ is open
2) If there are finitely many $\lambda$, $\bigcap_\lambda A_\lambda$ is open.
Proof. 1) To show \( \bigcup A_i \) is open, consider an arbitrary \( x \in \bigcup A_i \). Then there is some \( i \) with \( x \in A_i \). Since \( A_i \) is open, there is \( \varepsilon > 0 \) with \( D(x, \varepsilon) \subseteq A_i \). Then \( D(x, \varepsilon) \subseteq A_i \subseteq \bigcup A_i \) (why?) so this \( \varepsilon > 0 \) works for \( \bigcup A_i \), too. \( \square \)

2) To show \( A_1 \cap A_2 \cap \cdots \cap A_n \) is open, consider an arbitrary \( x \in A_1 \cap A_2 \cap \cdots \cap A_n \). Since each \( A_i \) is open and \( x \in A_i \), there is \( \varepsilon_i > 0 \) so that \( D(x, \varepsilon_i) \subseteq A_i \). Set \( \varepsilon = \min \{ \varepsilon_1, \ldots, \varepsilon_n \} \). Then \( D(x, \varepsilon) \subseteq D(x, \varepsilon_i) \subseteq A_i \). So (why?) \( D(x, \varepsilon) \subseteq A_1 \cap A_2 \cap \cdots \cap A_n \).

Defn. The complement of \( A \subseteq M \) is \( A^c = M \setminus A \).

Defn. A subset of a metric space is closed if its complement is open. (This is not Rudin's definition but we'll see they're equivalent.)

Proposition (De Morgan's laws)

1) \((A \cup B)^c = A^c \cap B^c \)
2) \((A \cap B)^c = A^c \cup B^c \)

Proof. This is really just the fact that not \( A \) or \( B \) if and only if \( \neg (A \cup B) \).

1) Show \((A \cup B)^c \subseteq A^c \cap B^c\) by showing \((A \cup B)^c \subseteq A^c \) and \((A \cup B)^c \subseteq B^c \).

Since the statement is symmetric in \( A \) and \( B \) we'll just show \((A \cup B)^c \subseteq A^c \).

Suppose \( x \in (A \cup B)^c \).

2 cases. \( x \in A \). Then \( x \in A \cup B \); so \( x \notin (A \cup B)^c \).

\( x \notin A \). Then \( x \in A^c \).
Now show \( A \cap B^c \subseteq (A \cup B)^c \). If \( x \in A \cap B^c \), then

\[ x \notin A \quad \text{and} \quad x \notin B. \]

**2 cases**: \( x \in A \cup B \). Then either \( x \in A \) or \( x \in B \).

\[ x \notin A \cup B. \]

Then done.

**Proposition** In any metric space \((M,d)\), \(\emptyset\) and \(M\) are open.

**Proof**: Show \( M \subseteq M \) is open. Let \( x \in M \), and choose \( \epsilon = 1 \).

Then \( D(x,1) = \{ y \in M | d(x,y) < 1 \} \subseteq M \).

To see that \( \emptyset \subseteq M \) is open, recall that an open set means:

for any \( x \in A \), there is \( \epsilon > 0 \) with \( D(x,\epsilon) \subseteq A \).

But this is true of the empty set since there aren't any such \( x \in \emptyset \)!

**Corollary** In any metric space \((M,d)\), \(\emptyset\) and \(M\) are closed.

**Proof**: To see that \( \emptyset \) is closed, show \( \emptyset^c \) is open.

But \( \emptyset^c = M \), which by the proposition is open.

To see that \( M \) is closed, show \( M^c \) is open.

But \( M^c = \emptyset \), which by the proposition is open.

**Definition**: A subset which is both open and closed is called clopen. (Yes, really.)

**Example**: Consider \((\mathbb{Q}, d_{ST})\) (is any subset of a metric space?)

The set \( A = \{ q \in \mathbb{Q} \mid q^2 < 2 \} \) is open since for any \( q \in A \), there is some \( p \in A \) with \( p > q \).

Since in an ordered field we can square both sides of an inequality, the rational interval \((0, p) = \{ r \in \mathbb{Q} \mid 0 < r < p \} \subseteq A \).

If we set \( \epsilon = \min \{ p, p - q^2 > 0 \} \), then \( D(q,\epsilon) \subseteq (0, p) \subseteq A \).

Therefore \( A \) is open.

On the other hand, \( A^c = \{ q \in \mathbb{Q} \mid q^2 > 2 \} \)

\[ = \{ q \in \mathbb{Q} \mid q^2 > 2 \} \quad \text{(since no rational has } q^2 = 2) \]

A similar argument shows \( A^c \) is open.

So \( A \) is closed.

**Luckily for us**, clopen sets are relatively rare in well-behaved metric spaces.
The reason to define "interior" is that not all subsets are open, and we'd like to have some measure "how open" a given subset is. This is a general theme in the course (and more broadly...)

Proposition \( \text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B) \)

**Proof.** \( \text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B) \)

If \( x \in \text{int}A \cap \text{int}B \), then there is some open \( U \subset A \cap B \), with \( x \in U \). Since \( A \cap B \subset A \), \( x \in \text{int}A \).

Similarly, \( x \in \text{int}B \). So \( x \in \text{int}A \cap \text{int}B \).

\( \text{int}A \cap \text{int}B \subset \text{int}(A \cap B) \)

If \( x \in \text{int}A \), then there is \( U_A \) open with \( x \in U_A, U_A \subset A \).
If \( x \in \text{int}B \), then there is \( U_B \) open with \( x \in U_B, U_B \subset B \).

Then \( U_A \cap U_B \subset A \cap B \) is open, and \( x \in U_A \cap U_B \).
So \( x \in \text{int}(A \cap B) \).

Example. What is the interior of \( \mathbb{R} \setminus \mathbb{Q} \subset (\mathbb{R},d_{\text{eucl}}) \)?

Given \( x \in \mathbb{R} \), for any \( \varepsilon > 0 \), there is \( q \in \mathbb{Q} \) with \( |x - q| < \varepsilon \). So every interval in \( \mathbb{R} \) contains...
Some rational numbers. Thus $\text{int}(\mathbb{Q}) = \emptyset$.

- What is the interior of $\mathbb{Q}$?

Given any $x \in \mathbb{R}$ and $\varepsilon > 0$, there is $r \in \mathbb{Q}$ with $x - \varepsilon < r < x + \varepsilon$. So every interval in $\mathbb{R}$ contains some irrational number. Thus $\text{int}(\mathbb{Q}) = \emptyset$.

Just like we measure how open a set is with its interior points, we measure how closed a set is by:

Definition: $x \in M$ is a limit point for $A \subseteq M$ if every open set $U \subseteq M$ which contains $x$ also contains some $a \in A$ with $x \neq a$.

Note: $x$ need not be an element of $A$ to be a limit point for $A$.

Example: Every $x \in \mathbb{R}$ is a limit point for $\mathbb{Q} \subseteq \mathbb{R}$. Every $x \in \mathbb{R}$ is a limit point for $\mathbb{R} \subseteq \mathbb{R}$.

**Theorem.** Suppose $A \subseteq (\mathbb{R}, d)$ and $x$ is a limit point for $A$. Then for any $\varepsilon > 0$, $D(x, \varepsilon) \cap A$ contains infinitely many points of $A$.

**Proof.** Suppose there were some $\varepsilon > 0$ with $D(x, \varepsilon) \cap A = \{a_1, \ldots, a_n\}$.

Letting $\varepsilon = \frac{1}{2} \min \{d(x, a_i) | a_i \neq x\}$, we see that $D(x, \varepsilon)$ is an open set containing $x$, but which does not contain any point of $A$, except possibly $x$ itself. So $x$ is not a limit point for $A$. \(\Box\)

**Corollary.** Finite subsets have no limit points.

**Theorem.** A subset $A \subseteq (\mathbb{R}, d)$ is closed if it contains all of its limit points.

**Proof.** ($\Rightarrow$) Suppose $A$ is closed. Then $M \setminus A$ is open.

Let $x$ be a limit point for $A$.

2 cases: $x \in A$ Then we're done.

- $x \notin A$. Since $M \setminus A$ is open, there is $\varepsilon > 0$ with $D(x, \varepsilon) \subseteq M \setminus A$. But this contradicts...
the assumption that \( x \) is a limit point for \( A \). So this case doesn't occur.

\[ \iff \]

To show \( A \) is closed, show \( M \setminus A \) is open.

Let \( x \in M \setminus A \). Since \( A \) contains all its limit points, \( x \) can't be a limit point. So \( \exists \varepsilon > 0 \) with \( D(x, \varepsilon) \cap A = \emptyset \), i.e. \( D(x, \varepsilon) \subseteq M \setminus A \).

Thus \( M \setminus A \) is open.

\[ \Box \]

**Defn.** The closure of \( A \subset (M, \tau) \) is \( \text{cl}(A) = \overline{A} = \{ x \in M : x \text{ is a limit point of } A \} \).

**Proof.**

1) \( A \subseteq \text{cl}(A) \)

2) \( \text{cl}(A) \) is closed

3) \( A = \text{cl}(A) \) iff \( A \) is closed.

**Proof.**

1) is clear from defn.

2) Consider \( M \setminus \text{cl}(A) = \{ x \in M : x \notin A \text{ and } x \text{ is not a limit point for } A \} \).

Let \( x \in M \setminus \text{cl}(A) \).

Then \( \exists \varepsilon > 0 \) so that \( D(x, \varepsilon) \cap A = \emptyset \)

(same argument as in the previous theorem.)

**Claim.** \( D(x, \varepsilon) = M \setminus \text{cl}(A) \)

**Proof.** We already have that no point of \( D(x, \varepsilon) \) lies in \( A \); what remains to be shown is that no point of \( D(x, \varepsilon) \) is a limit point for \( A \).

Let \( y \in D(x, \varepsilon) \) and let \( \varepsilon > 0 \) be such that \( D(y, \varepsilon) \subseteq D(x, \varepsilon) \). No point of \( D(y, \varepsilon) \) lies in \( A \), (not even \( y \)), so \( y \) is not a limit point for \( A \).

3) \( \iff \) Since \( \text{cl}(A) \) is closed, \( A = \text{cl}(A) \Rightarrow A \) is closed

\[ \iff \] Suppose \( A \) is closed. Then \( A \) contains its limit points, so \( A = A \cup \{ \text{limit points for } A \} = \text{cl}(A) \).
Theorem. Given \( A \subset (M,d) \), consider the collection

\( F_A = \{ F \subset M \mid F \text{ is closed and } A \subset F \} \)

of all closed subsets of \( M \) which contain \( A \). \( \overline{A} = \bigcap F \).

Proof. (\( \subset \)) We'll show that for any \( F \in F_A \), \( \overline{A} \subset F \).

Since \( \overline{A} = \bigcup \{ A \cup \{ \text{limit points of } A \} \} \), and \( A \subset F \) by assumption, just need to show each limit point for \( A \) is in \( F \).

Since \( A \subset F \), every limit point for \( A \) is also a limit point for \( F \).

Since \( F \) is closed, \( F \) contains all its limit points.

So each limit point for \( A \) lies in \( F \).

(\( \supset \)) Suppose \( x \in M \) is neither an element of \( A \) nor a limit point for \( A \). Then, there is some \( \varepsilon > 0 \) so that \( D(x, \varepsilon) \) contains no point of \( A \). Note that \( D(x, \varepsilon)^c \) is a closed set which contains \( A \). But \( x \notin D(x, \varepsilon)^c \).

So \( x \notin \overline{F} \).

Thus \( (\overline{A})^c = \bigcap F^c \), so \( \overline{A} \subset \overline{F} \).

This any closed set which contains \( A \) also contains \( \overline{A} \).

\( \overline{A} \) is the minimal closed set containing \( A \).

Defn. Given \( A \subset B \subset (M,d) \), say \( A \) is dense in \( B \) if \( B = \overline{A} \). Equivalently, \( A \) is dense in \( B \) it for every \( \varepsilon > 0 \) and every \( b \in B \), \( D(b, \varepsilon) \) contains some element of \( A \).

Ex. The rationals are dense in \( \mathbb{R} \).

Stone-Weierstrass Theorem: Polynomials are dense in \( C[0,1] \).

(We don't quite know what this means—yet.)
Boundary

Definition: The boundary of a subset \( A \subseteq \mathbb{R}^n \) is

\[ \partial A = \text{cl}(A) \cap \text{cl}(\overline{A}) \]

Immediate consequences:
1) \( \partial A \) is closed
2) \( \partial A = \partial (\overline{A}) \)

Proposition: \( x \in \partial A \) if every open set \( U \) containing \( x \) also contains some point of \( A \) and some point not in \( A \).

Proof:
\[ \partial A = \text{cl}(A) \cap \text{cl}(\overline{A}) = (A \cup \text{lim for } A) \cap (\overline{A} \cup \text{lim for } \overline{A}) = (A \cap \overline{A}) \cup (A \cap \text{lim for } \overline{A}^c) \cup (\overline{A} \cap \text{lim for } A) \cup (\overline{A} \cap \text{lim for } A^c) \]

(Intersection distributes over union, so we can FOIL)

\[ (B \cup C) \cap (D \cup F) \text{ has for possible parts } \]

\[ A \cap (\overline{A} \cap \text{lim for } \overline{A}^c) \]

So any such \( x \in (\text{lim for } A \cap \overline{A}^c) \cup (\overline{A} \cap \text{lim for } A) \subseteq \partial A. \]

\[ A \cap (\overline{A} \cap \text{lim for } \overline{A}^c) \]

Consider the cases for \( x \in \partial A \):

1) \( x \in A \) and \( x \) is a limit point for \( \overline{A} \).
   Then any open \( U \) containing \( x \) contains a point of \( A \) (namely \( x \)) and contains (infinitely many) points of \( \overline{A} \).

2) \( x \in \overline{A} \) and \( x \) is a limit point for \( A \).
   Same as 1.

3) \( x \) is a limit point for \( \overline{A} \) and for \( A \).
   Then every open set \( U \) containing \( x \) contains (infinitely many) points of \( A \) and of \( \overline{A} \).

\((\Rightarrow)\) Two cases:
1) \( x \in A \) then \( x \in A \). By hypothesis every open \( U \) contains some point of \( \overline{A} \), which can't be \( x \). So \( x \) is a limit point for \( \overline{A} \).

2) \( x \in \partial A \) Save as 1.

So any such \( x \in (\text{lim for } A \cap \overline{A}^c) \cup (\overline{A} \cap \text{lim for } A) \subseteq \partial A. \)
For $Q = \text{cl}(Q) \cap \text{cl}(R\setminus Q) = R \cap R = R$.

Let's see what can be said about the closure and boundary of $D(x, R)$.

$D(x, R) = \{y \in M \mid d(xy) < R\}$

$D(x, R) = \{y \in M \mid d(xy) \geq R\}$

What is $\text{cl}(D(x, R))$?

Claim: $B(x, R) = \{y \in M \mid d(xy) < R\}$ is closed.

Note: $\partial A$ is not just $\text{cl}(A) \setminus A$, e.g., $A = \emptyset$, then $\partial A = \emptyset$, $\text{cl}(A) = A$, $\text{cl}(A) \setminus A = \emptyset$.

Lemma: $\text{cl}(A) \setminus A < A$

Proof: Let $x \not\in \text{cl}(A) \setminus A$.

Proposition. $x \in \text{cl}(A)$ if $\inf \{d(x,y) \mid y \in A\} = 0$.

Proof. $(\Rightarrow)$ Suppose $x \in \text{cl}(A)$. Two cases:

$x \not\in A$. Then $d(x,x) = 0$, so $\inf \{d(x,y) \mid y \in A\} = 0$.

$x$ is a limit point for $A$. Then for each $\varepsilon > 0$ there is some $y \in A$ with $d(x,y) < \varepsilon$.

So $\inf \{d(x,y) \mid y \in A\} < \varepsilon$. Since $\varepsilon$ was arbitrary, $\inf \{d(x,y) \mid y \in A\} = 0$.

$(\Leftarrow)$ By the $\varepsilon$-characterization of $\inf$, for each $\varepsilon > 0$ there is $y \in A$ with $0 \leq d(x, y) < \varepsilon$.

If some $\varepsilon$ has $y = x$, then $x \in A \subset \text{cl}(A)$.

If no $\varepsilon$ has $y = x$, then $y$ is an element of $D(x, \varepsilon) \cap A$ not equal to $x$, so $x$ is a limit point for $A$. $\blacksquare$