Sequences in Metric Spaces

Defn A *sequence* in a metric space (M,d) is a choice, for each $n\in\mathbb{N}$, of an element $x_n\in M$. Write $(x_n)_{n\in\mathbb{N}}$.

Defn A sequence $(x_n)_{n\in\mathbb{N}}$ *converges to* $x\in M$ if, for every open set U containing x, there is $N\in\mathbb{N}$ so that $n>N$ guarantees $x_n\in U$.

Prop'n $cl(A) = \left\{ x \;\middle|\; \begin{array}{l}\text{there is a sequence}\\ (x_n)_{n\in\mathbb{N}} \text{ with } x_n\in A\\ \text{and } x_n\to x\end{array}\right\}$

Proof If $x\in cl(A)$, two cases:

$x\in A$ Set $x_n = x$ for all $n\in\mathbb{N}$.

x is a limit point for A Since x is a limit point, for each n there is some $x_n\in A$ with $x_n\in D(x,\frac{1}{n})$.

Claim: $x_n\to x$.

If there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points of A limiting to x, then every open set U containing x contains the tail of $(x_n)_{n\in\mathbb{N}}$. So $x\in cl(A)$. ☒

Hence the name "limit point"

Corollary The boundary of a set A is exactly the collection of points x with the property that there are sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ so that:
1). $\forall n$, $a_n\in A$
2) $\forall n$, $b_n\notin A$
3) $\lim a_n = x = \lim b_n$.

Proposition A is closed iff every sequence $(a_n)_{n\in\mathbb{N}}$ of points of A which converges has $x=\lim a_n\in A$.

Note: The condition is not that every sequence in A must converge, only that if a sequence of points of A happens to converge, its limit must lie in A.

This proposition is the origin of the term "closed." Think of "closed" as shorthand for "closed under taking limits".

Proof: If $a_n\to x$, then x is a limit point for A.
If x is a limit point for A, then $\exists\, a_n\to x$.
Finish as an exercise.

Proposition $x_n \to x$ iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ so that $n \geq N$ guarantees $d(x_n, x) < \varepsilon$.

Proof ($\Rightarrow$) Since $x_n \to x$, we can take $U = D(x, \varepsilon)$ in the definition of convergence. ▨

($\Leftarrow$) Given an open set U containing x, there is $\varepsilon > 0$ with $D(x, \varepsilon) \subset U$. By hypothesis, there is $N \in \mathbb{N}$ with $n \geq N \Rightarrow x_n \in D(x, \varepsilon) \subset U$. ▨

Cauchy Sequences and Complete Metric Spaces

Notions like "monotone" and "least" require an ordering, which doesn't exist in a general metric space. Nevertheless, metric spaces like

$$\mathbb{Q}^2 = \{(q^1, q^2) \mid q^1, q^2 \in \mathbb{Q}\}$$

$$\mathbb{R}^2 \setminus \mathbb{R}$$

have "holes". How to formulate (and then fix!) this? We can talk about the Cauchy property!

Defn A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (M, d) is **Cauchy** if $\forall \varepsilon > 0\ \exists N \ni m, n \geq N$ guarantees $d(x_m, x_n) < \varepsilon$.

Defn A metric space (M, d) is **complete** if every Cauchy sequence in (M, d) converges in (M, d).

E.g. $\mathbb{R}^2 \setminus \mathbb{R} = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^2 \neq 0\}$ is not complete.

$\left(\left(0, \frac{1}{n}\right)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence. But if $\left(0, \frac{1}{n}\right) \to (x^1, x^2)$, we'd have $x^2 = 0$. So the limit of this sequence could not be in $\mathbb{R}^2 \setminus \mathbb{R}$.

Def'n A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (M,d) is **bounded** if there is $y\in M$ and $R\in\mathbb{R}$ so that for all $n\in\mathbb{N}$,

$$d(y,x_n)\leq R.$$

Proposition
- Every convergent sequence is Cauchy.
- Every convergent sequence is bounded.
- Every Cauchy sequence is bounded.
- A Cauchy sequence can have at most one subsequential limit.
- If a subsequence of a Cauchy sequence converges then the Cauchy sequence converges.

Prop'n Let N be a subset of a complete metric space (M,d). Then (N,d) is complete iff $N\subset(M,d)$ is closed.

Proof (closed in $M \Rightarrow$ complete)

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in N. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M, so by completeness $\exists\, x\in M$ with $x_n\to x$ in (M,d). Since N is closed, $x\in N$. ∎

(complete $\Rightarrow$ closed in M)

Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in N which converges in (M,d), say to $y\in M$. This sequence is therefore Cauchy in (M,d) hence Cauchy in (N,d), so by completeness of (N,d) there is $x\in N$ with $x_n\to x$. Then $x=y$, so any sequence in N which converges in M converges in N, so N is closed. ∎

Theorem Given any metric space (M,d), there is a metric space $(\bar{M},\bar{d})$ so that:

1) $(\bar{M},\bar{d})$ is complete.
2) There is a one-to-one map $\phi: M\to\bar{M}$ with $\bar{d}(\phi(x),\phi(y))=d(x,y)$ "ϕ is isometric"
3) If $\psi: M\to(N,\rho)$ is any isometric injection into a complete metric space, then there is $\tilde{\psi}:\bar{M}\to N$ an isometric injection with $\tilde{\psi}\circ\phi=\psi$.

$$\begin{tikzcd} & \bar{M} \arrow[d, "\tilde{\psi}"] \\ M \arrow[ur, "\phi"] \arrow[r, "\psi"'] & N \end{tikzcd}$$

Def'n We call $(\bar{M},\bar{d})$ the completion of (M,d).

Proof of Theorem

We'll construct $(\bar{M},\bar{d})$ by taking (M,d) and throwing in all of the Cauchy sequences.

Def'n Two Cauchy sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are equivalent $\left((x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \right)$ if $\lim_{n\to\infty} d(x_n,y_n)=0$.

Lemma $\sim$ is an equivalence relation. That is:

1) $(x_n)_{n\in\mathbb{N}} \sim (x_n)_{n\in\mathbb{N}}$ reflexive
2) $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \Rightarrow (y_n)_{n\in\mathbb{N}} \sim (x_n)_{n\in\mathbb{N}}$ symmetric
3) If $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \sim (z_n)_{n\in\mathbb{N}}$, then $(x_n)_{n\in\mathbb{N}} \sim (z_n)_{n\in\mathbb{N}}$. transitive

Proof The first two are exercises.

3) $0 \le d(x_n,z_n) \le d(x_n,y_n) + d(y_n,z_n)$ so

$$0 \le \lim d(x_n,z_n) \le \lim d(x_n,y_n) + \lim d(y_n,z_n) = 0.$$

Now consider the set of equivalence classes of Cauchy sequences in M, $\bar{M} = \{ [(x_n)_{n\in\mathbb{N}}] \mid (x_n)_{n\in\mathbb{N}} \text{ is Cauchy in } (M,d) \}$

Recall: $[(x_n)_{n\in\mathbb{N}}] = \{ (y_n)_{n\in\mathbb{N}} \mid (y_n)_{n\in\mathbb{N}} \text{ is Cauchy and } (x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \}$

Def'n Given $X, Y \in \bar{M}$, pick any $(x_n)_{n\in\mathbb{N}} \in X$, $(y_n)_{n\in\mathbb{N}} \in Y$, and set $\bar{d}(X,Y) = \lim d(x_n,y_n)$

Lemma This $\bar{d}(X,Y)$ is well-defined, ie does not depend on the choices $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$.

Proof Let $(x_n)_{n\in\mathbb{N}}, (x_n')_{n\in\mathbb{N}}$ be two representatives of X,
$(y_n)_{n\in\mathbb{N}}, (y_n')_{n\in\mathbb{N}}$ " " Y.

We need to show $\lim d(x_n,y_n) = \lim d(x_n',y_n')$.

$$\lim d(x_n,y_n) \le \lim d(x_n,x_n') + d(x_n',y_n') + d(y_n',y_n)$$
$$= \lim d(x_n',y_n')$$

and

$$\lim d(x_n',y_n') \le \lim d(x_n',x_n) + d(x_n,y_n) + d(y_n,y_n')$$
$$= \lim d(x_n,y_n). \quad \blacksquare$$

Lemma $(\bar{M},\bar{d})$ is a metric space.

Proof. Let $X, Y, Z \in \bar{M}$, with representatives $(x_n)_{n\in\mathbb{N}} \in X$, $(y_n)_{n\in\mathbb{N}} \in Y$, $(z_n)_{n\in\mathbb{N}} \in Z$.

- $\bar{d}(X,Y) = \lim d(x_n,y_n) \geq 0$ ✓
- $\bar{d}(X,X) = \lim d(x_n,x_n) = 0$ ✓
- If $\bar{d}(X,Y)=0$, then $\lim d(x_n,y_n)=0$, so by def'n of $\sim$, $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$, so $X=Y$. ✓
- $\bar{d}(X,Z) = \lim d(x_n,z_n)$

$$\leq \lim d(x_n,y_n) + d(y_n,z_n) \quad ✓$$
$$= \bar{d}(X,Y) + \bar{d}(Y,Z)$$

Actually all of the above assumed that the limits $\bar{d}(X,Y) = \lim d(x_n,y_n)$ existed! To justify this:

For $m,n \in \mathbb{N}$,

$$d(x_n,y_n) \leq d(x_n,x_m) + d(x_m,y_m) + d(y_m,y_n)$$

So $d(x_n,y_n) - d(x_m,y_m) \leq d(x_n,x_m) + d(y_m,y_n)$.

Swapping m and n,

$$|d(x_n,y_n) - d(x_m,y_m)| \leq d(x_n,x_m) + d(y_m,y_n).$$

Let $\varepsilon > 0$.

Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, $\exists N_x$ with $m,n \geq N_x$ guaranteeing $d(x_n,x_m) < \frac{\varepsilon}{2}$.

Sim. $\exists N_y$ so $m,n \geq N_y$ guarantees $d(y_m,y_n) < \frac{\varepsilon}{2}$.

So if $m,n \geq N = N_x + N_y$, we have

$$|d(x_n,y_n) - d(x_m,y_m)| < \varepsilon.$$

That is, $(d(x_k,y_k))_{k\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\lim d(x_k,y_k)$ exists. ∎

Lemma (Second-to-last Nail)

$(\bar{M}, \bar{d})$ is a complete metric space.

Proof Let $(X_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\bar{M},\bar{d})$. Let $(x_n^k)_{n\in\mathbb{N}}$ be a representative of X_k.

Since $(X_k)_{k\in\mathbb{N}}$ is Cauchy, $\forall \varepsilon$ there is N so that $k,l \geq N$ guarantees $\underset{\lim_n d(x_n^k,x_n^l)}{\underset{\parallel}{\bar{d}(X_k,X_l)}} < \frac{\varepsilon}{2}$

Since each $(x_n^k)_{n\in\mathbb{N}}$ is Cauchy there is N_k so that $m,n \geq N_k$ guarantees $d(x_m^k, x_n^k) < \frac{1}{k}$.

Define a diagonal sequence $x_k = x_{N_k}^k$.

Claim $(X_k)_{k\in\mathbb{N}}$ is Cauchy in (M,d)

Proof For any $p\in\mathbb{N}$,

$$d(x_k,x_\ell)=d(x^k_{N_k},x^\ell_{N_\ell})\le d(x^k_{N_k},x^k_p)+d(x^k_p,x^\ell_p)+d(x^\ell_p,x^\ell_{N_\ell}).$$

If $p\ge\max\{N_k,N_\ell\}$, then

$$\le\frac{1}{k}+d(x^k_p,x^\ell_p)+\frac{1}{\ell}$$

Now take the limit in p

$$d(x_k,x_\ell)=\lim_p d(x_k,x_\ell)\le\lim_p \frac{1}{k}+d(x^k_p,x^\ell_p)+\frac{1}{\ell}$$
$$=\frac{1}{k}+\frac{1}{\ell}+\overline{d}(X^k,X^\ell)$$

If $k,\ell\ge\max\{N,\frac{4}{\varepsilon}\}$, then

$$<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon.\ \blacksquare$$

So $(x_k)_{k\in\mathbb{N}}$ defines an equivalence class $X=[(x_k)]\in\overline{M}$.

Claim $X_k\to X$ in $(\overline{M},\overline{d})$.

Proof

$$\overline{d}(X_k,X)=\lim_p d(x^k_p,x_p)=\lim_p d(x^k_p,x^p_{N_p})$$
$$\le\lim_p\Big[\overset{(1)}{d(x^k_p,x^k_{N_k})}+\overset{(2)}{d(x^k_{N_k},x^k_q)}+\overset{(3)}{d(x^k_q,x^p_q)}+\overset{(4)}{d(x^p_q,x^p_{N_p})}\Big]$$

If $q\ge N_k$, $(2)<\frac{1}{k}$. If $p\ge N_k$, $(1)<\frac{1}{k}$

If $q\ge N_p$, $(4)<\frac{1}{p}$. So for large p,q,

$$d(x^k_p,x^p_{N_p})<\frac{2}{k}+\frac{1}{p}+d(x^k_q,x^p_q)$$

Taking $\lim_q$,

$$d(x^k_p,x^p_{N_p})<\frac{2}{k}+\frac{1}{p}+\overline{d}(X_k,X_p).$$

But $(X_k)_{k\in\mathbb{N}}$ is Cauchy, so choosing $p\ge\max\{N,\frac{2}{\varepsilon}\}$,

$$d(x^k_p,x^p_{N_p})<\frac{2}{k}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\frac{2}{k}+\varepsilon$$

Taking $\lim_p$, $\overline{d}(X_k,X)=\lim_p d(x^k_p,x^p_{N_p})<\frac{2}{k}+\varepsilon$

But ε was arbitrary, so $\overline{d}(X_k,X)\le\frac{2}{k}$.

Thus $X_k\to X$ in $(\overline{M},\overline{d})$. $\blacksquare$

Last Nail There is an isometric injection $\phi: M\to\overline{M}$.

Proof For each $x\in M$, the constant sequence $x_n=x$ is Cauchy. Define $\phi(x)=[(x_n=x)]\in\overline{M}$.

Showing this is injective and isometric is an exercise.

Functions (finally!)

Def'n A *function* or *map* from the set A to the set B is a way of associating, for each $a \in A$, some $f(a) \in B$.

We write $f: A \to B$

Def'n Given (M,d), (N,ρ) metric spaces, $A \subset M$, $f: A \to N$, x_0 a limit point for A, we say $b \in N$ is the limit of f at x_0, or

$$\lim_{a \to x_0} f(a) = b$$

if for any $\varepsilon > 0$ there is $\delta > 0$ so that for all $a \in A$ with $a \neq x_0$ and $d(a,x_0) < \delta$, $\rho(f(a), b) < \varepsilon$.

Q. Why do we require x_0 to be a limit point for A?

Q. Why did we exclude the centre in the definition of "limit point"?

Exercise Rewrite the definition of "$\lim_{a \to x_0} f(a) = b$" in terms of open balls. Then rewrite it in terms of open sets.

Proposition Limits of functions are unique, i.e. if $\lim_{a \to x_0} f(a) = b_1$ and $\lim_{a \to x_0} f(a) = b_2$, then $b_1 = b_2$.

Proof Let $\varepsilon > 0$. By def'n of $\lim_{a \to x_0} f(a) = b_i$, $\exists\ \delta_1 > 0$ and $\delta_2 > 0$ so that if $a \neq x_0$ has $d(a,x_0) < \min\{\delta_1, \delta_2\}$, then $\rho(f(a), b_1) < \frac{\varepsilon}{2}$ and $\rho(f(a), b_2) < \frac{\varepsilon}{2}$.

So $\rho(b_1, b_2) \leq \rho(f(a), b_1) + \rho(f(a), b_2) < \varepsilon$.

This works so long as we can find a with $d(a, x_0) < \min\{\delta_1, \delta_2\}$ and $a \neq x_0$. Why does such a exist?

So for all ε, we have $\rho(b_1, b_2) < \varepsilon$.

Thus $\rho(b_1, b_2) = 0$ so $b_1 = b_2$. ∎

E.g. If $A = (x_0, \infty) \subset (\mathbb{R}, d_{std})$, then x_0 is a limit point for a. We usually write $\lim_{x \to x_0^+} f(x)$, or $f(x_0^+)$.

E.g. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Consider $f: \mathbb{Q} \to \mathbb{R}$. Then $\lim_{x \to 0} f(x) = 0$

Consider $f: \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$. Then $\lim_{x \to 0} f(x) = 1$.

Proposition ("Two-Path Test")

$\lim_{a\to x_0} f(a) = b$ iff for any sequence $(x_n)_{n\in\mathbb{N}}$ in $A\setminus\{x_0\}$ with $x_n \to x_0$, we have $f(x_n)\to b$.

Proof ($\Rightarrow$) Suppose $\lim_{a\to x_0} f(a)=b$ and $x_n\to x_0$.

To show $f(x_n)\to b$, given $\varepsilon>0$,

By defn of limit, $\forall \varepsilon>0\ \exists\, \delta>0$ so that if $d(a,x_0)<\delta$ and $a\neq x_0$, then $\rho(f(a),b)<\varepsilon$.

Since $x_n\to x_0$, for any δ there is $N\in\mathbb{N}$ so $n\geq N$ guarantees $d(x_n,x_0)<\delta$. Then $n\geq N$ implies $\rho(f(x_n),b)<\varepsilon$.

Since ε was arbitrary, $f(x_n)\to b$. //

($\Leftarrow$) Prove the contrapositive: suppose $\lim_{a\to x_0} f(a)=b$ is false.

Then there is some $\varepsilon>0$ so that for any $\delta>0$, there is some $a\in A\setminus\{x_0\}$ with $0<d(a,x_0)<\delta$ but $\rho(f(a),b)\geq\varepsilon$. Let x_k be the x produced if $\delta=\frac{1}{k}$. Then $x_k\to x_0$, but $f(x_k)$ stays ε away from b. So $f(x_k)\to b$ is false. ■

Defn Given $A\subset M$, $f: M\to(N,\rho)$, $x_0\in A$, we say f is **continuous at** x_0 if either

1) x_0 is not a limit point for A. or
2) $\lim_{a\to x_0} f(a) = f(x_0)$.

Proposition f is continuous at x_0 iff for every $(x_k)_{k\in\mathbb{N}}$ with $x_k\to x_0$, we have $f(x_k)\to f(x_0)$

That is, f is continuous at x_0 if $\lim_{a\to x_0} f(a) = f\left(\lim_{a\to x_0} a\right)$ (x_0 is not a limit point for A or)

Defn We say $f: A\to N$ **is continuous on** $A'\subset A$ if f is continuous at x_0 for each $x_0\in A'$

We say $f: A\to N$ **is continuous** if f is continuous on A.

Defn If $f: A\to B$, $U\subset A$, $V\subset B$, define:

the **preimage (or inverse image) of V under f**

$$f^{-1}(V) = \{a\in A \mid f(a)\in V\}\subset A$$

the **image of U under f**

$$f(U) = \{f(a) \mid a\in U\}\subset B$$

Theorem If $f:(M,d)\to(N,\rho)$ is a map between metric spaces, the following are equivalent:

① f is continuous

② For each open $V\subset(N,\rho)$, $f^{-1}(V)\subset(M,d)$ is open.

③ For each closed $W\subset(N,\rho)$, $f^{-1}(W)\subset(M,d)$ is closed.

② "the preimage of an open set is open"
"f pulls back open sets to open sets"

③ "preimages of closed sets are closed."

Proof ③ ⇒ ②

Given an open $V\subset N$, $V^c\subset N$ is closed.

So
$$\begin{aligned} f^{-1}(V^c) &= \{a\in M \mid f(a)\in V^c\}\\ &= \{a\in M \mid f(a)\notin V\}\\ &= \{a\in M \mid f(a)\in V\}^c = [f^{-1}(V)]^c \end{aligned}$$
is closed. Hence $f^{-1}(V)$ is open.

② ⇒ ①

To show f is continuous, let $x_0\in M$. Given $\varepsilon>0$, we need to find δ so that if $a\neq x_0$ and $d(a,x_0)<\delta$, we are guaranteed $\rho(f(a),f(x_0))<\varepsilon$.

The set $D_\rho(f(x_0),\varepsilon)$ is open in (N,ρ). So $f^{-1}(D_\rho(f(x_0),\varepsilon))$ is open.

$x_0\in f^{-1}(D_\rho(f(x_0),\varepsilon))$, so there is $\delta>0$ with $D_d(x_0,\delta)\subset f^{-1}(D_\rho(f(x_0),\varepsilon))$.

Thus $d(x_0,a)<\delta \Rightarrow a\in D(x_0,\delta)\Rightarrow a\in f^{-1}(D_\rho(f(x_0),\varepsilon))$
$\Rightarrow f(a)\in D_\rho(f(x_0),\varepsilon)\Rightarrow \rho(f(a),f(x_0))<\varepsilon$.

(What about "$a\neq x_0$"?)

① ⇒ ③

Given a closed $V\subset(N,\rho)$ we want to show $f^{-1}(V)$ is closed. So let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of $f^{-1}(V)$ which converges to x_0.

Then by continuity of f, $f(x_0)=f(\lim x_n)=\lim f(x_n)$

Since V is closed and $f(x_n)\in V$, we have $\lim f(x_n)\in V$.

So $f(x_0)\in V \Leftrightarrow x_0\in f^{-1}(V)$. ∎

Our second calculus theorem:

Max-Min Theorem

Suppose $f: [a,b] \to \mathbb{R}$ is a continuous function on a closed interval. Then

1) f is bounded
2) There exist $c_-, c_+ \in [a,b]$ with
$$f(c_-) = \inf\{f(x) \mid x \in [a,b]\}$$
$$f(c_+) = \sup\{f(x) \mid x \in [a,b]\}$$

"Continuous functions achieve their sup and inf on closed intervals."

Quick proof of 2

Suppose we've proved 1), so that $\sup\{f(x) \mid x \in [a,b]\} = S$ is finite. Then there exists a sequence $(y_k)_{k \in \mathbb{N}}$ in $\{f(x) \mid x \in [a,b]\}$ with $S - \frac{1}{k} < y_k < S$.
Each $y_k = f(x_k)$ for some $x_k \in [a,b]$. The theorem of Bolzano-Weierstraß says some subsequence of $(x_k)_{k \in \mathbb{N}}$, say $(x_{k_p})_{p \in \mathbb{N}}$, converges to $x_+ \in [a,b]$.
Since f is continuous, $f(x_+) = f(\lim_p x_{k_p}) = \lim_p f(x_{k_p}) = S$. $\blacksquare$

But this doesn't prove 1), and in any case we'd like a max-min theorem which handles questions like

"Is there a point on the earth which is hottest?"

Intuitively, the closed interval $[a,b]$ is finite in some sense. It has finite length, for example. But (a,b) has finite length, too, yet

E.g. Consider $f: (0,1) \to \mathbb{R}$.
$$x \mapsto \frac{1}{x}$$
f is continuous, but $\sup\{f(x) \mid x \in (0,1)\} = \infty$.

So "finite length" isn't enough. We need another topological property:

COMPACTNESS

Def'n Given a subset A of a metric space (M,d), an open cover for A is a collection $\{U_\lambda\}_{\lambda\in\Lambda}$ of open sets so that $A \subset \bigcup_\lambda U_\lambda$.

E.g.

probably an uncountable collection → • Let (M,d) be any metric space, and set $U_x = D(x,1)$. Then $\{U_x\}_{x\in M}$ is an open cover for M

countable collection → • Fix some $x_0 \in M$. For each $n\in\mathbb{N}$, set $U_n = D(x_0,n)$. Then $\{U_n\}_{n\in\mathbb{N}}$ is an open cover for M.

finite collection → • The collection $\{M\}$ is an open cover!

• $(\mathbb{R}^2, d_{std})$, $U_n = \left\{(x,y) \;\middle|\; \begin{matrix} x\in(n-1,n+1) \\ y\in\left(-\frac{1}{n},\frac{1}{n}\right)\end{matrix}\right\}$

Def'n Given an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ for a set $A\subset(M,d)$, a subcover is a subset of $\{U_\lambda\}_{\lambda\in\Lambda}$ which is an open cover for A.

E.g.

• $\{D(x,1)\mid x\in M\}$ is a subcover of $\{D(x,r) \mid x\in M, r\in(0,\infty)\}$

• $\{D(x_0,n)\mid n\in\mathbb{N}\}$ is a subcover of $\{D(x,r) \mid x\in M, r\in(0,\infty)\}$

• $\{D(x,1)\mid x\in M\}$ is not a subcover of $\{D(x_0,n)\mid n\in\mathbb{N}\}$ (because it's not a subset!)

• $\{D(x_0,1)\}$ is not a subcover of $\{D(x,1)\mid x\in M\}$ (because it does not cover!)

Def'n A subset A of a metric space (M,d) is compact if every open cover for A has finite subcover.

E.g. $\mathbb{R}$ is not compact (with respect to the standard metric)

We only need to find one open cover with no finite sub cover. Try: $U_n = (-n,n)$.

Then for any finite subset $\{U_{n_1},\dots,U_{n_K}\}$,

$$\bigcup_{i=1}^{K} U_{n_i} = (-\max n_i, \max n_i)$$ which misses a lot.

Proposition Closed intervals are compact in $(\mathbb{R}, d_{std})$.

Proof To simplify things, we'll prove the interval $[0,1]$ is compact.

Consider an open cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ for $[0,1]$.

Define the set

$$C = \{x \mid \text{some finite collection } U_1, \dots, U_k \text{ covers } [0,x]\}$$

$0 \in C$ (take any of the U_λ which contain 0)

Also each $x \in C$ is ≤ 1. So C is a nonempty bounded subset, hence $\sup C \leq 1$.

Let c be a limit point for C.

Since $c \in [0,1]$, some U_{λ_0} contains c. Since U_{λ_0} is open, there is $\varepsilon > 0$ so that $(c-\varepsilon, c+\varepsilon)$ lies in U_{λ_0}. Since $c = \sup C$, there is some $x \in C$ with $x \in (c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2})$. So some finite collection $U_{\lambda_1}, \dots, U_{\lambda_k}$ covers $[0,x]$.

Then $U_{\lambda_0}, U_{\lambda_1}, \dots, U_{\lambda_k}$ covers $[0, c + \frac{\varepsilon}{2}]$

So $c \in C$. Thus C contains its limit points, i.e. is closed.

So $\sup C \in C$.

Indeed, the $\frac{\varepsilon}{2}$ allows us to extend C past $\sup C$ (which is absurd) so $\sup C = 1$.

Thus $1 \in C$, i.e. $[0,1]$ is covered by finitely many of the U_λ. ∎

That was kind of a mess. Can we find a more intuitive criterion?

Defn A subset A of (M,d) is *sequentially compact* if every sequence $(a_k)_{k \in \mathbb{N}}$ of points of A has a convergent subsequence whose limit lies in A.

E.g.
- $\mathbb{R}$ is not sequentially compact.
  Use $a_n = n$.
- $[a,b]$ is sequentially compact. This is the Bolzano Weierstraß Theorem.

Bolzano-Weierstraß Theorem

A subset A of a metric space (M,d) is compact iff it is sequentially compact.

We'll need some lemmata (that will help us understand compactness better, too).

Lemma 1 Any compact set is closed.

Proof Let $A \subset (M,d)$ be a compact set.

Case 1 $A = M$. Then A is closed.

Case 2 $A \neq M$. Then $\exists x_0 \in M \setminus A$

Consider the cover $U_n = \{y \mid d(x_0,y) > \frac{1}{n}\}$.

Then $\{U_n\}_{n\in\mathbb{N}}$ covers $M \setminus \{x_0\}$, hence A.

Since A is compact, there is some finite subcover.

So $\exists N$ with $y \in A \Rightarrow d(x_0,y) > \frac{1}{N}$.

Thus $D(x_0, \frac{1}{N}) \subset M \setminus A$.

Now $x_0 \in M \setminus A$ was arbitrary. So $M \setminus A$ is open. $\boxtimes$

Lemma 2 If (M,d) has M compact and $B \subset M$ is closed. Then B is compact.

Proof If $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover for B.

Then $\{U_\lambda\}_{\lambda\in\Lambda} \cup \{M \setminus B\}$ is an open cover for M.

Since M is compact, there is a finite subcover $\{U_1, \dots, U_N, M\setminus B\}$ for M. Then $\{U_1, \dots, U_N\}$ is a finite subcover of $\mathcal{U}$. $\boxtimes$

Proof of Bolzano-Weierstraß:

(compact $\Rightarrow$ sequentially compact)

Given $A \subset (M,d)$ compact, $(a_n)_{n\in\mathbb{N}}$ a sequence in A.

Case 1 $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Since A is closed, the subsequential limit lies in A.

Case 2 $(a_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence.

Consider the set $\{a_n\}_{n\in\mathbb{N}}$. By hypothesis, this set has no limit points. So it's closed.

For each $n \in \mathbb{N}$, there is some ε_n so that $m \neq n \Rightarrow x_m \notin D(x_n, \varepsilon_n)$. Since $\{a_n\}_{n\in\mathbb{N}} \subset A$ is closed and A is compact, $\{a_n\}_{n\in\mathbb{N}}$ is compact.

So finitely many of the $D(x_n, \varepsilon_n)$ suffice. $\rightarrow\leftarrow$

(Sequentially compact $\Rightarrow$ compact)

Suppose $A \subset (M,d)$ is sequentially compact and $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$

Lemma 3 If A is sequentially compact, then for any $\varepsilon > 0$, there are finitely many $x_1, \dots, x_N$ with

$$A \subset D(x_1,\varepsilon) \cup D(x_2,\varepsilon) \cup \dots \cup D(x_N,\varepsilon)$$

Proof If not, $\exists\, \varepsilon > 0$ so that no finite collection of ε-balls covers A. Let $y_0 \in A$, $y_1 \in A \setminus D(y_0,\varepsilon)$, $y_2 \in A \setminus (D(y_0,\varepsilon) \cup D(y_1,\varepsilon))$, etc. Then $(y_n)_{n\in\mathbb{N}}$ is a sequence in A, so by sequential compactness $(y_n)_{n\in\mathbb{N}}$ converges. But the construction of $(y_n)_{n\in\mathbb{N}}$ makes this impossible! ▨

Lemma 3 says that sequentially compact sets are "compact with respect to covers by balls."

Lemma 4 Given any open cover $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of a sequentially compact set A, there is some $r > 0$ so that, for each $y \in A$, $D(y,r)$ is contained in a particular U_λ.

Note the order of quantifiers: $\exists r\ \forall y$, i.e. the same r works for all y. This is (very) different from $\forall y\ \exists r$.

Proof If not, then for each m there is y_m with $D(y_m, \frac{1}{m})$ not contained in any U_λ.

This sequence $(y_m)_{m\in\mathbb{N}}$ has a convergent subsequence (y_{m_k}) with $y_{m_k} \xrightarrow[k]{} y \in A$. There is some λ with $y \in U_\lambda$, and since U_λ is open, $\exists\, \varepsilon > 0$ with $D(y,\varepsilon) \subset U_\lambda$.

Since $y_{m_k} \to y$, eventually $D(y_{m_k}, \frac{\varepsilon}{2}) \subset D(y,\varepsilon)$.

For m_k large enough, $D(y_{m_k}, \frac{1}{m_k}) \subset D(y_{m_k}, \frac{\varepsilon}{2})$

$$\subset D(y,\varepsilon) \subset U_\lambda$$

which contradicts our choice of $(y_m)_{m\in\mathbb{N}}$. $\rightarrow\leftarrow$

To show that A is compact, take our arbitrary open cover $\mathcal{U} = \{U_\lambda\}$ and let r be given by Lemma 4.

By Lemma 3, finitely many r-balls suffice:

$$A \subset D(x_1,r) \cup D(x_2,r) \cup \dots \cup D(x_N,r)$$

Each $D(x_i,r)$ is contained in some U_{λ_i}. So

$$A \subset U_{\lambda_1} \cup \dots \cup U_{\lambda_N},$$ i.e. $\{U_{\lambda_i}\}_{i=1}^N$ is a subcover. ▨

Max-Min Theorem

Let (M,d) be a metric space, $A\subset(M,d)$ compact, $f: A\to(\mathbb{R}, d_{std})$ continuous.

Then 1) f is bounded, i.e. there is $B\in\mathbb{R}$ s.t. $\forall x\in A$, $|f(x)|\leq B$.

2) There are $x_+, x_-\in A$ so that

$$f(x_+) = \sup\{f(x)\mid x\in A\}$$
$$f(x_-) = \inf\{f(x)\mid x\in A\}$$

Proof 1. For each $\alpha,\beta\in\mathbb{R}$, the set $U_{\alpha,\beta}=f^{-1}((\alpha,\beta))$ is open. Moreover, $\mathcal{U}=\{U_{\alpha,\beta}\mid \alpha,\beta\in\mathbb{R}\}$ is an open cover for A. So compactness guarantees finitely many $\alpha_1,\dots,\alpha_N$, $\beta_1,\dots,\beta_N$ suffice. Let $m=\min\alpha_i$, $M=\max\beta_i$.

Then $A\subset f^{-1}(m,M)$, i.e. $f(A)\subset(m,M)\subset[m,M]$.

So f is bounded.

2. (The same as ② before)

Let $y_k=f(x_k)$ be a sequence of values of f which converges to $\sup\{f(x)\mid x\in A\}$.

Then $(x_k)_{k\in\mathbb{N}}$ is a sequence in a compact set, so there is a convergent subsequence $(x_{k_p})_{p\in\mathbb{N}}$ whose limit x_+ lies in A.

Then $y_{k_p}\to\sup\{f(x)\mid x\in A\}$ and $y_{k_p}\to f(x_+)$.

So $f(x_+)=\sup\{f(x)\mid x\in A\}$.

Similarly with x_-. ∎

Heine-Borel Theorem

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof $(\Rightarrow)$ We showed in the last proof that compact sets are closed. To show compact sets are bounded, pick any $x\in M$ and consider the open cover

$$\mathcal{U}=\{D(x,R)\mid R>0\}$$

By compactness, finitely many R suffice. Pick the largest.

This side has nothing to do with $\mathbb{R}^n$.

$(\Leftarrow)$ We'll show a closed, bounded subset of $\mathbb{R}^n$ is sequentially compact, i.e. that any sequence in a closed, bounded set A has a subsequential limit which lies in A.

But this is HW6 #10.

E.g. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$

$$= \{(x^1, \dots, x^{n+1}) \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1\}$$

is bounded (by definition!).

S^n is closed: suppose $(x_k)_{k\in\mathbb{N}}$ is a sequence in S^n converging to $x \in \mathbb{R}^{n+1}$. Then $\|x\| = \|\lim x_k\| = \lim \|x_k\| = \lim 1 = 1$.

The norm is ____.

So $x \in S^n$.

Thus S^n is compact by Heine-Borel.

Corollary At any time, there are hottest and coldest points on the earth's surface.

Theorem If $A \subset (M,d)$ is compact and $f: A \to (N,\rho)$ is continuous, then $f(A) \subset (N,\rho)$ is compact.

Proof Let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover for $f(A)$.

Then $f^*\mathcal{U} = \{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover for A.

So finitely many $\lambda_1, \dots, \lambda_N$ suffice to cover A.

Claim: $\{U_{\lambda_1}, \dots, U_{\lambda_N}\}$ covers $f(A)$.

Proof Let $x \in f(A)$. Then $\exists a \in A$ with $x = f(a)$.

$a \in f^{-1}(U_{\lambda_i})$ for some i. So $x = f(a) \in U_{\lambda_i}$. ▨

"Continuous maps push compactness forward."

Intrinsic vs. Extrinsic

If (M,d) is a metric space and $A \subset (M,d)$ is a subset, then $(A, d|_A)$ is a metric space.

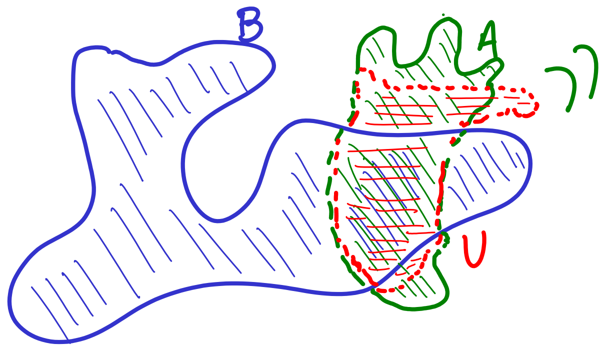
As a subset of $(A, d|_A)$, A is open. And closed.

As a subset of (M,d), maybe not.

We say openness and closedness are extrinsic properties.

Def'n If (M,d) is a metric space and $B \subset (M,d)$ a subset, we say $A \subset (M,d)$ is open relative to B if $A \cap B$ is open in $(B, d|_B)$.

Proposition For any A, B subsets of (M,d), A is open relative to B iff there is some open $U \subset (M,d)$ with $A \cap B = U \cap B$.



Proof First we make the following claim, which is an exercise.

Claim For any $R>0$, $x\in B$, the ball $D_B(x,R)=\{y\in B \mid d|_B(x,y)<R\}$ is just $B\cap D(x,R)$.

($\Rightarrow$) Since A is open relative to B, for any $y\in A\cap B$, there is $\varepsilon_y>0$ with $D_B(y,\varepsilon_y)\subset B\cap A$.

Set $U=\bigcup_{y\in B\cap A} D(y,\varepsilon_y)$. Then U is open in (M,d).

Also, $U\cap B=\bigcup_{y\in B\cap A}\big(D(y,\varepsilon_y)\cap B\big)=\bigcup_{y\in B\cap A} D_B(y,\varepsilon_y)\subset B\cap A$.

and $B\cap A\subset\bigcup_{y\in B\cap A} D_B(y,\varepsilon_y)$. ∎

($\Leftarrow$) Suppose $B\cap A=B\cap U$ for some open $U\subset(M,d)$.

For each $y\in B\cap A$, there is $\varepsilon>0$ with $D(y,\varepsilon)\subset U$.

Then $D_B(y,\varepsilon)=B\cap D(y,\varepsilon)\subset B\cap U=B\cap A$ so $B\cap A$ is open in $(B,d|_B)$. ∎

Theorem Let $A\subset B$ be subsets of a metric space (M,d). A is compact in (M,d) iff A is compact in $(B,d|_B)$.

Proof ($\Rightarrow$) To show A is compact in $(B,d|_B)$, consider an open cover $\{V_\lambda\}_{\lambda\in\Lambda}$. Each $V_\lambda=U_\lambda\cap B$, so that $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover for $A\subset(M,d)$. A is compact, so finitely many $\{U_{\lambda_1},\dots,U_{\lambda_N}\}$ suffice. Then $\{V_{\lambda_1},\dots,V_{\lambda_N}\}$ are a finite subcover for $\{V_\lambda\}_{\lambda\in\Lambda}$.

($\Leftarrow$) To show A is compact in (M,d), let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover. Set $V_\lambda=B\cap U_\lambda$. Then $\{V_\lambda\}_{\lambda\in\Lambda}$ is an open cover for A with respect to $(B,d|_B)$, so finitely many $\{V_{\lambda_1},\dots,V_{\lambda_N}\}$ suffice. Then $\{U_{\lambda_1},\dots,U_{\lambda_N}\}$ are a finite subcover for $\{U_\lambda\}_{\lambda\in\Lambda}$.

Thus compactness is intrinsic. It makes sense to refer to a compact metric space.

E.g. Consider $S^\infty=\{x\in\mathbb{R}^\infty \mid \|x\|=1\}\subset\mathbb{R}^\infty$.

Then S^∞ is bounded in the standard metric.

Also if $(x_n)_{n\in\mathbb{N}}$ is a sequence in S^∞ which converges to some x, must have $\|x\|=1$, so S^∞ is closed.

But... Consider

$$x_n=(0,0,\dots,0,\overset{n^{\text{th}}\text{ slot}}{1},0,\dots)$$

Then $\|x_n-x_m\|=\sqrt{2}$ for any $m\neq n$, so no subsequence can possibly converge! S^∞ is not compact!

The Intermediate Value Theorem

Our third theorem from calculus:

Intermediate Value Theorem

Suppose $f: [a,b] \to \mathbb{R}$ is a continuous function. For any y between $f(a)$ and $f(b)$, there is $c \in [a,b]$ with $f(c) = y$.

Corollary The image of a closed interval under a continuous function is a closed interval.

Proof By Max-Min, there are finite m, M with $f([a,b]) \subset [m, M]$. Moreover $\exists\, x_+, x_- \in [a,b]$ with $f(x_+) = M$, $f(x_-) = m$.

By IVT, for any $y \in [m, M]$, there is c between x_+ and x_- with $f(c) = y$. So

$$[m, M] \subset f([x_+, x_-]) \subset f([a,b]) \subset [m, M] \quad \blacksquare$$

Again, in order to prove this theorem (and properly state its very interesting generalisation) we need more topological technology.

Connectedness

Def'n A path in a metric space (M,d) is a continuous map into M from a closed interval (with the standard metric)

$$\varphi : [a,b] \to M.$$

We call $\varphi(a)$ and $\varphi(b)$ the endpoints of the path φ.

Def'n A subset $A \subset (M,d)$ is path-connected if for every pair $x, y \in A$, there is a path $\varphi_{xy} : [0,1] \to A$ with $\varphi_{xy}(0) = x$, $\varphi_{xy}(1) = y$, and for all $t \in [0,1]$, $\varphi_{xy}(t) \in A$.

E.g. Intervals in $\mathbb{R}$ are path-connected.

Consider $[a,b]$. For any $x, y \in [a,b]$, define φ_{xy} by

$$\varphi_{xy} : t \mapsto ty + (1-t)x$$ ← "Convex combination" of x and y

Then $\varphi_{xy}(0) = x$, $\varphi_{xy}(1) = y$, and:

Case $x<y$ Then for $t \in [0,1]$,

$$x \leq x + t(y-x) = (1-t)x + ty \leq (1-t)y + ty = y$$

So $\varphi_{xy}(t) \in [x,y]$.

Case $y<x$ Reverse the inequalities.

Theorem If $A\subset(M,d)$ is path-connected and $f:A\to(N,\rho)$ is continuous, then $f(A)\subset(N,\rho)$ is path-connected.

"Continuous maps push connectedness forward"

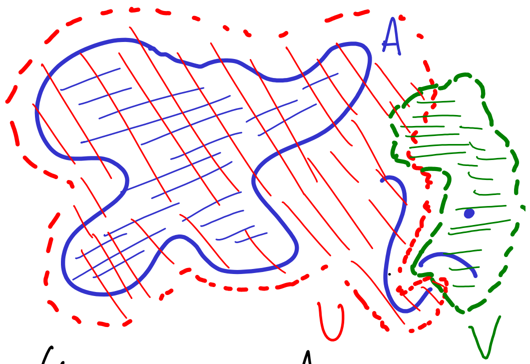
Proof Given $f(x), f(y)\in f(A)$, by hypothesis, there is a path $\varphi_{xy}:[0,1]\to A$ with $\varphi_{xy}(0)=x$, $\varphi_{xy}(1)=y$.

Then $f\circ\varphi_{xy}:[0,1]\to N$ is a continuous map with $(f\circ\varphi_{xy})(0)=f(x)$, $(f\circ\varphi_{xy})(1)=f(y)$, and $(f\circ\varphi_{xy})(t)=f(\varphi_{xy}(t))\in f(A)$. ∎

Unfortunately, path-connectedness isn't all that.

Defn A **disconnection** of $A\subset(M,d)$ is a pair of open sets $U,V\subset(M,d)$ such that:

1. $A\subset U\cup V$
2. $U\cap V\cap A=\emptyset$
3. Neither $U\cap A$ nor $V\cap A$ is empty.



If a set A has no disconnections we say A is **connected**.

E.g. Intervals in $\mathbb{R}$ are connected.

Suppose U,V are disjoint, open sets which cover $[a,b]$. Let $\alpha\in U$, $\beta\in V$, assume WLOG $\alpha<\beta$. Restrict our attention to $U\cap[\alpha,\beta]$ and $V\cap[\alpha,\beta]$, which are both clopen sets in $[\alpha,\beta]$. Consider $\sup U$. Since U is closed, $\sup U\in U$, so $\sup U<\beta$. Since U is open in $[\alpha,\beta]$, there is $\varepsilon>0$ so that $(\sup U-\varepsilon, \sup U+\varepsilon)\subset U\cap[\alpha,\beta]$ $\rightarrow\leftarrow$ def'n of supremum.

The only way out is if there are no $\beta\in V$, i.e. if V is empty. ∎

Theorem Path-connected sets are connected.

Proof Let $A\subset(M,d)$ be a path-connected set, and U,V a disconnection of A. Let $x\in A\cap U$, $y\in A\cap V$. Then $\tilde U=\varphi_{xy}^{-1}(A\cap U)$ and $\tilde V=\varphi_{xy}^{-1}(A\cap V)$ are open, disjoint, and cover $[0,1]$. But $[0,1]$ is connected! ∎

Proposition Any connected subset of $\mathbb{R}$ is an interval.

Proof If $A \subset \mathbb{R}$ is not an interval, then there are $x, y \in A$ and $r \in \mathbb{R}$ with $x < r < y$.
Then $U = (-\infty, r)$ and $V = (r, \infty)$ is a disconnection of A. $\blacksquare$

Cor Any path-connected subset of $\mathbb{R}$ is an interval.

Important Corollary

The only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ and ϕ.

Proof A nontrivial clopen set U gives a disconnection, since U and U^c are open, disjoint, nontrivial, and cover.

Intermediate Value Theorem

If $f : A \to \mathbb{R}$ is a continuous real-valued function, $K \subset A$ is connected, $x, y \in K$, and $c \in \mathbb{R}$ is between $f(x)$ and $f(y)$, then there is $z \in K$ with $f(z) = c$.

Proof If $c \notin f(K)$, then $f^{-1}((-\infty, c))$ and $f^{-1}((c, \infty))$ are a disconnection of K. $\blacksquare$

Application: Right now, there are two antipodal points on the Earth's surface with the same temperature.

Proof The Earth's surface S is connected. The map
$$\sim : S \to S, \quad p \mapsto \tilde{p}$$
which takes each point to its antipode is continuous. The temperature function f is continuous. So $g(p) = f(p) - f(\tilde{p})$ is continuous map $S \to \mathbb{R}$.

Pick any $p_0 \in S$. If $g(p_0) = 0$, then we're done. Otherwise, $g(p_0) \neq 0$. $g(p_0) = -g(\tilde{p}_0)$. One is positive and one is negative, so by IVT $\exists\, \bar{p}$ with $g(\bar{p}) = 0$ and we are done. $\blacksquare$

Defn A maximal connected subset $A_0 \subset A$ is called a **connected component**.

. Here "maximal" means: if B is a connected subset of A which intersects A_0, then $B \subset A_0$.

Proposition Let $A \subset (M,d)$ be any subset. Then each $x \in A$ lies in a unique connected component $A_0(x)$.

Proof. We'll construct the connected components of A.

For $x,y \in A$, say $x \sim y$ if $\exists C \subset A$ connected with $x,y \in C$

Claim $\sim$ is an equivalence relation.

Proof. $x \sim x$ since $\{x\}$ is connected.

$x \sim y \Leftrightarrow y \sim x$ (use the same C)

If $x \sim y$ and $y \sim z$, then $\exists\, C_{xy}$ and C_{yz} connected with $x,y \in C_{xy}$, $y,z \in C_{yz}$. Set $C = C_{xy} \cup C_{yz}$.

Claim C is connected.

Suppose U,V disconnect C. Then

$$C \cap U = (C_{xy} \cap U) \cup (C_{yz} \cap U)$$
$$C \cap V = (C_{xy} \cap V) \cup (C_{yz} \cap V) \text{ are disjoint.}$$

So $C_{xy} \cap U$ and $C_{xy} \cap V$ are disjoint.

C_{xy} is connected, so one must be empty.

Same with $C_{yz} \cap U$ and $C_{yz} \cap V$.

Both $C_{xy} \cap U$ and $C_{yz} \cap U$ can't be empty.

So it must be that $C_{xy} \cap U \neq \emptyset$ and $C_{yz} \cap V \neq \emptyset$.

But y is in both, so U and V aren't disjoint! $\rightarrow\leftarrow$

Claim $[x]$ is connected.

Claim $[x]$ is maximal. ∎