Differentiation

Now that we know something about functions, we'll start in on the calculus...

Def'n Let $f:[a,b]\to\mathbb{R}$. For any $x\in[a,b]$, define
$$f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x} \quad \text{(if the limit exists.)}$$

Note that this is the limit of a function $\phi_x(t)=\frac{f(t)-f(x)}{t-x}$, whose domain is $[a,b]\setminus\{x\}$.

Def'n If $f'(x)$ exists, we say f is *differentiable at* x.
If f is differentiable at x for all $x\in[a,b]$, then we say f is *differentiable*.

Let's rewrite this definition in terms of ε and δ:

" For any $\varepsilon>0$, there is $\delta>0$ so that $0<|t-x|<\delta$ guarantees
$$\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|=\left|\frac{f(t)-f(x)-f'(x)(t-x)}{t-x}\right|<\varepsilon$$
$$|f(t)-f(x)-f'(x)(t-x)|<\varepsilon|t-x|$$
$$|f(t)-(f(x)+f'(x)(t-x))|<\varepsilon|t-x|$$

Food for thought:

"$t-x$" makes sense if x,t are in a vector space.
"$f(x)-f(t)$" makes sense if $f(x),f(t)$ are in a vector space.
So we need, in general, for
"$f'(x)(t-x)$" to be in the same vector space
as $f(x)$ and $f(t)$. Thus:

$f'(x)$ should be the sort of thing that takes things in the domain vector space to things in the target vector space...

Theorem Differentiability implies continuity, i.e. if f is diff'ble at x_0, then f is continuous at x_0.

Proof
$$\lim_{x\to x_0}f(x)=\lim_{x\to x_0}[f(x)-f(x_0)+f(x_0)]$$
$$=\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)+f(x_0)\right]$$
$$=\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\right)\lim_{x\to x_0}(x-x_0)+\lim_{x\to x_0}f(x_0)$$
$$=f'(x_0)\cdot 0+f(x_0)=f(x_0) \quad \blacksquare$$

Since you've learned all about derivatives already, there's no need to replicate that work here.

Suffice to say, the derivative rules and formulas you know and love are true.

For the homework, you will prove Leibniz' rule (the "product rule"):

$$(fg)'(x) = f'(x)\,g(x) + g'(x)\,f(x).$$

We could also prove some formulas like the "power rule" $\frac{d}{dx}x^p = p\,x^{p-1}$.

But it turns out basically all differentiation formulas and rules follow from Leibniz' rule together with the most important rule of derivatives ...

<u>Theorem</u> (<u>the Chain Rule</u>)

If f, g are real-valued functions of real variables with compatible domains and targets, and f, g are differentiable, then $h(t) = f(g(t))$ is differentiable and

$$h'(x) = f'(g(x))\,g'(x)$$

<u>Proof</u> Since g is diff'ble at x, g is cont's at x, so $\forall \varepsilon_1 > 0\ \exists \delta_1$ so that $|x-t| < \delta_1 \Rightarrow |g(x) - g(t)| < \varepsilon_1$

Since f is diff'ble at $g(x)$, $\forall \varepsilon_2 > 0\ \exists \delta_2 > 0$ so that $|y - g(x)| < \delta_2 \Rightarrow$

$$|f(y) - f(g(x)) - f'(g(x))(y - g(x))| \leq \varepsilon_2\,|y - g(x)|$$

Using $\varepsilon_1 = \delta_2$, we have $|x-t| < \delta_1 \Rightarrow$

$$|f(g(t)) - f(g(x)) - f'(g(x))(g(t) - g(x))| \leq \varepsilon_2\,|g(t) - g(x)|.$$

Also g is diff'ble at x, $\forall \varepsilon_3\ \exists \delta_3$ so $|x-t| < \delta_3 \Rightarrow$

$$|g(t) - g(x) - g'(x)(t-x)| \leq \varepsilon_3\,|t-x|.$$

So if $|x-t| \leq \min\{\delta_1, \delta_3\}$,

$$f(g(t)) - f(g(x)) - f'(g(x))g'(x)(t-x)$$
$$\leq f'(g(x))\left(g(t)-g(x)\right) + \varepsilon_2|g(t)-g(x)| - f'(g(x))g'(x)(t-x)$$
$$\leq f'(g(x))\left(g'(x)(t-x) + \varepsilon_3|t-x|\right) + \varepsilon_2\left(|g'(x)|\,|t-x| + \varepsilon_3|t-x|\right)$$
$$- f'(g(x))g'(x)(t-x)$$

$\boxed{\text{Assume } f'(g(x))>0}$

$$= \left[f'(g(x))\,\varepsilon_3 + |g'(x)|\,\varepsilon_2 + \varepsilon_2\varepsilon_3\right]|t-x|$$

Given $\varepsilon>0$, Set
$$\varepsilon_2 = \varepsilon_3 = \min\left\{\frac{\varepsilon}{3|f'(g(x))|}, \frac{\varepsilon}{3|g'(x)|}, \frac{\sqrt{\varepsilon}}{\sqrt{3}}\right\}$$

Then if $|x-t| < \min\{\delta_1, \delta_2\}$,
$$f(g(t)) - f(g(x)) - f'(g(x))g'(x)(t-x) \leq \varepsilon|t-x|.$$

We can similarly bound
$$-\varepsilon|t-x| \leq f(g(t)) - f(g(x)) - f'(g(x))g'(x)(t-x)$$

The case $f'(g(x))<0$ is similar. ∎

Def: We say $f: A\to\mathbb{R}$ has a local maximum at $a\in A$ if there is some open $U\subset A$ containing a, so that $x\in U$ guarantees $f(a)\geq f(x)$.

—— local minimum —— $f(a)\leq f(x)$

Theorem Suppose $f:[a,b]\to\mathbb{R}$ has a local maximum at $c\in(a,b)$, and f is differentiable at c. Then $f'(c)=0$.

Proof. If $t<c$, then $\dfrac{f(t)-f(c)}{t-c}\geq 0$.

So $\displaystyle\lim_{t\to c}\frac{f(t)-f(c)}{t-c}\geq 0$.

If $t>c$, then $\dfrac{f(t)-f(c)}{t-c}\leq 0$.

So $\displaystyle\lim_{t\to c}\frac{f(t)-f(c)}{t-c}\leq 0$.

Thus $\displaystyle f'(c) = \lim_{t\to c}\frac{f(t)-f(c)}{t-c} = 0$. ∎

Note that we need some $t<c$ and some $t>c$, so this argument fails if $c=a$ or $c=b$.

Now we are ready for the most important theorem concerning derivatives:

Mean Value Theorem (for Derivatives)

Let $f, g : [a,b] \to \mathbb{R}$ be functions which are continuous on $[a,b]$ and diff'ble on (a,b). Then there is $c \in (a,b)$ with

$$[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c)$$

Proof Set $h(t) = [f(b)-f(a)]g(t) - [g(b)-g(a)]f(t)$.

Then h is cont's on $[a,b]$ and diff'ble on (a,b).

$$h(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a)$$
$$= f(b)g(a) - g(b)f(a).$$

$$h(b) = f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b)$$
$$= -f(a)g(b) + f(b)g(a).$$

So $h(a) = h(b)$.

Case $h(t) \equiv h(a)$ Then $h'(t) = 0$ for all $t \in (a,b)$

Case otherwise. By Max-Min, h achieves a max and a min on $[a,b]$. At most one of them can happen at the endpoints, so h has a local extreme.

Thus by the previous theorem, $\exists\, c \in (a,b)$ with $h'(c) = 0$.

In both cases, there is some $c \in (a,b)$ with

$$h'(c) = 0$$
$$[f(b)-f(a)]g'(c) - [g(b)-g(a)]f'(c) = 0$$
$$[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c) \quad \blacksquare$$

Mean Value Theorem (more familiar version)

Suppose $f : [a,b] \to \mathbb{R}$ is cont's on $[a,b]$ and diff'ble on (a,b). There is $c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \leftarrow \text{average or "mean" rate of change of } f.$$

Proof Apply the above theorem with $g(x) = x$. $\blacksquare$

Cor Suppose $f: [a,b] \to \mathbb{R}$ is diff'ble on (a,b). Then:

① If $\forall x \in (a,b)$ $f'(x) \geq 0$, then f is nondecreasing.

①' If $\forall x \in (a,b)$ $f'(x) > 0$, then f is strictly increasing

② If $\forall x \in (a,b)$ $f'(x) \leq 0$, then f is nonincreasing.

②' If $\forall x \in (a,b)$ $f'(x) < 0$, then f is strictly decreasing

③ If $\forall x \in (a,b)$, $f'(x) = 0$, then f is constant.

Proof. Let $a < x_1 < x_2 < b$.

By MVT $\exists x \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$$

and the various statements follow from the sign of $f'(x)$ since $x_2 - x_1 > 0$. ∎

Cor If f, g are as above and $f' \equiv g'$, then $f(x) = g(x) + c$ for some constant c.

Cor (Inverse Function Theorem)

Suppose $f: [a,b] \to \mathbb{R}$ is as above, and $f'(x) > 0$ for all $x \in (a,b)$. Then f has an inverse, i.e. there is $g: f((a,b)) \to (a,b)$ with $f(g(y)) = y$ and $g(f(x)) = x$.

Moreover, g is diff'ble and

$$g'(f(x)) = \frac{1}{f'(x)}$$

Proof

Lemma 1 Strictly increasing functions are one-to-one.

So there is $g: f((a,b)) \to (a,b)$ with $f(g(y)) = y$ and

$$g(f(x)) = x$$

Notice Since f is increasing and continuous,

$$f((a,b)) = (f(a), f(b)).$$

Lemma 2. Strictly increasing continuous maps are open maps, i.e. push open sets forward.

Lemma 3 $g:(f(a),f(b)) \to (a,b)$ is continuous.

Proof Suppose $U \subset (a,b)$ is open. Then

$$\begin{aligned} g^{-1}(U) &= \{y \in (f(a),f(b)) \mid g(y) \in U\} \\ &= \{f(x) \in (f(a),f(b)) \mid x \in U\} \\ &= f(U) \end{aligned}$$

which is open since f is an open map. ∎

Now consider diff'bility of g. Let $y_0 \in (f(a),f(b))$, $x_0 = g(y_0)$ (so that $f(x_0)=y_0$).

$$g'(y_0) = \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0} = \lim_{y\to y_0} \frac{x-x_0}{f(x)-f(x_0)} = \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)}$$

$$= \lim_{x\to x_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}$$

where the limit exists because $f'(x_0)$ exists and is nonzero. ∎

Intermediate Value Property for derivatives

Suppose $f:[a,b]\to\mathbb{R}$ is diff'ble on $[a,b]$ and λ is a real number between $f'(a)$ and $f'(b)$.

Then $\exists\, x \in [a,b]$ with $f'(x)=\lambda$.

Proof WLOG assume $\exists c$ with $f'(a) < c < f'(b)$.

Set $g(t) = f(t) - ct$.

Then g is diff'ble on $[a,b]$, hence cont's on $[a,b]$.

Since $g'(a) = f'(a) - c < 0$, g decreases from $g(a)$.

Sim. g increases into $g(b)$.

So it cannot be the case that $g(a)$ or $g(b)$ is a minimum for g on $[a,b]$, so g has an interior minimum, say at $x \in (a,b)$.

Then $g'(x) = f'(x) - c = 0$. ∎

Higher Derivatives

If $f:[a,b]\to\mathbb{R}$ is diff'ble, then $f':[a,b]\to\mathbb{R}$ is a function, of which we may ask: is f' diff'ble?

If so, we write f'' for $(f')'$.

Sim. get f''', f'''', etc. Write $f^{(0)}=f$, $f^{(1)}=f'$, $f^{(2)}=f''$, …, $f^{(n+1)}=(f^{(n)})'$

Def'n $f:[a,b]\to\mathbb{R}$ is of class C^k if $f^{(k)}$ exists and is cont's on $[a,b]$. Write $C^k([a,b])$ for the (vector space of all) C^k functions on $[a,b]$.

Note $C^k\subset C^{k-1}\subset\cdots\subset C^1\subset C^0=C$

Taylor's Theorem

Suppose $f:[a,b]\to\mathbb{R}$ has, for some $n\in\mathbb{N}$, $f^{(n)}$ exists on (a,b) and $f^{(n-1)}$ is cont's on $[a,b]$. Let $a<\alpha<\beta<b$.

Then there is $c\in(\alpha,\beta)$ so that

$$\left.\begin{aligned}f(\beta)=f(\alpha)&+f'(\alpha)(\beta-\alpha)+\tfrac{1}{2!}f''(\alpha)(\beta-\alpha)^2+\dots\\&+\tfrac{1}{k!}f^{(k)}(\alpha)(\beta-\alpha)^k+\dots+\tfrac{1}{(n-1)!}f^{(n-1)}(\alpha)(\beta-\alpha)^{n-1}+\end{aligned}\right]\text{deg. } n-1 \text{ Taylor poly.}$$

deg $n-1$ Taylor error $\to$ $\left[\tfrac{1}{n!}f^{(n)}(c)(\beta-\alpha)^n\right.$

Proof The case $n=1$ is MVT. Prove like MVT.

Set $M=\dfrac{f(\beta)-\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k}{(\beta-\alpha)^n}$ and

$$g(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k-M(t-\alpha)^n.$$

Then $g(\alpha)=0$

$g'(\alpha)=f'(\alpha)-f'(\alpha)=0$

$\vdots$

$g^{(n-1)}(\alpha)=0$

and $g(\beta)=0$.

Since $g(\alpha)=g(\beta)=0$, $\exists\, x_1\in(\alpha,\beta)$ with $g'(x_1)=0$.

Then $g'(\alpha)=g'(x_1)=0$, so $\exists\, x_2\in(\alpha,x_1)$ with $(g')'(x_2)=0$.

$\vdots$

$\exists\, x_n\in(\alpha,x_{n-1})$ with $g^{(n)}(x_n)=0$.

OTOH, $g^{(n)}(x_n)=f^{(n)}(x_n)-n!M$ so

$M=\frac{1}{n!}f^{(n)}(x_n)$, which is what we were to show. ∎

L'Hôpital's Rule

Suppose $f,g:[a,b]\to\mathbb{R}$ are diff'ble, for all $x\in[a,b]$ $g'(x)\neq 0$, and $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L$. If $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=L.$$

Proof Let $q>L$. For any $r\in(L,q)$, $\exists\ \varepsilon>0$ so that

$$x_0<x<x_0+\varepsilon \Rightarrow \frac{f'(x)}{g'(x)}<r.$$

If $x_0<x<y<x_0+\varepsilon$, $\exists\, t\in(x,y)$ with

$$\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(t)}{g'(t)}<r.$$

Fix y, let $x\to x_0$. Then $\frac{f(y)}{g(y)}=\lim_{x\to x_0}\frac{f(x)-f(y)}{g(x)-g(y)}\le r$.

So $\lim_{y\to x_0}\frac{f(y)}{g(y)}\le r<q$.

So $\lim_{y\to x_0}\frac{f(y)}{g(y)}\le q$. But $q>L$ was arbitrary, so

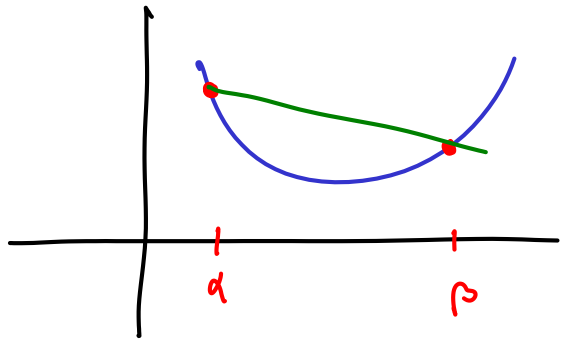
$\lim_{y\to x_0}\frac{f(y)}{g(y)}\le L$. By similar analysis to the left of x_0,

$$\lim_{y\to x_0}\frac{f(y)}{g(y)}\ge L. \blacksquare$$

Def'n A function $f:[a,b]\to\mathbb{R}$ is convex if, for any $\alpha,\beta\in[a,b]$, and any $t\in[0,1]$,

$$f((1-t)\alpha+t\beta)\le(1-t)f(\alpha)+tf(\beta)$$

f of the convex combination is at most the convex combination of f.



Note: "convex" makes sense if α and β are in a vector space.

Theorem Suppose $f:[a,b]\to\mathbb{R}$ is twice diff'ble on (a,b) and for all $x\in(a,b)$, $f''(x)\ge 0$. Then f is convex.

Lemma Suppose $g:[a,b]\to\mathbb{R}$ has $g(a)=g(b)$ and $g''\ge 0$. Then g achieves its maximum at a and b.

Proof By Max-Min, g has a maximum at some $c\in[a,b]$. If $c\in(a,b)$, then $g'(c)=0$. Then Taylor says for any t, $\exists\, t_0$ between c and t so that

$$g(t) = g(c) + g'(c)(t-c) + \tfrac{1}{2}g''(t_0)(t-c)^2$$
$$\geq g(c)$$

So $g(c)$ is a minimum for the values of g. ∎

Now given $f:[a,b]\to\mathbb{R}$ with $f''\geq 0$, consider

$$g(t) = f((1-t)\alpha + t\beta) - ((1-t)f(\alpha) + tf(\beta)).$$

$$g(0) = f(\alpha) - f(\alpha) = 0$$

$$g(1) = f(\beta) - f(\beta) = 0$$

$$g'(t) = (\beta-\alpha)f'((1-t)\alpha + t\beta) - (f(\beta) - f(\alpha))$$

$$g''(t) = (\beta-\alpha)^2 f''((1-t)\alpha + t\beta)$$

So $g:[0,1]\to\mathbb{R}$ has $g(0)=g(1)$ and $g''\geq 0$.

Thus $g(t)\leq g(0)$ for all $t\in[0,1]$, i.e.

$$f((1-t)\alpha + t\beta) \leq (1-t)f(\alpha) + tf(\beta)$$ ∎

Integration!

Defn Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

A partition P is a choice of finitely many elements of $[a,b]$

$$a\leq x_0\leq x_1\leq \dots \leq x_{n-1}\leq x_n = b$$

The Darboux upper sum of f over P is

$$U(f,P) = \sum_{k=0}^{n-1}\sup\{f(x)\mid x\in[x_k,x_{k+1}]\}(x_{k+1}-x_k)$$

The Darboux lower sum of f over P is

$$L(f,P) = \sum_{k=0}^{n-1}\inf\{f(x)\mid x\in[x_k,x_{k+1}]\}(x_{k+1}-x_k)$$

E.g. Let $f:\mathbb{R}\to\mathbb{R}$

$$x\mapsto\begin{cases}1 & \text{if } x\notin\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{Q}\end{cases}$$

Then for any $\alpha<\beta$, $\sup_{[\alpha,\beta]} f = 1$ $\quad \inf_{[\alpha,\beta]} f = 0$

So for any $[a,b]$,

$$U(f,P) = \sum_{k=0}^{n-1}\sup_{[x_k,x_{k+1}]} f\,(x_{k+1}-x_k) = x_1 - a + x_2 - x_1 + \dots + b - x_{n-1}$$
$$= b-a$$

$$L(f,P) = 0$$

E.g. Let $g(x) = \begin{cases} x & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$

Then $U(g,P) = \sum_{k=0}^{n-1} x_{k+1}(x_{k+1} - x_k)$

$L(g,P) = 0$

E.g. Let $h(x) = x$.

Then $U(h,P) = \sum_{k=0}^{n-1} x_{k+1}(x_{k+1} - x_k)$

$L(h,P) = \sum_{k=0}^{n-1} x_k(x_{k+1} - x_k)$

Each upper sum gives an overestimate of the area under the graph; each lower sum gives an underestimate.

Def'n The **upper integral** of f on $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) \mid P \text{ is any partition of } [a,b]\}$$

The **lower integral** is

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) \mid P \text{ is any partition of } [a,b]\}$$

Some people write: $\overline{\int_{[a,b]}} f(x)\,dx \qquad \overline{\int_a^b} f \qquad \overline{\int_{[a,b]}} f$

E.g. $\overline{\int_a^b} f(x)dx = \inf\{b-a \mid P \text{ is a partition}\} = b-a$

$\underline{\int_a^b} f(x)dx = \sup\{0 \mid P \text{ is a partition}\} = 0$

Def'n We say f is **integrable** on $[a,b]$ if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

and call their common value the **integral** of f on $[a,b]$,

$$\int_a^b f(x)\,dx \quad \left(\text{or } \int_{[a,b]} f(x)dx \text{ or } \int_a^b f\right)$$

Proposition For any bounded $f:[a,b] \to \mathbb{R}$,

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

(Note that $L(f,P) \le U(f,P)$ for any particular P but this alone does not guarantee the proposition.)

Def'n Given a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, a partition P' *refines* P if $P \subset P'$. (Remember that a partition is just a finite subset of $[a,b]$.)

Lemma If P' is a refinement of P, then

$$L(f,P') \geq L(f,P) \quad \text{and} \quad U(f,P') \leq U(f,P)$$

Proof Case $P' = P \cup \{\alpha\}$ for $\alpha \notin P$. Let $P = \{x_0, \dots, x_n\}$.

Then there is $i \in \mathbb{N}$ so that $x_i < \alpha < x_{i+1}$.

$$L(f,P') = \sum_{k=0}^{i-1} \inf_{[x_k, x_{k+1}]} f\,(x_{k+1} - x_k) + \inf_{[x_i, \alpha]} f\,(\alpha - x_i) + \inf_{[\alpha, x_{i+1}]} f\,(x_{i+1} - \alpha) + \sum_{k=i+1}^{n-1} \inf_{[x_k, x_{k+1}]} f\,(x_{k+1} - x_k)$$

$$\geq \sum_{k=0}^{i-1} \inf_{[x_k, x_{k+1}]} f\,(x_{k+1} - x_k) + \sum_{k=i+1}^{n-1} \inf_{[x_k, x_{k+1}]} f\,(x_{k+1} - x_k) + \inf_{[x_i, x_{i+1}]} f\,(x_{i+1} - \alpha) + \inf_{[x_i, x_{i+1}]} f\,(\alpha - x_i)$$

$$= L(f,P)$$

General Case Since P, P' are finite, $P' = P \cup \{\alpha_1, \dots, \alpha_p\}$ for some $\alpha_i \notin P$, and

$$L(f,P) \leq L(f, P \cup \{\alpha_1\}) \leq \dots \leq L(f,P') \quad \blacksquare$$

Now we prove the proposition.

Suppose P_1, P_2 are partitions. Then

$$L(f,P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2)$$

since $P_1 \cup P_2$ refines both P_1 and P_2.

Thus any lower sum is less than any upper sum.

$U(f,P_2)$ is an upper bound for $\{L(f,P) \mid P \text{ is a partition}\}$

So $\underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) \mid P \text{ is a partition}\} \leq U(f,P_2)$.

P_2 is arbitrary, so $\underline{\int_a^b} f(x)\,dx$ is a lower bound for $\{U(f,P_2) \mid P_2 \text{ is a partition}\}$, so

$$\underline{\int_a^b} f(x)\,dx \leq \inf\{U(f,P_2) \mid P_2 \text{ is a partition}\} = \overline{\int_a^b} f(x)\,dx \quad \text{Pf}$$

Properties of the Integral

① Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded, integrable functions and $\lambda \in \mathbb{R}$. Then $f \pm g$ and λf are integrable, and

$$\int_{[a,b]} f \pm g = \int_{[a,b]} f \pm \int_{[a,b]} g, \quad \int_{[a,b]} \lambda f = \lambda \int_{[a,b]} f$$

② Suppose $f : [a,b] \to \mathbb{R}$ is bounded and integrable, and $c \in (a,b)$. Then f is integrable on $[a,c]$ and on $[c,b]$, and

$$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f$$

③ If $f \geq 0$, then $\int_{[a,b]} f \geq 0$.

④ Any function is integrable on $[a,a]$, with integral 0.

Proof of sum rule

Consider a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$.

$$U(f+g, P) = \sum_{k=0}^{n-1} \sup\{f(x)+g(x) \mid x_k < x < x_{k+1}\}(x_{k+1} - x_k)$$

$$\leq \sum_{k=0}^{n-1} \left[\sup\{f(x) \mid x_k < x < x_{k+1}\} + \sup\{g(x) \mid x_k < x < x_{k+1}\}\right](x_{k+1} - x_k)$$

$$= U(f,P) + U(g,P).$$

So $\overline{\int_{[a,b]}} f+g \leq U(f,P) + U(g,P)$

For any $\varepsilon > 0$, $\exists\ P_1, P_2$ so that:

$$U(f,P_1) \leq \overline{\int_{[a,b]}} f + \frac{\varepsilon}{2}$$
$$U(g,P_2) \leq \overline{\int_{[a,b]}} f + \frac{\varepsilon}{2}$$

Then $U(f, P_1 \cup P_2) \leq \overline{\int_{[a,b]}} f + \frac{\varepsilon}{2}$

$U(g, P_1 \cup P_2) \leq \overline{\int_{[a,b]}} g + \frac{\varepsilon}{2}$ so

$$\overline{\int_{[a,b]}} f+g \leq \overline{\int_{[a,b]}} f + \overline{\int_{[a,b]}} g + \varepsilon$$ and letting $\varepsilon \to 0$,

$$\overline{\int_{[a,b]}} f+g \leq \overline{\int_{[a,b]}} f + \overline{\int_{[a,b]}} g$$

A similar argument shows

$$\underline{\int_{[a,b]}} f + \underline{\int_{[a,b]}} g \leq \underline{\int_{[a,b]}} f+g \leq \overline{\int_{[a,b]}} f+g \leq \overline{\int_{[a,b]}} f + \overline{\int_{[a,b]}} g$$

and since f and g are both integrable, all these inequalities are actually equalities.

Proof of constant multiple rule is easier since $\sup \lambda S = \lambda \sup S$

Proof of domain sum rule

Claim $\overline{\int_{[a,b]}} f = \overline{\int_{[a,c]}} f + \overline{\int_{[c,b]}} f$; $\underline{\int_{[a,b]}} f = \underline{\int_{[a,c]}} f + \underline{\int_{[c,b]}} f$

Proof. Suppose P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$. Then $P_1 \cup P_2$ is a partition of $[a,b]$. Moreover,

$$U(f, P_1 \cup P_2) = U(f, P_1) + U(f, P_2).$$

Given $\varepsilon > 0$, $\exists P_1, P_2$ with

$$U(f, P_1) \leq \overline{\int_{[a,c]}} f + \frac{\varepsilon}{2}$$

$$U(f, P_2) \leq \overline{\int_{[c,b]}} f + \frac{\varepsilon}{2}$$

So $U(f, P_1 \cup P_2) \leq \overline{\int_{[a,c]}} f + \overline{\int_{[c,b]}} f + \varepsilon$

hence $\overline{\int_{[a,b]}} f \leq \overline{\int_{[a,c]}} f + \overline{\int_{[c,b]}} f$.

Now given P a partition of $[a,b]$, $P \cup \{c\}$ is a partition. Let P_1 and P_2 be the partitions

$$P_1 = (P \cup \{c\}) \cap [a,c] \quad P_2 = (P \cup \{c\}) \cap [c,b]$$

Then $U(f, P_1) + U(f, P_2) = U(f, P \cup \{c\}) \leq U(f, P)$

Given $\varepsilon > 0$, $\exists P$ with $U(f, P) \leq \overline{\int_{[a,b]}} f + \varepsilon$.

So $U(f, P_1) + U(f, P_2) \leq \overline{\int_{[a,b]}} f + \varepsilon$.

So $\inf\limits_{P_1, P_2 \text{ coming from such } P} \{U(f, P_1) + U(f, P_2)\}$

$$= \inf_{P_1 \text{ coming from such } P} U(f, P_1) + \inf_{P_2 \text{ coming from such } P} U(f, P_2)$$

$$\leq \overline{\int_{[a,b]}} f + \varepsilon$$

So $\overline{\int_{[a,c]}} f + \overline{\int_{[c,b]}} f \leq \overline{\int_{[a,b]}} f$ ∎

Claim $\underline{\int_{[a,c]}} f + \underline{\int_{[c,b]}} f = \underline{\int_{[a,b]}} f$

Now consider:

$$0 = \overline{\int_{[a,b]}} f - \underline{\int_{[a,b]}} f = \underbrace{\overline{\int_{[a,c]}} f - \underline{\int_{[a,c]}} f}_{} + \underbrace{\overline{\int_{[c,b]}} f - \underline{\int_{[c,b]}} f}_{}$$

non negative by the proposition

So $\overline{\int_{[a,c]}} f = \underline{\int_{[a,c]}} f$ and $\overline{\int_{[c,b]}} f = \underline{\int_{[c,b]}} f$, hence f is integrable on the subintervals. The claims then give:

$$\int_{[a,b]} f = \overline{\int_{[a,b]}} f = \overline{\int_{[a,c]}} f + \overline{\int_{[c,b]}} f = \int_{[a,c]} f + \int_{[c,b]} f \quad \blacksquare$$

The other two properties are clear from the definitions.

Fundamental Theorem of Calculus

We'll see later this is redundant.

Suppose $f:[a,b]\to\mathbb{R}$ is continuous and integrable.

Set $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$.

Proof: For any $a<y<x<b$, consider the difference quotient

$$\frac{F(x)-F(y)}{x-y} = \frac{\int_a^x f(t)\,dt - \int_a^y f(t)\,dt}{x-y}$$

$$= \int_y^x \frac{f(t)}{x-y}\,dt$$

Now since f is continuous, $\forall \varepsilon>0\ \exists \delta>0$ so that $x-\delta<y<x$ guarantees $|f(x)-f(y)|<\varepsilon$.

Continuing with such y:

$$= \int_y^x \frac{f(t)-f(x)+f(x)}{x-y}\,dt < \int_y^x \frac{\varepsilon}{x-y}\,dt + \int_y^x \frac{f(x)}{x-y}\,dt$$

$$= \varepsilon + f(x).$$

Sim, $\int_y^x \frac{f(t)-f(x)+f(x)}{x-y}\,dt > -\varepsilon + f(x)$

So $\left|\frac{F(x)-F(y)}{x-y} - f(x)\right| < \varepsilon$ as long as $y \in (x-\delta, x)$

The same analysis shows

$$\left|\frac{F(x)-F(y)}{x-y} - f(x)\right| < \varepsilon \text{ as long as } y \in (x, x+\delta).$$

So $F'(x) = \lim_{y\to x} \frac{F(x)-F(y)}{x-y} = f(x)$. ∎

Notice that this argument used only the formal properties of integration! We didn't have to look under the hood at all.

Further, MVT says any function with $g'(x) = f(x)$ has $g(x) = F(x) + C$.

$$= \int_a^x f(t)\,dt + C$$

Cor. (Mean Value Theorem for Integrals)

Suppose $f:[a,b]\to\mathbb{R}$ is continuous and integrable. Then $\exists\, c \in [a,b]$ so that $\int_{[a,b]} f = f(c)(b-a)$

Proof Apply MVT for derivatives to $F(x) = \int_a^x f(t)\,dt$ ∎

Theorem If $F:[a,b]\to\mathbb{R}$ has F' integrable, then

$$\int_{[a,b]} F' = F(b) - F(a)$$

Proof For any partition $P: x_0 < x_1 < \cdots < x_n$, the MVT says $\exists\, c_k \in [x_k, x_{k+1}]$ so that $F'(c_k)(x_{k+1}-x_k) = F(x_{k+1}) - F(x_k)$

Thus

$$\begin{aligned} F(b)-F(a) &= F(x_n) - F(x_{n-1}) \\ &\quad + F(x_{n-1}) - F(x_{n-2}) \\ &\quad + \cdots - F(x_0) \\ &= \sum_{k=0}^{n-1} F'(c_k)(x_{k+1}-x_k) \end{aligned}$$

OTOH, this sum lies between $L(F',P)$ and $U(F',P)$. Taking sup and inf,

$$\underline{\int_{[a,b]}} F' \le F(b)-F(a) \le \overline{\int_{[a,b]}} F'$$

So if F' is integrable, we are done. ∎

A useful criterion for integrability

$f:[a,b]\to\mathbb{R}$ is integrable iff $\forall \varepsilon>0\ \exists$ a partition P_ε with $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$.

Proof. ($\Rightarrow$) If f is integrable, given $\varepsilon>0\ \exists\ P_1, P_2$ with

$$U(f,P_1)<\int_{[a,b]} f+\frac{\varepsilon}{2} \qquad L(f,P_2)>\int_{[a,b]} f-\frac{\varepsilon}{2}$$

Set $P=P_1\cup P_2$; then

$$U(f,P)-L(f,P)\le U(f,P_1)-L(f,P_2)$$
$$<\int_{[a,b]} f+\frac{\varepsilon}{2}-\left(\int_{[a,b]} f-\frac{\varepsilon}{2}\right)=\varepsilon$$

($\Leftarrow$) $0\le \overline{\int_{[a,b]}} f-\underline{\int_{[a,b]}} f\le U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon.$

So $\overline{\int_{[a,b]}} f-\underline{\int_{[a,b]}} f$ is a nonnegative number smaller than any $\varepsilon>0$, hence $=0$. ∎

This will allow us to show integrability by constructing just one nice partition for each $\varepsilon>0$.

Theorem Bounded monotone functions are integrable.

Proof. Let $f:[a,b]\to\mathbb{R}$ be a bounded monotone function. We'll consider the case when f is nondecreasing i.e.

$$x\le y\Rightarrow f(x)\le f(y).$$

Since f is monotone, $\sup_{[\alpha,\beta]} f=f(\beta)$ $\inf_{[\alpha,\beta]} f=f(\alpha)$.

Consider, for each $n\in\mathbb{N}$, the partition P_n given by

$x_i=a+i\,\frac{b-a}{n}$ (this is the partition you're probably familiar with from intro calculus...)

Then $U(f,P_n)-L(f,P_n)$

$$=\sum_{k=0}^{n-1}\Big(\sup_{[x_k,x_{k+1}]} f-\inf_{[x_k,x_{k+1}]} f\Big)(x_{k+1}-x_k)$$
$$=\sum_{k=0}^{n-1}\big(f(x_{k+1})-f(x_k)\big)\left(\frac{b-a}{n}\right)$$
$$=\frac{b-a}{n}\big(f(b)-f(a)\big)$$

which by choosing n large can be made arbitrarily small. The useful criterion gives integrability of f. ∎

Theorem Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Then $|f|$ is integrable and $\left| \int_{[a,b]} f \right| \leq \int_{[a,b]} |f|$.

Proof Set
$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$
$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

Then $f_+(x) \geq 0$, $f_-(x) \geq 0$, and
$$f(x) = f_+(x) - f_-(x)$$
$$|f(x)| = f_+(x) + f_-(x)$$

Claim If $f : [a,b] \to \mathbb{R}$ is integrable, then f_+ is integrable.

Proof Given any partition P,
$$U(f_+, P) - L(f_+, P) = \sum_{k=0}^{n-1} \Big(\sup_{[x_k, x_{k+1}]} f - \inf_{[x_k, x_{k+1}]} f \Big)(x_{k+1} - x_k)$$

For any α, β,
$$\sup_{[\alpha,\beta]} f_+ - \inf_{[\alpha,\beta]} f_+ = \begin{cases} \sup_{[\alpha,\beta]} f - \inf_{[\alpha,\beta]} f & \text{if } \inf_{[\alpha,\beta]} f > 0 \\ \sup_{[\alpha,\beta]} f & \text{if } \inf_{[\alpha,\beta]} f \leq 0,\ \sup f > 0 \\ 0 & \text{if } \sup f \leq 0 \end{cases}$$

In any case, $\sup_{[\alpha,\beta]} f_+ - \inf_{[\alpha,\beta]} f_+ \leq \sup_{[\alpha,\beta]} f - \inf_{[\alpha,\beta]} f$.

So $U(f_+, P) - L(f_+, P) \leq U(f, P) - L(f, P)$ and integrability of f allows us to make this arbitrarily small. Thus f_+ is integrable. ∎

Then $f_- = f_+ - f$ is integrable and so $|f| = f_+ - f_-$ is integrable.

Then
$$\left| \int_{[a,b]} f \right| = \left| \int_{[a,b]} f_+ - \int_{[a,b]} f_- \right|$$
$$\leq \left| \int_{[a,b]} f_+ \right| + \left| \int_{[a,b]} f_- \right|$$
$$= \int_{[a,b]} f_+ + \int_{[a,b]} f_- = \int_{[a,b]} f_+ + f_- = \int_{[a,b]} |f|$$

Uniform Continuity

We'd like to prove the following theorem:

Theorem Continuous functions are integrable.

We'll need some more topological technology...

Def'n Given $A \subset (M,d)$, we say $f: A \to (N,\rho)$ is **uniformly continuous** if for every $\varepsilon > 0$ $\exists \delta > 0$ so that

$$d(x,y) < \delta \Rightarrow \rho(f(x),f(y)) < \varepsilon.$$

How is this different from just "continuous"?

Continuity: $\forall x \in A\ \forall \varepsilon > 0\ \exists \delta > 0 \ni \forall y: d(x,y) < \delta,\ \rho(f(x),f(y)) < \varepsilon$

unif. continuity: $\forall \varepsilon > 0\ \exists \delta > 0 \ni \forall x\ \forall y: d(x,y) < \delta,\ \rho(f(x),f(y)) < \varepsilon$

It's "just" the order of quantifiers.

E.g. The identity function $g: A \to A$, $x \mapsto x$. Then we could take $\delta = \varepsilon$, so g is uniformly continuous.

E.g. Let $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$.

To guarantee $|x^2 - y^2| < \varepsilon$,

$$-\varepsilon < x^2 - y^2 < \varepsilon$$
$$x^2 - \varepsilon < y^2 < \varepsilon + x^2$$
$$\sqrt{x^2-\varepsilon} < y < \sqrt{\varepsilon + x^2}$$
$$\sqrt{x^2-\varepsilon} - x < y - x < \sqrt{\varepsilon + x^2} - x$$

So we should use $\delta = \min\{x - \sqrt{x^2-\varepsilon}, \sqrt{x^2+\varepsilon} - x\}$.

Note that as $x \to \infty$, $\delta \to 0$.

So we couldn't choose any one $\delta > 0$ that works for all $x \in \mathbb{R}$. f is not uniformly cont's on $\mathbb{R}$.

But if we only consider $x \in [0,b)$ for some $b \in \mathbb{R}$, then we could pick one particular δ. (Which one?)

So f is uniformly cont's on $[0,b)$.

Why is uniform continuity important?

Theorem Uniformly continuous functions are integrable.

Proof Given $f: [a,b] \to \mathbb{R}$ uniformly continuous:

Let P_n be the even partition $x_i = a + i\frac{b-a}{n}$.

For any $\varepsilon > 0$, uniform continuity gives $\delta > 0$ so that

$|x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$.

Choose n large enough that $\frac{b-a}{n} < \delta$.

Then $U(f,P_n) - L(f,P_n)$

$$= \sum_{k=0}^{n-1} \left(\sup_{[x_k,x_{k+1}]} f - \inf_{[x_k,x_{k+1}]} f \right) \left(\frac{b-a}{n} \right)$$

$$\stackrel{*}{=} \sum_{k=0}^{n-1} \left(f(x_k^+) - f(x_k^-) \right) \left(\frac{b-a}{n} \right)$$

where $x_k^+ \in [x_k, x_{k+1}]$, $x_k^- \in [x_k, x_{k+1}]$

$$< \sum_{k=0}^{n-1} \frac{\varepsilon}{b-a} \frac{b-a}{n} = \varepsilon$$

So f is integrable. ∎

* Lemma: A uniformly continuous function is continuous.

(Yes, you need to prove this!)

How to go from "continuous" to "uniformly continuous"?

We need to change the order of quantifiers...

Theorem Suppose $f: A \to (N,\rho)$ is continuous and $K \subset A$ is compact. Then f is uniformly continuous on K.

Proof Given $\varepsilon > 0$, $x \in K$, there is $\delta_x > 0$ so that $d(y,x) < \delta_x$ $\Rightarrow \rho(f(x), f(y)) < \frac{\varepsilon}{2}$.

The collection $\{D(x, \frac{1}{2}\delta_x)\}_{x \in K}$ is an open cover for K. By compactness, there is a finite subcover

$D(x_1, \frac{1}{2}\delta_1), \ldots, D(x_n, \frac{1}{2}\delta_n)$

Let $\delta = \min_{i=1,\ldots,n} \frac{1}{2}\delta_i$. Now if x,y are two points of K with $d(x,y) < \delta$, there is some x_i with $d(x_i,x) < \frac{1}{2}\delta_i$.

Then $d(x_i,y) \le d(x_i,x) + d(x,y) < \frac{1}{2}\delta_i + \delta \le \delta_i$

So $\rho(f(x_i), f(y)) < \frac{1}{2}\varepsilon$ and $\rho(f(x_i), f(x)) < \frac{1}{2}\varepsilon$.

So $\rho(f(x), f(y)) < \varepsilon$. ∎

So we finally have:

Theorem Continuous functions are integrable on closed intervals.

Aside: improper integrals

For an exam problem (alt. Rudin's Theorem 6.20), we know that if f is integrable on $[a,b]$, then for any $c \in [a,b]$,

$$\int_a^c f = \lim_{x \to c} \int_a^x f$$

So we can define, without fooling the theory,

Defn $f : [a,b) \to \mathbb{R}$ is integrable on $[a,b]$ if

1. for all $c \in [a,b]$, f is integrable on $[a,c]$.
2. $\lim_{c \to b} \int_a^c f$ exists.