Sequences of Functions

As promised, we're going to apply some of our topological technology to metric spaces of functions.

The first question to ask is: which metric? Equivalently, we can ask: what should it mean for a sequence of functions to converge?

Eg. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$x \mapsto x^n$$

Each function is continuous, and indeed uniformly continuous. Let's try and take a limit of $(f_n)_{n \in \mathbb{N}}$.

Fix $x \in [0,1]$. What is $\lim_n f_n(x)$?

Case: $x = 1$. Then $f_n(x) = 1$. So $\lim_n f_n(x) = 1$

Case: $x < 1$. Then $f_n(x) = x^n$, so $\lim_n f_n(x) = 0$

So $\lim_n f_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$

Defn Given a sequence of functions $f_n : A \to (N, \rho)$, if for each $x \in A$, we have $\lim_n f_n(x) = f(x)$, we say $(f_n)_{n \in \mathbb{N}}$ **converges pointwise** to f.

Write $f_n \to f$ on A

Our example shows that pointwise convergence isn't all that.

Pointwise limits of sequences of continuous functions need not be continuous!

"$C[0,1]$ is not closed under pointwise limits."

We can understand the problem with this family as a problem of interchange of limits.

$$\lim_{x \to 1} \lim_{n \to \infty} x^n = \lim_{x \to 1} 0 = 0$$

$$\lim_{n \to \infty} \lim_{x \to 1} x^n = \lim_{n \to \infty} 1 = 1$$

The homework has several similar examples.

Let's unpack the quantifiers in the definition of pointwise convergence.

$f_n \to f$ on A if

$$\forall x \in A\ \forall \varepsilon > 0\ \exists N = N(\varepsilon, x) \in \mathbb{N} \ni \forall n \geq N,$$
$$\rho(f_n(x), f(x)) < \varepsilon.$$

If we swap a $\forall$ and a $\exists$, we get:

$$\forall \varepsilon > 0\ \exists N = N(\varepsilon) \in \mathbb{N} \ni \forall x \in A,\ \forall n \geq N,$$
$$\rho(f_n(x), f(x)) < \varepsilon.$$

Defn Given $f_n : A \to (N, \rho)$, we say the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f on A if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that $n \geq N$ guarantees $\rho(f_n(x), f(x)) < \varepsilon$.

We write $f_n \rightrightarrows f$ on A or

$f_n \to f$ (unif.) on A

E.g. The functions $f_n : x \mapsto x^n$ do not converge uniformly on $[0,1]$.

For each $n \in \mathbb{N}$, let $y_n = \sqrt[n]{\frac{1}{2}}$. Then $f_n(y_n) = \frac{1}{2}$.

So each f_n takes the value $\frac{1}{2}$, which is $\frac{1}{2}$ away from both 0 and 1 (which are the only candidates for the values of a uniform limit).

Proposition ("uniform convergence implies pointwise convergence")

$$f_n \rightrightarrows f \text{ on } A \Rightarrow f_n \to f \text{ on } A$$

Proposition (uniform Cauchy criterion)

Suppose (N, ρ) is complete. Then $f_n : A \to (N, \rho)$ have

$$f_n \rightrightarrows f \text{ iff } \forall \varepsilon > 0\ \exists N \in \mathbb{N} \ni \forall m, n \geq N,\ \forall x \in A,$$
$$\rho(f_n(x), f_m(x)) < \varepsilon.$$

Theorem Suppose $f_n \rightrightarrows f$ and f_n are all continuous. Then f is continuous.

This will follow from the following theorem about limits:

Theorem Suppose $f_n : A \to (N,\rho)$ are a sequence of maps

- $\lim_{a \to x_0} f_n(a) = L_n$
- $f_n \rightrightarrows f$ on A
- $\lim_n L_n = L$

Then $\lim_{a \to x_0} f(a) = L$.

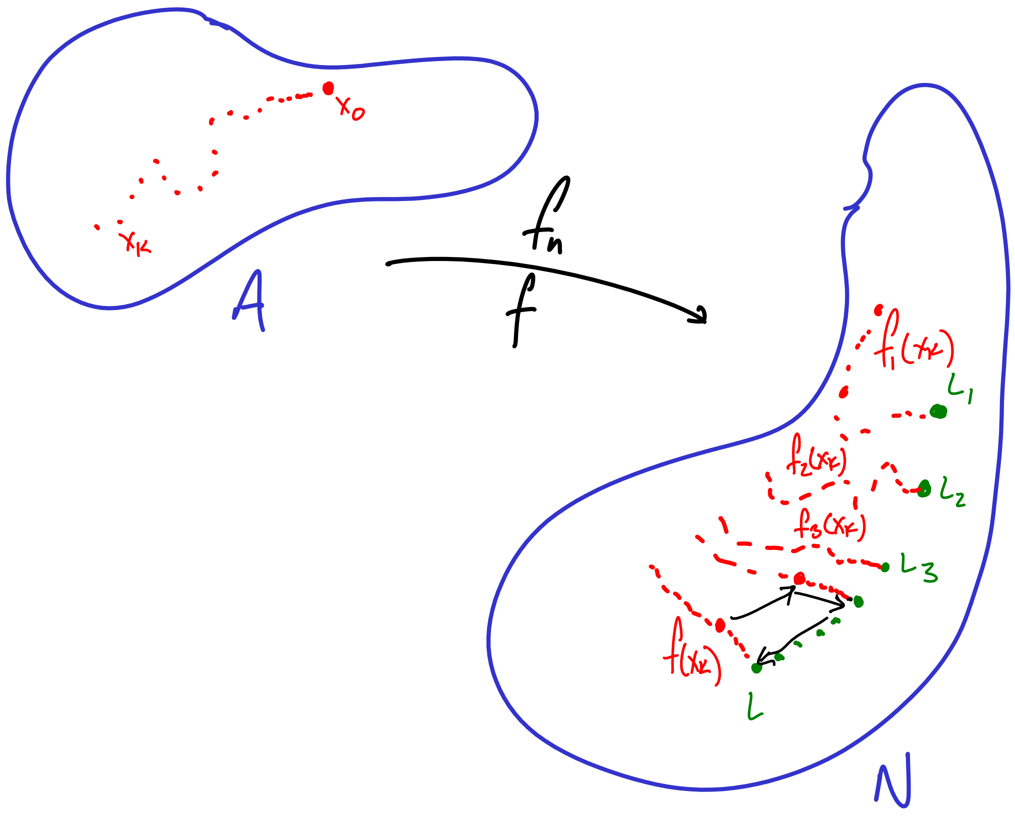
That is, $\lim_n \lim_{a\to x_0} f_n(a) = \lim_{a \to x_0} \lim_n f_n(a)$.

Proof Recall the sequential characterisation of the limit of a function:

Proposition ("Two-Path Test")

$\lim_{a \to x_0} f(a) = b$ iff for any sequence $(x_n)_{n\in\mathbb{N}}$ in $A \setminus \{x_0\}$ with $x_n \to x_0$ we have $f(x_n) \to b$.

We want to show that $\lim_{a\to x_0} f(a) = L$, so we'll let $(x_k)_{k\in\mathbb{N}}$ be a sequence in A with $x_k \to x_0$, and show that $f(x_k) \to L$.



We need to estimate $\rho(f(x_k), L)$. The triangle (quadrilateral?) inequality says

$$\rho(f(x_k), L) \le \rho(f(x_k), f_n(x_k)) + \rho(f_n(x_k), L_n) + \rho(L_n, L)$$

(as shown with black arrows in the figure)

$\rho(f(x_K), f_n(x_K))$

Given $\varepsilon>0$, by uniform convergence there is $N_1 = N_1(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq N_1, \forall a \in A$, $\rho(f_n(a), f(a)) < \frac{\varepsilon}{3}$.

$\rho(f_n(x_K), L_n)$

Since $f_n(x_K) \xrightarrow{K} L_n$, given $\varepsilon>0$ there is $M(n,\varepsilon)$ so that for all $K \geq M(n,\varepsilon)$, $\rho(f_n(x_K), L_n) < \frac{\varepsilon}{3}$

$\rho(L_n, L)$

Since $L_n \to L$, given $\varepsilon>0$ there is $N_2 = N_2(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq N_2$ $\rho(L_n, L) < \frac{\varepsilon}{3}$.

Now set $K = N_1 + N_2$ and $N = M(K, \varepsilon)$

Then $k \geq K$ guarantees that

$$\rho(f(x_k), L) \leq \rho(f(x_k), f_K(x_k)) + \rho(f_K(x_k), L_K) + \rho(L_K, L)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \blacksquare$$

More niceties of uniform convergence

Theorem Suppose $f_n : [a,b] \to \mathbb{R}$ are integrable and $f_n \rightrightarrows f$ on $[a,b]$. Then f is integrable and

$$\lim_n \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Proof We're asked to show f is integrable.

So given $\varepsilon>0$, $\exists N \ni n \geq N$ guarantees, $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$.

So for such n, $\sup_{[\alpha,\beta]} f \leq \sup_{[\alpha,\beta]} f_n + \frac{\varepsilon}{4(b-a)}$ on any interval $[\alpha,\beta] \subset [a,b]$. So for any partition P, $n \geq N$ guarantees

$$U(f,P) \leq U(f_n, P) + \frac{\varepsilon}{4} \quad \text{and sim.}$$
$$L(f,P) \geq L(f_n, P) - \frac{\varepsilon}{4}$$

So $U(f,P) - L(f,P) \leq U(f_n,P) - L(f_n,P) + \frac{\varepsilon}{2}$.

Now fixing a particular n, there is some $P(n,\varepsilon)$ so that $U(f_n, P(n,\varepsilon)) - L(f_n, P(n,\varepsilon)) \leq \frac{\varepsilon}{2}$.

Thus $U(f, P(n,\varepsilon)) - L(f, P(n,\varepsilon)) \leq \varepsilon$.

So f is integrable.

Now consider the function $f(x)-f_n(x)$, which is integrable. If $n \geq N$, we have

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b (f_n(x)-f(x))dx\right|$$
$$\leq \int_a^b |f_n(x)-f(x)|\,dx \leq \int_a^b \frac{\varepsilon}{4(b-a)}dx$$
$$= \frac{\varepsilon}{4} < \varepsilon.$$

Thus $\lim \int_a^b f_n(x)dx = \int_a^b f(x)dx$. ■

Note. Uniform convergence is necessary here.

Eg. Let $f_n(x) = \begin{cases} n^2x & x\in[0,\frac{1}{n}] \\ 2n-n^2x & x\in(\frac{1}{n},\frac{2}{n}] \\ 0 & x\in(\frac{2}{n},1] \end{cases}$

Then each f_n is continuous and $\int_0^1 f_n(x)dx = 1$.

Moreover, $f_n(0)=0$ and if $x>0$ there is n large enough that $\frac{2}{n}<x$, so $f_n(x)=0$.

Thus $f_n \to 0$ pointwise.

$$\lim_n \int_0^1 f_n(x)dx = \lim 1 = 1$$
$$\int_0^1 \lim_n f_n(x)dx = \int_0^1 0\,dx = 0$$

$0 \neq 1$!

Theorem Suppose $f_n:[a,b]\to\mathbb{R}$ are diff'ble and $f_n \to f$. Suppose also f_n' are cont's and $f_n' \rightrightarrows g$. Then f is diff'ble and $f'=g$.

Proof Since f_n' are cont's, we have, for any particular x_0,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt$$

Since $f_n \to f$ pointwise, we have

$$f(x) = \lim f_n(x) = \lim \left(f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt \right)$$
$$= \lim f_n(x_0) + \lim \int_{x_0}^{x} f_n'(t)\,dt$$
$$= f(x_0) + \int_{x_0}^{x} g(t)\,dt$$

Since each f_n' is cont's, g is cont's. So f is diff'ble, and $f'(x) = g(x)$ ■

<u>Note</u> It is possible to have f_n diff'ble, $f_n \rightrightarrows f$, but f not diff'ble.

<u>E.g.</u> Set $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $f(x) = |x|$.

Then $f_n(x) = \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2} + \sqrt{\frac{1}{n}} = |x| + \frac{1}{\sqrt{n}}$

So $|f_n(x) - f(x)| = f_n(x) - f(x) \leq \frac{1}{\sqrt{n}}$.

Given ε, choose $N = \frac{1}{\varepsilon^2} + 1$. Then $n \geq N$ guarantees $|f_n(x) - f(x)| < \varepsilon$. Thus $f_n \rightrightarrows f$ on $\mathbb{R}$.

Each f_n is diff'ble. (In fact, $f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$ are uniformly bounded!)

But f is not diff'ble.

<u>Proposition</u> (Weierstraß M-test for sequences)

Given $f_n : A \to \mathbb{R}$ with $f_n \to f$ pointwise on A,

Set $M_n = \sup_{x \in A} |f_n(x) - f(x)|$.

Then $f_n \rightrightarrows f$ iff $M_n \to 0$.

What does $\sup_{x \in A} |f_n(x) - f(x)|$ measure?

<u>Stone-Weierstraß Theorem (baby version)</u>

Let $f : [a,b] \to \mathbb{R}$ be a continuous function.

Then there is a sequence $(P_n)_{n \in \mathbb{N}}$ of polynomials with $P_n \rightrightarrows f$ on $[a,b]$.

<u>Proof.</u> We'll just prove it for $a=0$, $b=1$.

WLOG we may assume $f(1) = f(0) = 0$.

Extend f to $\mathbb{R}$ by setting $f \equiv 0$ outside $[0,1]$.

Then $f : \mathbb{R} \to \mathbb{R}$ is unif. cont's.

Now define $Q_n(x) = c_n (1 - x^2)^n$, where

$c_n = \frac{1}{\int_{-1}^{1} (1-x^2)^n\,dx}$, so that $\int_{-1}^{1} Q_n(x)\,dx = 1$.

Now it's not hard to compute that $c_n < \sqrt{n}$.

Set $P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt$.

Problem: this isn't a polynomial in x (at least, not obviously so.)

$$\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt \qquad \text{Use sub. } s = x+t$$

$$= \int_{0}^{1} f(s)\, Q_n(s-x)\, ds$$ which <u>is</u> obviously a polynomial in x.

Now we want to show $P_n \rightrightarrows f$. Given $\varepsilon > 0$, choose $\delta > 0$ so that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$. Set $M = \sup f$.

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) Q_n(t)\, dt - \int_{-1}^{1} f(x)\, Q_n(t)\, dt \right|$$

$$= \left| \int_{-1}^{1} (f(x+t) - f(x))\, Q_n(t)\, dt \right| \le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$= \int_{-1}^{-\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$+ \int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

Now when $|t| < \delta$, we have $|f(x+t) - f(x)| < \frac{\varepsilon}{2}$.

So $$\int_{-\delta}^{\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt \le \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\, dt \le \frac{\varepsilon}{2}$$

The other terms can be estimated since $|f| \le M$ and $x^2 \ge \delta^2$:

$$\int_{-1}^{-\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt \le 2M c_n (1-\delta^2)^n \le 2M\sqrt{n}(1-\delta^2)^n$$

$$\int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt \le 2M c_n (1-\delta^2)^n \le 2M\sqrt{n}(1-\delta^2)^n$$

<u>Lemma</u> For $\delta > 0$, $\sqrt{n}(1-\delta^2)^n \to 0$ as $n \to \infty$.

So we can take n large enough that

$4M\sqrt{n}(1-\delta^2)^n < \frac{\varepsilon}{2}$. Then $|P_n(x) - f(x)| < \varepsilon$. ∎

The Space of Continuous Maps

Defn Given $A \subset (M,d)$, set $\mathcal{B}(A) = \{f : A \to \mathbb{R} \mid f \text{ is bounded}\}$.

We proved (in the special case $A = [0,1]$) in homework that $\mathcal{B}(A)$ has a norm:

$$\|f\|_\infty = \sup_{x \in A} |f(x)|$$

(We also showed for HW that this norm doesn't come from an inner product.)

Given a normed space $(N, \|\cdot\|)$, we can define $\mathcal{B}(A;N) = \{f : A \to N \mid f \text{ is bounded}\}$ and equip this space with $\|f\|_\infty = \sup_{x \in A} \|f(x)\|$ to get a normed space.

Theorem Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of bounded functions $A \to N$. Then $f_n \to f$ in $\mathcal{B}(A;N)$ iff $f_n \rightrightarrows f$ on A.

Proof. It's just the M-test (which you'll prove for HW.)

$$f_n \rightrightarrows f \Leftrightarrow M_n \to 0$$

$$\Leftrightarrow \forall \varepsilon > 0 \; \exists N \text{ s.t. } n \geq N \Rightarrow M_n < \varepsilon$$

$$\Leftrightarrow f_n \to f \text{ in } \mathcal{B}(A;N). \blacksquare$$

Defn $\mathcal{C}_b(A;N) = \{f : A \to N \mid f \text{ is cont's and bounded}\}$.

Then $\mathcal{C}_b(A;N)$ is a vector subspace of $\mathcal{B}(A;N)$.

(If A is compact, $\mathcal{C}(A;N) = \mathcal{C}_b(A;N)$. Why?)

From here, we'll assume that $\mathcal{C}(A;N)$ is equipped with the metric $d_\infty(f,g) = \|f-g\|_\infty$, the uniform metric.

Some rephrasings

Stone-Weierstraß Polynomials are dense in $\mathcal{C}([a,b];\mathbb{R})$

Another Theorem (which?) $\mathcal{C}_b(A;N)$ is a closed subset of $\mathcal{B}(A;N)$

Another Theorem (which?) $I : \mathcal{C}([a,b];\mathbb{R}) \to \mathbb{R}$ is continuous.

$$f \longmapsto \int_{[a,b]} f$$

Eg. Consider $B = \{ f \in \mathcal{C}([0,1];\mathbb{R}) \mid f(x) > 1 \text{ for all } x \in [0,1] \}$.

Claim B is open in $\mathcal{C}([0,1];\mathbb{R})$.

Proof. Let $f \in B$. By Max-Min, $\exists\ x_0 \in [0,1]$ with $f(x) \geq f(x_0)$ for all $x \in [0,1]$. Also notice $f(x_0) > 1$.

Set $\varepsilon = \frac{1}{2}(f(x_0) - 1)$. Then $\|f-g\|_\infty < \varepsilon$

$\Rightarrow \forall x,\ |f(x) - g(x)| < \varepsilon$

$$\Rightarrow \forall x, \quad g(x) > f(x) - \varepsilon \geq f(x_0) - \tfrac{1}{2}(f(x_0) - 1)$$
$$= \tfrac{1}{2}(f(x_0) + 1) > 1$$

So $\|f-g\|_\infty < \varepsilon \Rightarrow g \in B$. Thus B is open!

Eg. The unit sphere in $\mathcal{C}([0,1];\mathbb{R})$ is not compact.

Proof It suffices to exhibit a sequence of functions $(f_n)_{n\in\mathbb{N}}$, with $\|f_n\|_\infty = 1$ but no subsequence of $(f_n)_{n\in\mathbb{N}}$ converges uniformly.

Set $f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{n}) \\ 2 - nx & x \in [\frac{1}{n}, \frac{2}{n}) \\ 0 & x \in (\frac{2}{n}, 1] \end{cases}$

Then any subsequential pointwise limit is discont's.

Note that the unit sphere is by definition bounded, and since any norm is continuous in the metric it induces, the unit sphere is closed. Thus:

Heine-Borel fails for $\mathcal{C}([0,1];\mathbb{R})$!

How can we tell if a subset of $\mathcal{C}(A;N)$ is compact?

Defn A collection ("family") of functions $\mathcal{F}$ $A \to N$ is called *equicontinuous* if for any $\varepsilon > 0$ there is $\delta > 0$ so that for any $x, y \in A$, and any $f \in \mathcal{F}$, we have $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$

Immediate consequence

If $\mathcal{F}$ is an equicontinuous family, then each $f \in \mathcal{F}$ is uniformly continuous.

Eg. Let $\mathcal{L} = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is diff'ble and } \sup|f'| \leq 2 \}$

For each $f \in \mathcal{L}$, MVT gives $|f(x) - f(y)| \leq 2|x-y|$

So we can take $\delta = \frac{\varepsilon}{2}$, which depends neither on x, nor on f. Thus $\mathcal{L}$ is equicontinuous.

Def'n A family $\mathcal{F}$ of functions $A \to N$ is compact at $x \in A$ if the image set $\mathcal{F}_x = \{f(x) \mid f \in \mathcal{F}\} \subset N$ is compact.
$\mathcal{F}$ is pointwise compact if $\mathcal{F}$ is compact at x for all $x \in A$.

E.g. $\mathcal{L}$?

unit sphere of $\mathcal{C}([0,1];\mathbb{R})$?

$\{f \in \mathcal{C}([0,1];\mathbb{R}) \mid f(x) > 1 \text{ for all } x \in [0,1]\}$?

the set of all power functions $[0,1] \to \mathbb{R}$, $x \mapsto x^p$?

the set of polynomials with constant term 0 ?

Arzelà - Ascoli:

The Arzelà-Ascoli theorem gives a way to tell if sets of functions are compact with respect to $\|\cdot\|_\infty$

Arzelà-Ascoli Theorem

Suppose $A \subset (M,d)$ is compact, $(N,\|\cdot\|)$ complete. Then $\mathcal{F} \subset \mathcal{C}(A;N)$ is compact iff
- $\mathcal{F}$ is uniformly closed
- $\mathcal{F}$ is pointwise compact
- $\mathcal{F}$ is equicontinuous

Proof. We'll show $(\Leftarrow)$. Use Bolzano-Weierstraß and show such $\mathcal{F}$ is sequentially compact.
First recall the following lemma from the proof of Heine-Borel:

Lemma If A is compact, for each $\delta > 0$ there is a finite set $\{y_1, \dots, y_K\} \subset A$ so that $A \subset \bigcup_{i=1}^{K} D(y_i;\delta)$

Set $C_{\frac{1}{n}} = \{y_1^n, \dots, y_{K_n}^n\}$ to be the set given by the Lemma with $\delta = \frac{1}{n}$. Let $C = \bigcup_{n\in\mathbb{N}} C_{\frac{1}{n}}$. Note that C is a countable set.
Denumerate its elements $C = \{x_1, x_2, x_3, \dots\}$
Given a sequence $(f_n)_{n\in\mathbb{N}}$ of points of $\mathcal{F}$. For each $x \in C$, note that $(f_n(x))_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}_x$, which is

compact. So some subsequence $(f_{n_k}(x))_{k\in\mathbb{N}}$ converges.

Apply this first to x_1 to get a subsequence $f_{1n}(x_1) \to z_1$.
From f_{1n}, select a subsequence f_{2n} with $f_{2n}(x_2) \to z_2$.
Since f_{2n} is a subsequence of f_{1n}, $f_{2n}(x_1) \to z_1$ as well.
In this manner, set

$$\begin{array}{cccccc} f_{11}(x_1) & f_{12}(x_1) & f_{13}(x_1) & \cdots & \to & z_1 \\ f_{21}(x_2) & f_{22}(x_2) & f_{23}(x_2) & \cdots & \to & z_2 \\ \vdots & & & & & \end{array}$$

where each f_{pn} is a subsequence of $f_{p-1\,n}$ and each $f_{pn}(x_k) \to z_k$ for all $k \le p$.

Now consider the diagonal sequence $g_n = f_{nn}$.

Claim $(g_n)_{n\in\mathbb{N}}$ is uniformly Cauchy.

Proof Given $\varepsilon > 0$, let $\delta > 0$ be such that $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \frac{\varepsilon}{3}$. Choose p with $\frac{1}{p} < \delta$. Then there are $\{y_1, \dots, y_k\}$ so that every point $x \in A$ is at most $\frac{1}{p}$ from one of the y_i. Since each $(g_n(y_i))_{n\in\mathbb{N}}$ is convergent in N, it is Cauchy in N. So there are $N_1, \dots, N_k$ with $m,n \geq N_i$ guaranteeing $\rho(g_n(y_i), g_m(y_i)) < \frac{\varepsilon}{3}$. Let $N_\varepsilon = \max\{N_1, \dots, N_k\}$.
Then $m,n \geq N_\varepsilon$ guarantees

$$\rho(g_n(x), g_m(x)) \le \underbrace{\rho(g_n(x), g_n(y_i))}_{\mathcal{F}\text{ equicont's}} + \underbrace{\rho(g_n(y_i), g_m(y_i))}_{(g_n(y_i))_{n\in\mathbb{N}}\text{ Cauchy}} + \underbrace{\rho(g_m(y_i), g_m(x))}_{\mathcal{F}\text{ equicont's}}$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon.$$

Since N_ε depends only on ε, and not on x, we have shown $(g_n)_{n\in\mathbb{N}}$ is uniformly Cauchy. ■

Since $(N, \|\cdot\|)$ is complete, $(g_n)_{n\in\mathbb{N}}$ is uniformly convergent.
Since $\mathcal{F}$ is uniformly closed, $g_n \rightrightarrows g$ implies $g \in \mathcal{F}$.
So we have shown every sequence in $\mathcal{F}$ has a convergent subsequence, whose limit lies in $\mathcal{F}$, i.e. $\mathcal{F}$ is sequentially compact.