MATH 360 Homework 1

Due 18 January 2012

Properties of the Real Numbers

1. Show that the following properties hold in any field.
   i. (Unique identities) If $a + x = a$ for some a, then $x = 0$. If $ax = a$ for some a, then $x = 1$.
   ii. (Unique inverses) If $a + x = 0$, then $x = -a$. If $ax = 1$, then $x = a^{-1}$.
   iii. (No divisors of zero) If $xy = 0$, then $x = 0$ or $y = 0$.
   iv. For all x, $-(-x) = x$.
   v. For all x, $-x = (-1) \cdot x$.
   vi. If $x \neq 0$ then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
   vii. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.
2. Show that the following properties hold in any ordered field.
   i. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$. If $x \leq y$ and $z \leq 0$, then $yz \leq xz$.
   ii. If $x \leq 0$ and $y \leq 0$, then $xy \geq 0$. If $x \leq 0$ and $y \geq 0$, then $xy \leq 0$.
   iii. For any x, $x^2 \geq 0$.
   iv. If $x \leq 0$, then $-x \geq 0$. (Prove this without the fact that $-x = (-1) \cdot x$.)
3. In an ordered field, $0 \leq x < y$ implies $x^2 < y^2$.
4. (Partial converse to the above) If $x^2 < y^2$, then $|x| < |y|$
5. In an ordered field,
   i. $|x| \geq 0$ for every x.
   ii. $|x| = 0$ if and only if $x = 0$.
   iii. $|xy| = |x||y|$
   iv. $|x + y| \leq |x| + |y|$
   v. $||x| - |y|| \leq |x - y|$
6. Consider the set $S = \{0, 1\}$ with the operations $\oplus$ and $\otimes$ given by the following tables:

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

   i. Show that $(S, \oplus, \otimes)$ is a field.
   ii. Show that $(S, \oplus, \otimes)$ is *not* an ordered field. (*Hint.* There are only two possible orderings on S.)
   iii. (*Bonus*) Show that no ordered field is finite.
7. Let $(S, \leq)$ be an ordered field, and $A \subset S$ a subset. A *lower bound* for A is an element $b \in S$ for which $b \leq a$ for all $a \in A$. A lower bound b is *greatest* if, for any other lower bound b', we have $b' \leq b$. We say that S has the *greatest lower bound property* if any $A \subset S$ which has a lower bound has a greatest lower bound.
   i. Show that if S has the least upper bound property, then S has the greatest lower bound property.
   ii. Show that if S has the greatest lower bound property, then S has the least upper bound property.
8. (Marsden-Hoffman's Problem 1.2.10) Define (x_n) by $x_0 = 0$, $x_{n+1} = \sqrt{2 + x_n}$.
   i. Show that the sequence (x_n) converges in $\mathbb{R}$.
   ii. Let $\lambda = \lim x_n$. Show that $\lambda^2 - \lambda - 2 = 0$.
   iii. What is a more familiar name for λ?