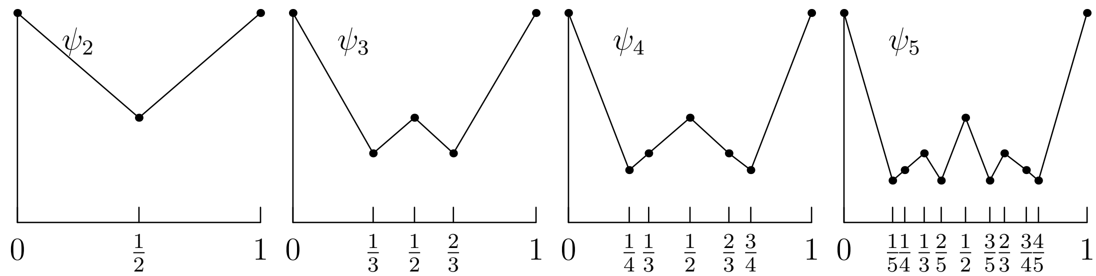
MATH 360 Homework 11

Due 23 April 2013

1. For a fixed $n \in \mathbb{N}$, consider the sequence of functions $\psi_n : [0,1] \to \mathbb{R}$, given by: if $x = \frac{p}{q}$ is a rational number in reduced form, with $q \leq n$, then $\psi_n(x) = \frac{1}{q^2}$; for other x (that is, the irrationals and any rational with reduced denominator $q > n$), let $\psi_n(x)$ be given by linear interpolation between the values of ψ_n at points with $q \leq n$. See the figure below.



   i. Show that each ψ_n is continuous.

   ii. For a given $x \in [0,1]$, consider the sequence $y_n = \psi_n(x)$. Show that $(y_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$.

   iii. What is the pointwise limit of the sequence $(\psi_n)_{n\in\mathbb{N}}$?

   iv. Is the convergence uniform?

2. Prove the Weierstraß M-test for sequences of functions:

   Theorem. *Let $A \subset (M,d)$ be a subset of a metric space, and $f_n : A \to (N,\rho)$ a sequence of maps which converge pointwise on A to $f : A \to (N,\rho)$. Set*

$$M_n = \sup_{x\in A} \rho(f_n(x), f(x))$$

   Show that $f_n \rightrightarrows f$ on A iff $M_n \to 0$.

3. Let $A \subset (M,d)$ be a compact subset of a metric space and $(N, \|\cdot\|)$ be a normed space. Let ρ be the metric on N induced by $\|\cdot\|$. For $f, g \in \mathcal{C}(A;N)$, define

$$d_\infty(f,g) = \sup_{x\in A} \|f(x) - g(x)\|$$

   Show that $(\mathcal{C}(A;N), d_\infty)$ is a complete metric space iff (N,ρ) is a complete metric space.

4. Show that the family of continuous functions $f : [0,1] \to \mathbb{R}$ with positive integrals is open in $\mathcal{C}([0,1];\mathbb{R})$.

5. Let $\mathcal{B} = \{f \in \mathcal{C}(\mathbb{R};\mathbb{R}) | f(x) > 0 \text{ for all } x \in \mathbb{R}\}$.

   i. Show that $f(x) = e^{-x^2}$ is in $\mathcal{B}$.

   ii. Is $\mathcal{B}$ open?

   iii. What is $\text{int}(\mathcal{B})$?

6. Suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence of (real) polynomial which converge uniformly on $\mathbb{R}$ to some $f : \mathbb{R} \to \mathbb{R}$.

   i. Show that f is a polynomial. (*Hint.* Use the Cauchy criterion for uniform convergence.)

   ii. Discuss (i) and the Stone-Weierstraß Theorem.