MATH 360 Homework 2

Due 25 January 2013

1. (Versions of the Archimedean property) Prove that the following properties are equivalent in an ordered field, i.e. any ordered field $\mathbb{F}$ which possesses one of the properties must possess the other two as well:
   i. If $x \in \mathbb{F}$, there is an integer n with $x < n$.
   ii. If $x, y \in \mathbb{F}$, and $0 < x < y$, then there is an integer n with $y < nx$.
   iii. If $x \in \mathbb{F}$, and $0 < x$, then there is an integer n with $0 < \frac{1}{n} < x$.
2. i. Suppose $(x_n)_n \in \mathbb{N}$ is a sequence and α is a number so that for all n, $x_n \leq \alpha$. Suppose $x_n \to L$. Show that $L \leq \alpha$.
   ii. Give an example where $x_n < \alpha$ for all n, but $L = \alpha$.
3. (No least positive real number) Let $0 \leq x$ be a real number so that $x \leq \epsilon$ for every $\epsilon > 0$. Show that $x = 0$.
4. (Marsden-Hoffman's Problem 1.5.5) Let (x_n) be a bounded sequence of real numbers. Is the following statement true or false?
   If $\limsup x_n = b$, then for large enough n, $x_n \leq b$.
   If it is true, provide a proof. If not, provide a counterexample.
5. State the *converse* of the statement in problem 4. Is it true or false? If it is true, provide a proof. If not, provide a counterexample.
6. Formulate and prove the proper sum rule for sup, i.e. given subsets $A, B \subseteq \mathbb{R}$, relate $\sup(A + B)$ to $\sup A$ and $\sup B$. You may assume both sets are nonempty. (Here $A+B$ is the set $\{a+b | a \in A, b \in B\}$.)
7. Formulate and prove the proper sum rule for lim sup, i.e. given sequences $(x_n), (y_n)$, relate $\limsup(x_n + y_n)$ to $\limsup x_n$ and $\limsup y_n$. You may assume both sequences are bounded below, but do not assume either is bounded above.
8. Is there a way to equip the complex field $\mathbb{C}$ with an ordering which makes it an ordered field?
9. i. Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with a convergent subsequence, i.e. $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_k x_{n_k} = L$. Show that $x_n \to L$.
   ii. Show that a Cauchy sequence has at most one subsequential limit.
   iii. Show that Cauchy sequences are bounded.
   iv. Conclude that Cauchy sequences converge.
10. (Rudin's Problem 3.1)
    i. If $(x_n)_{n\in\mathbb{N}}$ converges, show that $(|x_n|)_{n\in\mathbb{N}}$ converges.
    ii. Is the converse of the above true?
11. For a bounded sequence $(x_n)_{n\in\mathbb{N}}$, show that $\limsup x_n = \lim S_n$, where $S_n = \sup\{x_k | k \geq n\}$. (Note that you must justify why $\lim S_n$ exists.)
12. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[-B, B]$. Let $a_0 = -B$ and $b_0 = B$, and call the interval $[-B, B] = [a_0, b_0] = I_0$.
    i. Show that the following defines a subsequence. Let $n_0 = 0$. For $k > 0$, let I_k be whichever of the two halves of I_{k-1} (that is, whichever of $[a_{k-1}, \frac{1}{2}(a_{k-1}+b_{k-1})]$ and $[\frac{1}{2}(a_{k-1}+b_{k-1}), b_{k-1}]$) contains x_n for infinitely many n with $n > n_{k-1}$. Set n_k to be the first such n. Label the endpoints of by $I_k = [a_k, b_k]$.
    ii. Show that the sequence of right endpoints $(b_k)_{k\in\mathbb{N}}$ converges to some b.
    iii. Show that $\lim_k x_{n_k} = b$.
    iv. Congratulations! You have given an alternate proof of the Bolzano-Weierstrass Theorem.