1. (Versions of the Archimedean property) Prove that the following properties are equivalent in an ordered field, i.e. any ordered field $F$ which possesses one of the properties must possess the other two as well:
   i. If $x \in F$, there is an integer $n$ with $x < n$.
   ii. If $x, y \in F$, and $0 < x < y$, then there is an integer $n$ with $y < nx$.
   iii. If $x \in F$, and $0 < x$, then there is an integer $n$ with $0 < \frac{1}{n} < x$.

2. i. Suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence and $\alpha$ is a number so that for all $n$, $x_n \leq \alpha$. Suppose $x_n \to L$. Show that $L \leq \alpha$.
   ii. Give an example where $x_n < \alpha$ for all $n$, but $L = \alpha$.

3. (No least positive real number) Let $0 \leq A, B \subseteq \mathbb{R}$. Formulate and prove the proper sum rule for sup, i.e. given subsets $A$ and $B$, suppose $\sup(A + B)$ be whichever of the $A$ or $B$ is a number so that for all $x \in A$ and $y \in B$, $x + y \leq \sup(A + B)$. You may assume both sets are nonempty. (Here $A + B$ is the set $\{a + b | a \in A, b \in B\}$.)

4. (Marsden-Hoffman’s Problem 1.5.5) Let $(x_n)$ be a bounded sequence of real numbers. Is the following statement true or false?
   If $\limsup x_n = b$, then for large enough $n$, $x_n \leq b$.
   If it is true, provide a proof. If not, provide a counterexample.

5. State the converse of the statement in problem 4. Is it true or false? If it is true, provide a proof. If not, provide a counterexample.

6. Formulate and prove the proper sum rule for sup, i.e. given subsets $A, B \subseteq \mathbb{R}$, relate $\sup(A + B)$ to $\sup A$ and $\sup B$. You may assume both sets are nonempty. (Here $A + B$ is the set $\{a + b | a \in A, b \in B\}$.)

7. Formulate and prove the proper sum rule for $\limsup$, i.e. given sequences $(x_n), (y_n)$, relate $\limsup(x_n + y_n)$ to $\limsup x_n$ and $\limsup y_n$. You may assume both sequences are bounded below, but do not assume either is bounded above.

8. Is there a way to equip the complex field $\mathbb{C}$ with an ordering which makes it an ordered field?

9. i. Suppose $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with a convergent subsequence, i.e. $(x_{n_k})_{k \in \mathbb{N}}$ such that $\lim x_{n_k} = L$. Show that $x_n \to L$.
   ii. Show that a Cauchy sequence has at most one subsequential limit.
   iii. Show that Cauchy sequences are bounded.
   iv. Conclude that Cauchy sequences converge.

10. (Rudin’s Problem 3.1)
    i. If $(x_n)_{n \in \mathbb{N}}$ converges, show that $(|x_n|)_{n \in \mathbb{N}}$ converges.
    ii. Is the converse of the above true?

11. For a bounded sequence $(x_n)_{n \in \mathbb{N}}$, show that $\limsup x_n = \lim S_n$, where $S_n = \sup\{x_k | k \geq n\}$. (Note that you must justify why $\lim S_n$ exists.)

12. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[-B, B]$. Let $a_0 = -B$ and $b_0 = B$, and call the interval $[-B, B] = [a_0, b_0] = I_0$.
    i. Show that the following defines a subsequence. Let $n_0 = 0$. For $k > 0$, let $I_k$ be whichever of the two halves of $I_{k-1}$ (that is, whichever of $[a_{k-1}, \frac{1}{2}(a_{k-1} + b_{k-1})]$ and $[\frac{1}{2}(a_{k-1} + b_{k-1}), b_{k-1}]$) contains $x_n$ for infinitely many $n$ with $n > n_{k-1}$. Set $n_k$ to be the first such $n$. Label the endpoints of by $I_k = [a_k, b_k]$.
    ii. Show that the sequence of right endpoints $(b_k)_{k \in \mathbb{N}}$ converges to some $b$.
    iii. Show that $\lim x_{n_k} = b$.
    iv. Congratulations! You have given an alternate proof of the Bolzano-Weierstrass Theorem.