MATH 360 Homework 3

Due 1 February 2013

1. Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers with $a_n \to A$ and $A > 0$. Consider the sequence $b_n = \sqrt{n + a_n} - \sqrt{n}$.
   i. Let $0 < c < \frac{1}{2}$. Show that there is some $N \in \mathbb{N}$, depending on c and the sequence $(a_n)_{n\in\mathbb{N}}$, so that $n \geq N$ guarantees $2cn + c^2 a_n < n$.
   ii. Show that for large enough n, $n^2 + a_n n > (n + ca_n)^2$.
   iii. Show that $\lim b_n = 0$.
2. Given $A, B \subset \mathbb{R}$, define $A \preceq B$ to mean that for every $x \in A$ there is some $y \in B$ with $x \leq y$. Are the following statements true or false? If a statement is true, provide a proof. If it is false, give a counterexample.
   i. If $A \preceq B$, then $\sup A \leq \sup B$.
   ii. If $A \preceq B$, then $\inf A \leq \inf B$.
   iii. If $A \preceq B$ and $B \preceq A$, then $A = B$.
3. (Parallelogram Law) Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Let $\|\cdot\|$ be the norm that comes from the inner product, i.e. $\|x\| = \sqrt{\langle x, x\rangle}$. Show that for any $x, y \in V$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

4. Let $(V, \|\cdot\|)$ be a normed space, and let $d(x, y) = \|x - y\|$ be the metric that comes from the norm. Show that for any positive $N \in \mathbb{R}$, there are $x_N, y_N \in V$ with $d(x_N, y_N) > N$. (*Bonus.* Actually there is exactly one exception, that is, a normed space without arbitrarily-distant points. What is it?)
5. Let $V = \{f : [0, 1] \to \mathbb{R} | \exists B \ni . \forall x \in [0, 1], |f(x)| \leq B\}$ be the space of functions from $[0, 1]$ to $\mathbb{R}$ which are bounded.
   i. Define $\|f\|_\infty = \sup\{|f(x)| | x \in [0, 1]\}$.
   ii. Verify that $(V, \|\cdot\|_\infty)$ is a normed space with the standard operations of addition and scalar multiplication of functions. (You must prove that V is a $\mathbb{R}$-vector space and that $\|\cdot\|_\infty$ is a norm.)
   iii. Let $f(x) = x$, $g(x) = 1$. Find $\|f\|_\infty$, $\|g\|_\infty$, $\|f + g\|_\infty$, and $\|f - g\|_\infty$.
   iv. Does $\|\cdot\|_\infty$ come from an inner product on V? Justify your answer.
6. Sketch an open ball of radius $\frac{1}{2}$, an open ball of radius 1, and an open ball of radius 2 in $\mathbb{R}^2$ with respect to each of the following metrics:
   i. the *standard metric* $d_2(x, y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2}$.
   ii. the ∞*-metric* $d_\infty(x, y) = \max\{|x^1 - y^1|, |x^2 - y^2|\}$.
   iii. the *standard bounded metric* $\rho(x, y) = \dfrac{d_2(x, y)}{1 + d_2(x, y)}$.
   iv. the discrete metric.