1. Suppose \((a_n)_{n \in \mathbb{N}}\) is a sequence of real numbers with \(a_n \to A\) and \(A > 0\). Consider the sequence 
\[ b_n = \sqrt{n + a_n} - \sqrt{n}. \]

i. Let \(0 < c < \frac{1}{2}\). Show that there is some \(N \in \mathbb{N}\), depending on \(c\) and the sequence \((a_n)_{n \in \mathbb{N}}\), so that \(n \geq N\) guarantees \(2cn + c^2a_n < n\).

ii. Show that for large enough \(n\), \(n^2 + a_n > (n + ca_n)^2\).

iii. Show that \(\lim b_n = 0\).

2. Given \(A, B \subset \mathbb{R}\), define \(A \preceq B\) to mean that for every \(x \in A\) there is some \(y \in B\) with \(x \leq y\). Are the following statements true or false? If a statement is true, provide a proof. If it is false, give a counterexample.

i. If \(A \preceq B\), then \(\sup A \leq \sup B\).

ii. If \(A \preceq B\), then \(\inf A \leq \inf B\).

iii. If \(A \preceq B\) and \(B \preceq A\), then \(A = B\).

3. (Parallelogram Law) Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space. Let \(\|\cdot\|\) be the norm that comes from the inner product, i.e. \(\|x\| = \sqrt{\langle x, x \rangle}\). Show that for any \(x, y \in V\), we have
\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \]

4. Let \((V, \|\cdot\|)\) be a normed space, and let \(d(x, y) = \|x - y\|\) be the metric that comes from the norm. Show that for any positive \(N \in \mathbb{R}\), there are \(x_N, y_N \in V\) with \(d(x_N, y_N) > N\). (Bonus. Actually there is exactly one exception, that is, a normed space without arbitrarily-distant points. What is it?)

5. Let \(V = \{ f : [0, 1] \to \mathbb{R} \mid \exists B \in \mathbb{R} . \forall x \in [0, 1], |f(x)| \leq B \}\) be the space of functions from \([0, 1]\) to \(\mathbb{R}\) which are bounded.

i. Define \(\|f\|_\infty = \sup \{ |f(x)| \mid x \in [0, 1] \}\).

ii. Verify that \((V, \|\cdot\|_\infty)\) is a normed space with the standard operations of addition and scalar multiplication of functions. (You must prove that \(V\) is a \(\mathbb{R}\)-vector space and that \(\|\cdot\|_\infty\) is a norm.)

iii. Let \(f(x) = x, g(x) = 1\). Find \(\|f\|_\infty, \|g\|_\infty, \|f + g\|_\infty,\) and \(\|f - g\|_\infty\).

iv. Does \(\|\cdot\|_\infty\) come from an inner product on \(V\)? Justify your answer.

6. Sketch an open ball of radius \(\frac{1}{2}\), an open ball of radius 1, and an open ball of radius 2 in \(\mathbb{R}^2\) with respect to each of the following metrics:

i. the standard metric \(d_2(x, y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2}\).

ii. the \(\infty\)-metric \(d_\infty(x, y) = \max\{|x^1 - y^1|, |x^2 - y^2|\}\).

iii. the standard bounded metric \(\rho(x, y) = \frac{d_2(x, y)}{1 + d_2(x, y)}\).

iv. the discrete metric.