MATH 360 Homework 4

Due 8 February 2013

Recall that our definition of a *closed subset* differs from Rudin's:

Definition. A subset of a metric space $A \subset (M, d)$ is *closed* if its complement is open.

1. i. Show that the union of a finite number of closed sets is closed.
   ii. Show that the arbitrary intersection of closed sets is closed.
   iii. Give an example of an infinite family of closed sets whose union is not closed. (*Hint.* It's easier than you think.)
2. Let $(V, \|\cdot\|)$ be a normed space, and suppose $A \subset V$ is open with respect to the metric induced by the norm $\|\cdot\|$. Let $B \subset V$ be any subset. Show that $A + B = \{a + b | a \in A, b \in B\}$ is open.
3. (Marsden-Hoffman's problem 2.1.5) Let A be an open subset of $\mathbb{R}$ (with the standard metric) and B be any subset of $\mathbb{R}$.
   i. Give an example where $AB = \{ab | a \in A, b \in B\}$ is not open.
   ii. Give conditions on the sets A, B under which AB must be open. (*Hint.* Consider, for each $b \in B$, $Ab = \{ab | a \in A\}$.)
4. Given a subset of a metric space $A \subset (M, d)$, consider the collection $\mathcal{G}_A = \{U \subset M | U \subset A, U \text{ is open}\}$ of all open subsets of (M, d) which are subsets of A.
   i. Show that if $G \in \mathcal{G}_A$, then $G \subset \operatorname{int}(A)$.
   ii. Show that $\operatorname{int}(A) = \bigcup_{G \in \mathcal{G}_A} G$. (*Hint.* Show that for any $x \in \operatorname{int}(A)$, there is some $G \in \mathcal{G}_A$ with $x \in G$.
   iii. Show $\operatorname{int}(A)$ is open.
   iv. Explain why the preceding allows us to say that *the interior of A is the maximal open set that is a subset of A.*
5. Let (M, d_0) be a set with the discrete metric.
   i. Show that every subset $A \subset M$ is clopen.
   ii. Show that the set of limit points of any subset $A \subset M$ is empty.
   iii. Why is this not a problem for the following theorem?

   Theorem. *A set A is closed iff the limit points of A all belong to A.*
6. Let A and B be subsets of a metric space with $A \subset B$, and x a point in the metric space.
   i. If x is a limit point for B, is x necessarily a limit point for A?
   ii. If x is a limit point for A, is x necessarily a limit point for B?
7. (Rudin's problem 2.8)
   i. Is every point of every open (with respect to the standard metric) $E \subset \mathbb{R}^2$ a limit point of E?
   ii. Is every point of every closed (with respect to the standard metric) $E \subset \mathbb{R}^2$ a limit point of E?