Definition. A sequence \((x_n)_{n \in \mathbb{N}}\) in a metric space is called *eventually constant* if there is some \(N\) so that \(m, n \geq N\) guarantees \(x_n = x_m\).

1. Let \(A \subset B\) be subsets of a metric space \((M, d)\). Show that \(A\) is dense in \(B\) if and only if for any \(b \in B\) and any \(\varepsilon > 0\), there is \(a \in A\) with \(d(a, b) < \varepsilon\).

2. Let \((M, d_0)\) be an set with the discrete metric. Consider a sequence \((x_n)_{n \in \mathbb{N}}\).
   i. Show that \((x_n)_{n \in \mathbb{N}}\) converges if and only if \((x_n)_{n \in \mathbb{N}}\) is eventually constant.
   ii. Show that \((x_n)_{n \in \mathbb{N}}\) is Cauchy if and only if \((x_n)_{n \in \mathbb{N}}\) is eventually constant.

3. (Marsden-Hoffman's problem 2.21: open set characterisation of the Cauchy property) Prove that a sequence \((x_n)\) in a normed space \((V, \|\cdot\|)\) is Cauchy if and only if for every open set \(U\) containing \(0 \in V\), there is \(N \in \mathbb{N}\) so that \(m, n \geq N\) guarantees \(x_n - x_m \in U\).

4. Consider the standard metric on \(\mathbb{R}^n\), given by \(d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}\). Given a sequence \((x_k)_{k \in \mathbb{N}}\) of points in \(\mathbb{R}^n\), for each \(i \in \{1, \ldots, n\}\), let \(x^i_k\) be the \(i\)th coordinate of \(x_k\). Then \((x^i_k)_{k \in \mathbb{N}}\) is a sequence of real numbers.
   i. Show that the distance in \(\mathbb{R}^n\) is related to the distances between the coordinates of \(x = (x^1, \ldots, x^n)\) and \(y = (y^1, \ldots, y^n)\) as follows:
   \[
   \max\{ |x^1 - y^1|, \ldots, |x^n - y^n| \} \leq d(x, y) \leq (|x^1 - y^1| + \cdots + |x^n - y^n|)
   \]
   ii. Show that \(x_k \to x\) if and only if for each \(i \in \{1, \ldots, n\}\), \(x^i_k \to x^i\).
   iii. Show that \((x_k)_{k \in \mathbb{N}}\) is Cauchy if and only if for each \(i \in \{1, \ldots, n\}\), \((x^i_k)_{k \in \mathbb{N}}\) is Cauchy.
   iv. Show that \(\mathbb{Q}^n = \{(q^1, \ldots, q^n) | q^i \in \mathbb{Q}\} \subset \mathbb{R}^n\) is dense in \(\mathbb{R}^n\).

Bonus Recall that the *post office metric* on \(\mathbb{R}^2\) is defined by \(d_{PO}(x, x) = 0\) and \(d_{PO}(x, y) = \|x\| + \|y\|\) if \(x \neq y\).
   i. Show that a sequence \((x_n)_{n \in \mathbb{N}}\) converges if it either converges to \((0, 0) \in \mathbb{R}^2\) or is eventually constant.
   ii. Show that a sequence \((x_n)_{n \in \mathbb{N}}\) is Cauchy if it either converges to \((0, 0)\) or is eventually constant. 
   \(\text{(Hint. If } (x^1, x^2) \neq (0, 0), \text{ then there is a closest point to } (x^1, x^2) \text{ and it is } (0, 0).)\)