MATH 360 Homework 5

Due 22 February 2013

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space is called *eventually constant* if there is some N so that $m, n \geq N$ guarantees $x_n = x_m$.

1. Let $A \subset B$ be subsets of a metric space (M, d). Show that A is dense in B if and only if for any $b \in B$ and any $\epsilon > 0$, there is $a \in A$ with $d(a, b) < \epsilon$.
2. Let (M, d_0) be an set with the discrete metric. Consider a sequence $(x_n)_{n\in\mathbb{N}}$.
   i. Show that $(x_n)_{n\in\mathbb{N}}$ converges if and only if $(x_n)_{n\in\mathbb{N}}$ is eventually constant.
   ii. Show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(x_n)_{n\in\mathbb{N}}$ is eventually constant.
3. (Marsden-Hoffman's problem 2.21: open set characterisation of the Cauchy property) Prove that a sequence (x_n) in a normed space $(V, \|\cdot\|)$ is Cauchy if and only if for every open set U containing $0 \in V$, there is $N \in \mathbb{N}$ so that $m, n \geq N$ guarantees $x_n - x_m \in U$.
4. Consider the standard metric on $\mathbb{R}^n$, given by $d(x, y) = \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$. Given a sequence $(x_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$, for each $i \in \{1, \ldots, n\}$, let x_k^i be the ith coordinate of x_k. Then $(x_k^i)_{k\in\mathbb{N}}$ is a sequence of real numbers.
   i. Show that the distance in $\mathbb{R}^n$ is related to the distances between the coordinates of $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ as follows:
   $$\max\left\{|x^1 - y^1|, \ldots, |x^n - y^n|\right\} \leq d(x, y) \leq \left(|x^1 - y^1| + \cdots + |x^n - y^n|\right)$$
   ii. Show that $x_k \to x$ if and only if for each $i \in \{1, \ldots, n\}$, $x_k^i \to x^i$.
   iii. Show that $(x_k)_{k\in\mathbb{N}}$ is Cauchy if and only if for each $i \in \{1, \ldots, n\}$, $(x_k^i)_{k\in\mathbb{N}}$ is Cauchy.
   iv. Show that $\mathbb{Q}^n = \{(q^1, \ldots, q^n) | q^i \in \mathbb{Q}\} \subset \mathbb{R}^n$ is dense in $\mathbb{R}^n$.

Bonus Recall that the *post office metric* on $\mathbb{R}^2$ is defined by $d_{\text{PO}}(x, x) = 0$ and $d_{\text{PO}}(x, y) = \|x\| + \|y\|$ if $x \neq y$.
   i. Show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges iff it either converges to $(0, 0) \in \mathbb{R}^2$ or is eventually constant.
   ii. Show that a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy iff it either converges to $(0, 0)$ or is eventually constant. (*Hint.* If $(x^1, x^2) \neq (0, 0)$, then there is a closest point to (x^1, x^2) and it is $(0, 0)$.)