MATH 360 Homework 5 Notes and Selected Answers

Note 1. Some easy-to-prove facts you will find useful:

- Any eventually-constant sequence is convergent (hence Cauchy).
- A sequence $(x_n)_{n\in\mathbb{N}}$ is eventually constant iff there is some x and some N so that $n \geq N$ guarantees $x_n = x$.

Note 2. One thing that cannot be emphasised enough is that *not every metric space is complete.* Equivalently, "Cauchy" and "convergent" are not synonyms (unless we happen to know that we are in a complete metric space).

1. Let $A \subset B$ be subsets of a metric space (M, d). Show that A is dense in B if and only if for any $b \in B$ and any $\epsilon > 0$, there is $a \in A$ with $d(a, b) < \epsilon$.
2. Let (M, d_0) be an set with the discrete metric. Consider a sequence $(x_n)_{n\in\mathbb{N}}$.

   i. Show that $(x_n)_{n\in\mathbb{N}}$ converges if and only if $(x_n)_{n\in\mathbb{N}}$ is eventually constant.

   Solution (⇒). Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in M$. Then for $\epsilon = \frac{1}{2}$, there is N so that $n \geq N$ guarantees $d_0(x_n, x) < \frac{1}{2}$. But then for $n \geq N$, $x_n = x$ since the metric is discrete. So $(x_n)_{n\in\mathbb{N}}$ is eventually constant.

   ii. Show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(x_n)_{n\in\mathbb{N}}$ is eventually constant.

   Solution (⇒). Suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Then for $\epsilon = \frac{1}{2}$, there is N so that $m, n \geq N$ guarantees $d_0(x_n, x_m) < \frac{1}{2}$. But then for $m, n \geq N$, $x_n = x_m$ since the metric is discrete. So $(x_n)_{n\in\mathbb{N}}$ is eventually constant.

3. (Marsden-Hoffman's problem 2.21: open set characterisation of the Cauchy property) Prove that a sequence (x_n) in a normed space $(V, \|\cdot\|)$ is Cauchy if and only if for every open set U containing $0 \in V$, there is $N \in \mathbb{N}$ so that $m, n \geq N$ guarantees $x_n - x_m \in U$.
4. Consider the standard metric on $\mathbb{R}^n$, given by $d(x, y) = \sqrt{(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2}$. Given a sequence $(x_k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$, for each $i \in \{1, \ldots, n\}$, let x_k^i be the ith coordinate of x_k. Then $(x_k^i)_{k\in\mathbb{N}}$ is a sequence of real numbers.

   i. Show that the distance in $\mathbb{R}^n$ is related to the distances between the coordinates of $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ as follows:

   $$\max\left\{|x^1 - y^1|, \ldots, |x^n - y^n|\right\} \leq d(x, y) \leq \left(|x^1 - y^1| + \cdots + |x^n - y^n|\right)$$

   ii. Show that $x_k \to x$ if and only if for each $i \in \{1, \ldots, n\}$, $x_k^i \to x^i$.

   Solution (⇒). Consider a particular $i \in \{1, \ldots, n\}$. Given $\epsilon > 0$, since $x_k \to x$, there is some N so that $k \geq N$ guarantees $d(x_k, x) < \epsilon$. But $|x_k^i - x^i| \leq d(x_k, x) < \epsilon$, so this N works to show $x_k^i \to x^i$.

   Solution (⇐). Given $\epsilon > 0$, since each $x_k^i \to x^i$, there is some N_i so that $k \geq N_i$ guarantees $|x_k^i - x^i| < \frac{\epsilon}{n}$. Let $N = \max\{N_1, \ldots, N_n\}$. Then for $k \geq N$,

   $$d(x_k, x) \leq |x_k^1 - x^1| + \cdots + |x_k^n - x^n| < \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} = \epsilon$$

   iii. Show that $(x_k)_{k\in\mathbb{N}}$ is Cauchy if and only if for each $i \in \{1, \ldots, n\}$, $(x_k^i)_{k\in\mathbb{N}}$ is Cauchy.

   Solution (⇒). Consider a particular $i \in \{1, \ldots, n\}$. Given $\epsilon > 0$, since $(x_k)_{k\in\mathbb{N}}$ is Cauchy, there is some N so that $k, l \geq N$ guarantees $d(x_k, x_l) < \epsilon$. But $|x_k^i - x_l^i| \leq d(x_k, x_l) < \epsilon$, so this N works to show $(x_k^i)_{k\in\mathbb{N}}$ is Cauchy.

   Solution (⇐). Given $\epsilon > 0$, since each $(x_k^i)_{k\in\mathbb{N}}$ is Cauchy, there is some N_i so that $k, l \geq N_i$ guarantees $|x_k^i - x_l^i| < \frac{\epsilon}{n}$. Let $N = \max\{N_1, \ldots, N_n\}$. Then for $k, l \geq N$,

   $$d(x_k, x_l) \leq |x_k^1 - x_l^1| + \cdots + |x_k^n - x_l^n| < \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} = \epsilon$$

iv. Show that $\mathbb{Q}^n = \{(q^1, \dots, q^n) | q^i \in \mathbb{Q}\} \subset \mathbb{R}^n$ is dense in $\mathbb{R}^n$.

Solution. Let $x = (x^1, \dots, x^n)$ be a point of $\mathbb{R}^n$. We'll use problem 1. Given $\epsilon > 0$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each i there is $q^i \in \mathbb{Q}$ with $|q^i - x^i| < \frac{\epsilon}{n}$. Set $q = (q^1, \dots, q^n)$ and note that $q \in \mathbb{Q}^n$. Then

$$d(q, x) \leq |q^1 - x^1| + \cdots + |q^n - x^n| < \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} = \epsilon$$

Bonus Recall that the *post office metric* on $\mathbb{R}^2$ is defined by $d_{\text{PO}}(x, x) = 0$ and $d_{\text{PO}}(x, y) = \|x\| + \|y\|$ if $x \neq y$.

i. Show that a sequence $(x_n)_{n \in \mathbb{N}}$ converges iff it either converges to $(0, 0) \in \mathbb{R}^2$ or is eventually constant.

Solution ($\Rightarrow$)**.** Suppose that $(x_n)_{n \in \mathbb{N}}$ converges to some x. If $x = (0, 0)$, then we are done. So suppose $x \neq (0, 0)$. Then for $\epsilon = \frac{1}{2}\|x\|$, there is N so that $n \geq N$ guarantees $d_{\text{PO}}(x_n, x) < \frac{1}{2}\|x\|$. But then for $n \geq N$, $x_n = x$ by the definition of the post office metric. So $(x_n)_{n \in \mathbb{N}}$ is eventually constant.

ii. Show that a sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy iff it either converges to $(0, 0)$ or is eventually constant. (*Hint.* If $(x^1, x^2) \neq (0, 0)$, then there is a closest point to (x^1, x^2) and it is $(0, 0)$.)

Solution ($\Rightarrow$)**.** Suppose that $(x_n)_{n \in \mathbb{N}}$ is Cauchy.
Case 1: There is some $\eta > 0$ so that for infinitely many indices n, $\|x_n\| \geq \eta$. (Note that since $d_{\text{PO}}((0, 0), x_n) = \|x_n\|$, this means that the sequence $(x_n)_{n \in \mathbb{N}}$ is not converging to $(0, 0)$.)
Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, there is N so that $m, n \geq N$ guarantees $d_{\text{PO}}(x_n, x_m) < \frac{\eta}{2} < \|x_n\|$. By definition of the post office metric, this implies $x_n = x_m$. So $(x_n)_{n \in \mathbb{N}}$ is eventually constant.
Case 2: For all η, only finitely many n have $\|x_n\| \geq \eta$. Then for n larger than the largest such, $d_{\text{PO}}((0, 0), x_n) = \|x_n\| < \eta$. That is, $(x_n)_{n \in \mathbb{N}}$ converges to $(0, 0)$.