MATH 360 Homework 6.5
Not due, but do!

Define

$$\mathbb{R}^\infty = \left\{(x^1, x^2, \ldots, x^k, \ldots) \middle| x^i \in \mathbb{R}, \text{ all but finitely many } x^i = 0\right\}$$

Note that under pointwise operations, $\mathbb{R}^\infty$ is a vector space.

1. For $x, y \in \mathbb{R}^\infty$, show that $\langle x, y\rangle_2 = \sum_{i=1}^{\infty} x^i y^i$ defines an inner product on $\mathbb{R}^\infty$. Write d_2 for the metric induced by $\langle \cdot, \cdot \rangle_2$.
2. Show that $\|x\|_\infty = \max\left\{|x^i| \middle| i \in \mathbb{N}\right\}$ defines a norm on $\mathbb{R}^\infty$. Write d_∞ for the metric induced by $\|\cdot\|_\infty$.
3. Show that $d_{\text{taxi}}(x, y) = \sum_{i=1}^{\infty} |x^i - y^i|$ defines a metric on $\mathbb{R}^\infty$.
4. Consider the sequences $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^\infty$ defined by:

$$x_k^i = \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$$

$$y_k^i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

   Which of these sequences is bounded with respect to which of the metrics d_2, d_∞, d_{taxi}?
5. Show that $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ as above have no convergent subsequence with respect to any of the three metrics. (Compare to problem 9 of Homework 6.)
6. Show that no two of d_2, d_∞, and d_{taxi} are equivalent metrics.
7. Explain the role of the requirement "all but finitely many $x^i = 0$". (*Hint.* You may want to think about this before, during, and after the other problems.)