# MATH 360 Homework 6

Due 1 March 2013

**Definition.** Given two metrics $d$ and $\rho$ on the same set $M$, we say $d$ is *equivalent to* $\rho$ if there are positive real numbers $C_1$ and $C_2$ so that for any $x, y \in M$, we have

$$C_1\rho(x, y) \le d(x, y) \le C_2\rho(x, y)$$

1. Show that metric equivalence is an equivalence relation on the set of all metrics on a given set. That is, show for any metrics $d, \rho, \ell$ on the set $M$, we have:
   i. $d$ is equivalent to $d$.
   ii. If $d$ is equivalent to $\rho$, then $\rho$ is equivalent to $d$. (So we may say "$d$ and $\rho$ are equivalent".)
   iii. If $d$ is equivalent to $\rho$ and $\rho$ is equivalent to $\ell$, then $d$ is equivalent to $\ell$.
2. Show that $d$ is equivalent to $\rho$ iff there is some positive $C$ so that $\frac{1}{C}\rho(x, y) \le d(x, y) \le C\rho(x, y)$.
3. Show that on $\mathbb{R}^n$, the taxicab metric and the standard metric are equivalent. (*Hint.* The optimal $C$ as in problem 2 is $\sqrt{n}$.)
4. Suppose $d$ and $\rho$ are equivalent metrics. Show that $A \subset M$ is open in $(M, d)$ iff it is open in $(M, \rho)$.
5. Suppose $d$ and $\rho$ are equivalent metrics. Show that $(M, d)$ is complete iff $(M, \rho)$ is complete.
6. Are the following metric spaces complete? Give proofs. (*Hint.* Think about previous homework sets.)
   i. $\mathbb{R}^n$ with the standard metric.
   ii. $(\mathbb{R}^2, d_{\mathrm{PO}})$
   iii. $\mathbb{R}^n$ with the $\infty$-metric $d_\infty(x, y) = \max\{|x^1 - y^1|, \ldots, |x^n - y^n|\}$
   iv. Any set with the discrete metric.
7. Read Rudin's Theorem 4.8, its proof, and its corollary. Let $f : (M, d) \to (N, \rho)$ be a continuous map. For each of the following, either give a proof or a counterexample.
   i. If $U \subset M$ is open, need $f(U)$ be open?
   ii. If $U \subset M$ is closed, need $f(U)$ be closed?
8. Let $(M, d)$, $(N, \rho)$, and $(P, \ell)$ be metric spaces. Let $X \subset (M, d)$ and $Y \subset (N, \rho)$, and suppose $f : X \to N$ has $f(X) \subset Y$ and is continuous, and $g : Y \to P$ is continuous. Show that $g \circ f : X \to P$ is continuous as a map from $(X, d)$ to $(P, \ell)$. Read and understand, but do not use or mimic, the proof of this fact that Rudin gives on page 86.
9. i. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and $B \in \mathbb{R}$ be so that for all $k \in \mathbb{N}$, $\|x_k\| \le B$. Define a subsequence $(x_{k_p})_{p\in\mathbb{N}}$ as follows. $k_0 = 0$. Let $C_0$ be the cube $C_0 = [-B, B] \times \cdots \times [-B, B] \subset \mathbb{R}^n$. There are $2^n$ subcubes of $C_0$ given by choosing one half of each factor interval, such as:

   $$[-B, 0] \times [-B, 0] \times \cdots \times [-B, 0], \quad [0, B] \times [-B, 0] \times \cdots \times [-B, 0],$$
   $$[-B, 0] \times [0, B] \times \cdots \times [-B, 0], \quad [0, B] \times [0, B] \times \cdots \times [0, B], \text{ etc.}$$

   Let $C_1$ be one of the subcubes for which there are infinitely many $k$ with $x_k \in C_1$. (You must justify why there are any such subcubes.) Let $k_1$ be the first index $k$ for which $x_k \in C_1$. Repeat this process: $C_p$ is one of the $2^n$ subcubes of $C_{p-1}$ for which there are infinitely many $k$ with $x_k \in C_p$; $k_p$ is the first index greater than $k_{p-1}$ with $x_k \in C_p$. Show that $(x_{k_p})_{p\in\mathbb{N}}$ is Cauchy.

   ii. Conclude that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
10. If a set $A \subset \mathbb{R}^n$ is closed and bounded, any sequence $(x_k)_{k\in\mathbb{N}}$ of points of $A$ has a convergent subsequence whose limit is in $A$.