MATH 360 Homework 7

Due 15 March 2013

1. Suppose $(V, \|\cdot\|)$ is a normed space. Show that $\|\cdot\| : V \to \mathbb{R}$ is continuous with respect to the metric induced by $\|\cdot\|$ and the standard metric on $\mathbb{R}$. (*Hint.* Use the reverse triangle inequality.)
2. Suppose (M_1, d_1) and (M_2, d_2) are two metric spaces. On the product set $M_1 \times M_2 = \{(x, y) | x \in M_1, y \in M_2\}$, define the *product metric* $d_1 \times d_2$ by

$$(d_1 \times d_2)\left((x, y), (z, w)\right) = \sqrt{d_1(x, z)^2 + d_2(y, w)^2}$$

   i. Show that $d_1 \times d_2$ is a metric on $M_1 \times M_2$.

   ii. Show that a sequence $\left((x_k, y_k)\right)_{k \in \mathbb{N}}$ in $M_1 \times M_2$ converges to (x, y) iff $x_k \to x$ and $y_k \to y$.
3. If (M, d) is any metric space, show that $d : M \times M \to \mathbb{R}$ is continuous with respect to the product metric $d \times d$ and the standard metric on $\mathbb{R}$.
4. Given a metric space (M, d), fix $y \in M$, and define $D_y : M \to \mathbb{R}$ by $D_y(x) = d(x, y)$.

   i. Show that D_y is continuous.

   ii. Suppose A is a compact subset of M, and $y \notin A$. Show that $d(y, A) > 0$.
5. Let $A \subset M_1$ and $B \subset M_2$ be nonempty subsets. Show that $A \times B = \{(a, b) | a \in A, b \in B\}$ is compact in $(M_1 \times M_2, d_1 \times d_2)$ if and only if both $A \subset M_1$ and $B \subset M_2$ are compact. (*Hint.* Use Bolzano-Weierstrass.)
6. For any two subsets $A, B \subset (M, d)$, define $d(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$.

   i. Use (4) and (5) to show that if A and B are compact disjoint subsets of (M, d), then $d(A, B) > 0$.

   ii. Give an example of two closed, disjoint sets A, B in $\mathbb{R}^2$ which have $d(A, B) = 0$. You need not prove the sets are closed, but you must prove that they are disjoint and that $d(A, B) = 0$. (*Hint.* There is a bit of terminology from calculus (starting with an "a") which describes two subsets of $\mathbb{R}^2$ which get arbitrarily close to each other.)
7. If $(V, \|\cdot\|)$ is a normed space, we can consider the vector space operations as maps

$$\begin{aligned} \text{plus} &: V \times V \to V \\ \text{plus}(v_1, v_2) &= v_1 + v_2 \\ \text{times} &: \mathbb{R} \times V \to V \\ \text{times}(\lambda, v) &= \lambda v \end{aligned}$$

   Write d_{std} for the standard metric on $\mathbb{R}$ and $d_{\|\cdot\|}$ for the metric on V induced by $\|\cdot\|$.

   i. Show that $\text{times} : \mathbb{R} \times V \to V$ is continuous with respect to $d_{\text{std}} \times d_{\|\cdot\|}$ and $d_{\|\cdot\|}$.

   ii. Show that $\text{plus} : V \times V \to V$ is continuous with respect to $d_{\|\cdot\|} \times d_{\|\cdot\|}$ and $d_{\|\cdot\|}$.
8. Two norms $\|\cdot\|$ and $\{\!\{\cdot\}\!\}$ on the same vector space V are called *equivalent* if there are positive numbers C_1, C_2 so that for all $v \in V$, $C_1\|v\| \le \{\!\{v\}\!\} \le C_2\|v\|$.

   i. For any particular $v \in V$, show that there are some positive c_1, c_2 with $c_1\|v\| \le \{\!\{v\}\!\} \le c_2\|v\|$.

   ii. Given v, for c_1, c_2 as in part i, show that for any $\lambda \in \mathbb{R}$, we have $c_1\|\lambda v\| \le \{\!\{\lambda v\}\!\} \le c_2\|\lambda v\|$.

   iii. Let $\{\!\{\cdot\}\!\} : \mathbb{R}^n \to \mathbb{R}$ be any norm on $\mathbb{R}^n$ which is continuous with respect to the standard metrics on $\mathbb{R}^n$ and $\mathbb{R}$. Show that $\{\!\{\cdot\}\!\}$ is equivalent to the standard norm. (*Hint.* The sphere in the standard norm is compact and we have assumed that $\{\!\{\cdot\}\!\}$ is continuous.)

Bonus: Describe the interaction between compactness and order of quantifiers in parts i-iii.