MATH 360 Homework 8

Due 22 March 2013

1. i. Give a precise meaning to the statement, "Disconnections pull back." Is the statement true?

   ii. Show that the image of a connected set under a continuous map is connected.

2. Define $S^n = \left\{x \in \mathbb{R}^{n+1} \middle| \|x\| = 1\right\}$.

   i. Show that S^0 is disconnected.

   ii. Show that S^n for $n > 0$ is connected. (*Hint.* S^n is the image under a continuous map of a connected set in $\mathbb{R}^{n+1}$.)

3. (Marsden-Hoffman's 3.28) Suppose $A \subset (M, d)$ is connected, and A is not a singleton. Show that every point of A is a limit point for A.

4. i. What are the connected subsets of a discrete metric space?

   ii. What are the compact subsets of a discrete metric space?

5. (path components) Let $A \subset (M, d)$ be a subset of a metric space. For $x, y \in A$, write $x \sim y$ if there is a continuous path $\varphi : [0, 1] \to A$ with $\varphi(0) = x, \varphi(1) = y$.

   i. Show that $\sim$ is an equivalence relation, i.e. that

      a. ($\sim$ is reflexive) $x \sim x$ for all $x \in A$.

      b. ($\sim$ is symmetric) $x \sim y$ implies $y \sim x$.

      c. ($\sim$ is transitive) $x \sim y$ and $y \sim z$ implies $x \sim z$.

   ii. For each $x \in A$, let $[x] = \{y \in A | x \sim y\} \subset A$. Show that $[x]$ is path-connected. (*Hint.* This is more subtle than at first it seems.)

   iii. Show that $[x]$ is the maximal path-connected subset of A which contains x, i.e., that if $B \subset A$ is path-connected and has $x \in B$, then $B \subset [x]$.

6. Prove the product rule for derivatives of functions $f, g : [a, b] \to \mathbb{R}$. (*Hint.* Let $\phi_x(t)$ be the difference quotient for f and $\psi_x(t)$ be the difference quotient for g. Prove a product rule involving ϕ_x and ψ_x.)

7. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function so that $f'(x) = \frac{1}{x} f(x)$.

   i. Suppose $f(1) = 1$. Show that $f(x) = x$.

   ii. Suppose $f(1) = c$. What is $f(x)$?