MATH 360 Homework 9

Due 29 March 2013

1. Show that f is differentiable at x_0 iff $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. Moreover if this limit exists, it must be $f'(x_0)$.
2. Let $f : \mathbb{R} \to \mathbb{R}$.
   i. Suppose f is differentiable at $x_0 \in (a, b)$. Show that
   $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$$
   ii. Show that the limit in part i may exist even if f is not differentiable at x_0. (*Hint.* Consider a piecewise-linear function.)
   iii. (Rudin's problem 5.11) Use L'Hôpital's Rule to prove that if $f''(x_0)$ exists, then
   $$\lim_{h \to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} = f''(x_0)$$
   iv. Show that the limit in part iii may exist even if f is not twice-differentiable at x_0.
   v. Can the limit in part iii exist if f is not differentiable at x_0?
3. Let $0 < A < 1$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function so that for all $x \in \mathbb{R}$, $|g'(x)| \leq A$.
   i. Show that for all $x, y \in \mathbb{R}$, $|g(x) - g(y)| \leq A|x - y|$.
   ii. Pick some $x_0 \in \mathbb{R}$. For each $n \in \mathbb{N}$, define $x_{n+1} = g(x_n)$. Show that the sequence $(x_n)_{n \in \mathbb{N}}$ converges.
   iii. Let $\overline{x} = \lim x_n$. Show that $\overline{x}$ is a *fixed point* for g, i.e. that $g(\overline{x}) = \overline{x}$.
   iv. How many distinct fixed points could such a g have?
4. Let $g : [0, 1] \to [0, 1]$ be a continuous function. Show that g has a fixed point.
5. Prove the following:

   Inverse Function Theorem, ver. 2. *Let $g : [a, b] \to \mathbb{R}$ be differentiable on (a, b) and suppose that its derivative g' is continuous on (a, b). Suppose also that some $x \in (a, b)$ has $g'(x) > 0$. Then there are α, β with $a < \alpha < x < \beta < b$, such that g is invertible on (α, β).*