MATH 600 Notes

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These are my notes for MATH 600: Geometric Analysis and Topology, taught Fall 2012 at the University of Pennsylvania. The course texts were John Lee, *Introduction to Smooth Manifolds* and John Milnor, *Topology from the Differentiable Viewpoint*.

These are intended as lecturer's notes, and so are intended to be neither completely rigorous nor complete. The global structure is also not so clear, and there is no index. Nevertheless, it is my hope that these notes will at the very least not harm a student's understanding of this introduction to differential geometry.

If there are any mistakes, please let me know via email at `ancoop@math.upenn.edu`.

Known Errata

p. 60 The example should have $x(t) = (C - 2t)^{-\frac{1}{2}}$, so that the integral curve goes off to infinity as $t \to \frac{C}{2}$.

Smooth Manifolds

The idea of a manifold is to extend familiar properties of $\mathbb{R}^n$ to more general spaces.
The particular properties we choose become adjectives

smooth manifold
analytic manifold
Riemannian manifold

For geometric analysis, we want a manifold on which calculus makes sense, a smooth manifold.

Defn A topological space is a set X along with a collection of open subsets $\mathcal{O}$ with the following axioms:

- $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$
- Any union of open sets is open
- Any finite intersection of open sets is open.

Defn A map of topological spaces is continuous if the preimage of every open set is open.

E.g. $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$

For $i = 1, \ldots, n+1$, define $U_i^+ = \{x \in S^n \mid x^i > 0\}$
$U_i^- = \{x \in S^n \mid x^i < 0\}$

If $f: B^n \to \mathbb{R}$, $u \mapsto \sqrt{1 - |u|^2}$, then $U_i^{\pm} = \text{graph } \pm f(x^1, \ldots, \hat{x}^i, \ldots, x^{n+1})$

So $(U_i^{\pm}, \varphi_i^{\pm})$ give a manifold structure to S^n.

E.g. S^n. Let $N = (0, \ldots, 0, 1)$, $S = (0, \ldots, 0, -1)$

$U^+ = S^n \setminus \{S\}$, $U^- = S^n \setminus \{N\}$

Define $\varphi^+: U^+ \to \mathbb{R}^n$

$x \mapsto$ point in $\mathbb{R}^n \times \{0\}$ which the segment from S to x hits.

"stereographic projection"

A continuous map with a continuous inverse is a homeomorphism.

Def'n A topological manifold is a topological space which is:

1) Hausdorff — Given any $p, q \in X$ there are $U \ni p$, $V \ni q$ open with $U \cap V = \emptyset$. "points are separated"

2) second-countable — X has a countable base

3) locally Euclidean — For each $p \in X$, there is $U \ni p$ open, $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$. (U, φ) are a chart at p.

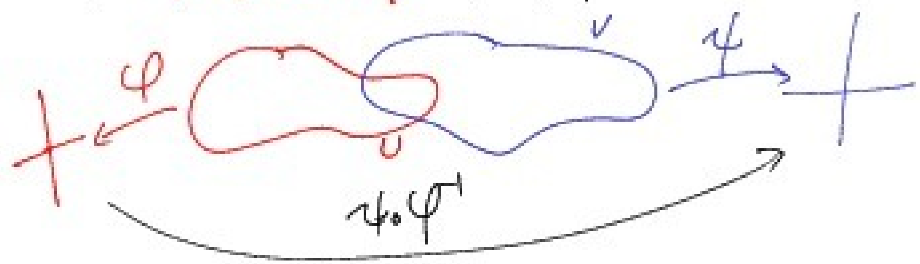
E.g. The surface of a cube in $\mathbb{R}^3$.

Non-eg. Cube with a spike.

Note If $\tilde{U} = \varphi(U)$ is contractible, we can homeomorph so that $\tilde{U} = \mathbb{B}$ and $\varphi(p) = \vec{0}$. Then we call (U, φ) a coordinate ball.

Note The coordinates of $\mathbb{R}^n$ give functions (coordinates) $x^1, \dots, x^n : U \to \mathbb{R}$.

Def'n Given charts $(U, \varphi), (V, \psi)$, if $U \cap V \neq \emptyset$, the transition map is $\psi \circ \varphi^{-1}$



Note the domain of $\psi \circ \varphi^{-1}$ is $\varphi(U \cap V)$, an open subset of $\mathbb{R}^n$, and that $\psi \circ \varphi^{-1}$ is a homeomorphism.

Def'n Two charts are C^k-compatible if their transition maps (both orders!) are C^k. ("C^k" makes sense because transition maps are in $\mathbb{R}^n$!)

Def'n A collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of charts which covers the manifold, and which are mutually C^k compatible, is called a C^k atlas for M.

E.g. For S^n with "graph coordinates" as above,

$\varphi_i^+ : (x^1, \dots, x^{n+1}) \mapsto (x^1, \dots, \hat{x}^i, \dots, x^{n+1})$

$(\varphi_i^+)^{-1} : (u^1, \dots, u^n) \mapsto (u^1, \dots, \underbrace{\sqrt{1-|u|^2}}_{i^{th} \text{ slot}}, \dots, u^n)$

So the components of $\varphi_j^+ \circ (\varphi_i^+)^{-1} : U \mapsto (\quad)$ are some u^k or $\sqrt{1-|u|^2}$.

Exercise Show that graph & stereographic coordinates are smoothly compatible.

Note This exercise implies that $\{(U_i^\pm, \varphi_i^\pm), (U^+, \sigma^\pm)\}$ is also a smooth atlas. We want to remove this ambiguity.

Defn An atlas is maximal if it contains every chart which is smoothly compatible with all of its charts.

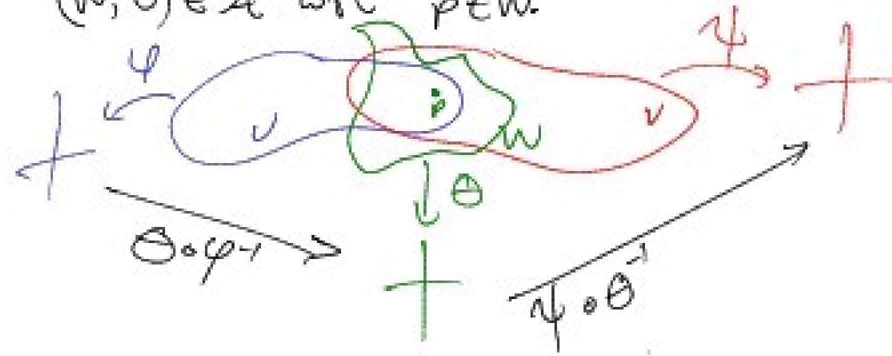
A smooth manifold is a pair $(M, \mathcal{A})$ of a topological manifold M and a maximal smooth atlas $\mathcal{A}$.

Lemma a) Every smooth atlas is contained in a unique maximal atlas.

b) Two atlases determine the same maximal atlas iff their union is also a smooth atlas.

Proof of b) is an exercise.

Proof of a) Given an atlas $\mathcal{A}$, let $\mathcal{A}^*$ be the set of all charts which are compatible with $\mathcal{A}$.
We'll show $\mathcal{A}^*$ is a smooth atlas.
Let $(U,\varphi), (V,\psi) \in \mathcal{A}^*$. If $U \cap V = \emptyset$ done.
If $p \in U \cap V$, then by defn of atlas there is $(W,\theta) \in \mathcal{A}$ with $p \in W$.



$$\psi\circ\varphi^{-1} = \psi\circ(\theta^{-1}\circ\theta)\circ\varphi^{-1} = \underbrace{(\psi\circ\theta^{-1})}_{\text{smooth}}\circ\underbrace{(\theta\circ\varphi^{-1})}_{\text{smooth}}$$

Now we claim $\mathcal{A}^*$ is maximal. Suppose (U,φ) is compatible with every chart of $\mathcal{A}^*$. Then since $\mathcal{A} \subset \mathcal{A}^*$, (U,φ) is compatible with $\mathcal{A}$. So $(U,\varphi) \in \mathcal{A}^*$.

To prove uniqueness let $\mathcal{B} \supseteq \mathcal{A}$ be a maximal atlas. Then $\mathcal{B} \subseteq \mathcal{A}^{+}$. But maximality of $\mathcal{B}$ means $\mathcal{A}^{+} \subseteq \mathcal{B}$. ■

E.g. $\mathbb{RP}^n = \{\text{lines in } \mathbb{R}^{n+1} \text{ containing } \vec{0} \in \mathbb{R}^{n+1}\}$

$$= \mathbb{R}^{n+1} \setminus \{0\} / \sim$$

where $x \sim \alpha x$ if $\alpha \in \mathbb{R} \setminus \{0\}$

Set $\tilde{U}_i = \{(x^1, \dots, x^{n+1}) \mid x^i \neq 0\}$. $\tilde{U}_i$ is open and saturated wrt $\sim$. So if $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, $x \longmapsto [x]$

then $U_i = \pi(\tilde{U}_i)$ is open.

Define $\tilde{\varphi}_i : \tilde{U}_i \to \mathbb{R}^n$

$$x \mapsto \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i} \right)$$

Note that $\tilde{\varphi}_i$ descends to $\varphi_i : U_i \to \mathbb{R}^n$

Check that $\varphi_i^{-1} : (u^1, \dots, u^n) \mapsto [(u^1, \dots, u^{i-1}, 1, u^{i+1}, \dots, u^n)]$

This example shows that manifolds need not come as subsets of some Euclidean space.

E.g. $(\mathbb{R}, \mathrm{id})$ is a chart ("global coordinates") which covers $\mathbb{R}$, so gives a smooth structure.

So does $(\mathbb{R}, \psi : x \mapsto x^3)$.

But what are the transition maps?

$$\psi \circ (\mathrm{id})^{-1} : x \mapsto x^3 \quad \text{smooth}$$

$$\mathrm{id} \circ \psi^{-1} : x \mapsto x^{1/3} \quad \text{not smooth!}$$

Thus $(\mathbb{R}, \mathrm{id})$ and $(\mathbb{R}, \psi)$ give rise to distinct smooth structures on $\mathbb{R}$!

Lemma Any open subset of a smooth manifold is a smooth manifold.

E.g. The general linear group $GL(n, \mathbb{R}) = \left\{ \begin{array}{l} n \times n \ \mathbb{R} \text{ matrices} \\ \text{which are} \\ \text{invertible} \end{array} \right\}$

$$GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\}) \subset \mathbb{R}^{n^2}$$

and since $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is cont's, this set is open.

Instead of constructing manifolds "top-down" by checking enough charts, we can also build them bottom-up.

Smooth Manifold Construction Lemma

A set M, together with a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ and injective maps $\varphi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^n$ such that:

1) $\varphi_\alpha(U_\alpha)$ is open
2) $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open for any $\alpha,\beta\in\Lambda$
3) $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha\cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth with a smooth inverse.
4) M is covered by finitely many U_α.
5) For any $p,q\in M$, either $\begin{cases} p,q\in U_\alpha \text{ or} \\ p\in U_\alpha,\ q\in U_\beta \text{ with } U_\alpha\cap U_\beta=\emptyset \end{cases}$

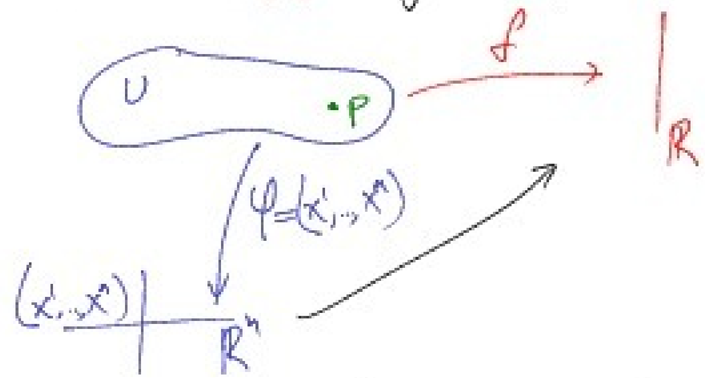
determines a smooth manifold on M.

Proof: The only thing to do is topologise M.

Smooth Maps

We want to do calculus.

Def'n: Given a map $f : M^n \to \mathbb{R}$, $p\in M$, and (U,φ) a chart at p, the coordinate representation of f in the chart (U,φ) is just $f\circ\varphi^{-1}$.



The coordinate representation is a map between Euclidean spaces, so we can define:

$$\left.\frac{\partial f}{\partial x^i}\right|_p = D_i(f\circ\varphi^{-1})_{\varphi(p)}$$

if such a derivative exists.

If every $p\in M$ lies in some chart so that the $\frac{\partial f}{\partial x^i}$ exist and are continuous, we say $f\in C^1(M;\mathbb{R})$

Lemma If $f \in C^1(M;\mathbb{R})$, then $\frac{\partial f}{\partial x^i}$ exists and is continuous for *any* coordinates $(x^1,\dots,x^n)$.

Proof Given $p \in M$, by defn of C^1, there is a coordinate system $(V, \psi = (y^1,\dots,y^n))$ with $\frac{\partial f}{\partial y^i}$ defined and cont's.

Let $(U, \varphi = (x^1,\dots,x^n))$ be some other coordinates at p.

Then
$$\frac{\partial f}{\partial x^i}\Big|_p = D_i\left(f\circ\varphi^{-1}\right)_{\varphi(p)}$$
$$= D_i\left(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1}\right)_{\varphi(p)}$$
exists by the chain rule. ∎

We'll need the formula:
$$\frac{\partial f}{\partial x^i}\Big|_p = D_i\left(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1}\right)_{\varphi(p)}$$
$$= \left(D(f\circ\psi^{-1})_{\psi(p)} \cdot D(\psi\circ\varphi^{-1})_{\varphi(p)}\right)_i$$
$$= \left(\frac{\partial f}{\partial y}\Big|_p \cdot \frac{\partial y}{\partial x}\Big|_p\right)_i$$
$$= \sum_{j=1}^n \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}$$

In other words, the notation $\frac{\partial f}{\partial x^i}$ is defined so that the chain rule works!

Aside: The Einstein Convention

Σs are tedious and ugly. We'll often have sums like the one in the chain rule, usually coming from matrix multiplication.

Note that
$$\frac{\partial f}{\partial x^i}\Big|_p = \sum_{j=1}^n \frac{\partial f}{\partial y^j}\Big|_p \frac{\partial y^j}{\partial x^i}\Big|_p$$
$$= \sum_{l=1}^n \frac{\partial f}{\partial y^l}\Big|_p \frac{\partial y^l}{\partial x^i}\Big|_p$$
ie the index j is 'dummy'.

We'll just omit the Σ, and write
$$\frac{\partial f}{\partial x^i}\Big|_p = \frac{\partial f}{\partial y^k}\Big|_p \frac{\partial y^k}{\partial x^i}\Big|_p$$
remembering to sum over any index which occurs twice, once above and once below.

Back to Smooth Maps

The definition of $C^k(M;\mathbb{R}^p)$ should now be clear.

What about maps between manifolds?

E.g. $S^n \longrightarrow S^n$

$x \mapsto -x$

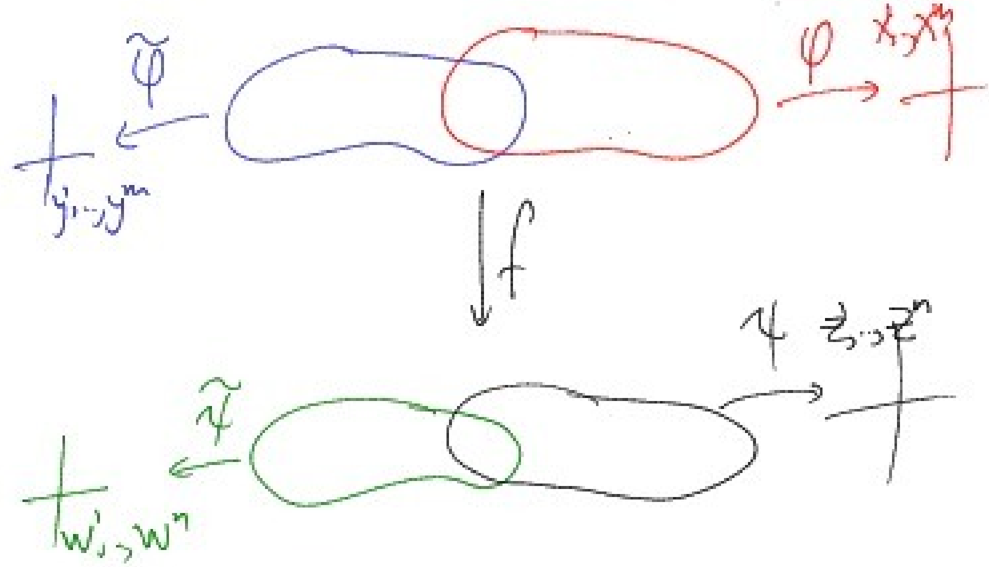
Def'n Given a map $f: M^m \longrightarrow N^n$, $p \in M$, (U,φ) a chart at p, (V,ψ) a chart at $f(p)$, the coordinate representation of f is $\psi\circ f\circ \varphi^{-1}$.

We say f is smooth at p if there is a chart at p and a chart at $f(p)$ for which the coordinate representation is smooth.

If $\overset{\varphi}{\overset{\shortparallel}{(x^1,\dots,x^m)}}$ are coords at p and $\overset{\psi}{\overset{\shortparallel}{(z^1,\dots,z^n)}}$ are coords at $f(p)$, we can write

$$\frac{\partial f^i}{\partial x^j} = D_j\left(z^i\circ f\circ \varphi^{-1}\right)$$

How do these transform if we pick other coordinates?



$$\frac{\partial \tilde{f}^k}{\partial y^i} = D_i\left(w^k\circ f\circ \tilde{\varphi}^{-1}\right)$$

$$= D_i\left(w^k\circ \psi^{-1}\circ \psi\circ f\circ \varphi^{-1}\circ \varphi\circ \tilde{\varphi}^{-1}\right)$$

$$= \frac{\partial w^k}{\partial z^p}\,\frac{\partial f^p}{\partial x^j}\,\frac{\partial x^j}{\partial y^i}$$

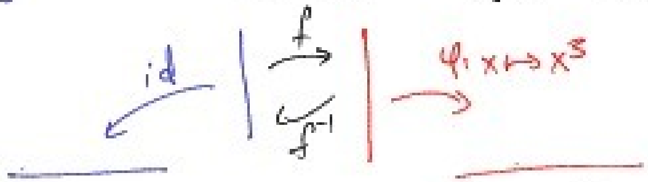
Def'n A smooth map with a smooth inverse is a **diffeomorphism**.

Exercise Show that $f: X \to Y$ is a diffeomorphism iff it is a homeomorphism and

(V, ψ) is a chart on Y iff

$(f^{-1}(V), \psi \circ f)$ is a chart on X.

E.g. Let $M = (\mathbb{R}, id)$, $N = (\mathbb{R}, x \mapsto x^3)$, $f: x \mapsto x^{1/3}$.



Coordinate representation of f:

$\varphi \circ f \circ id^{-1}: x \mapsto x \mapsto x^{1/3} \mapsto (x^{1/3})^3 = x$ smooth!

Coordinate representation of f^{-1}:

$id \circ f^{-1} \circ \varphi^{-1}: x \mapsto x^{1/3} \mapsto (x^{1/3})^3 \mapsto x$ smooth!

So $(\mathbb{R}, id)$ and $(\mathbb{R}, x \mapsto x^3)$ are **diffeomorphic**.

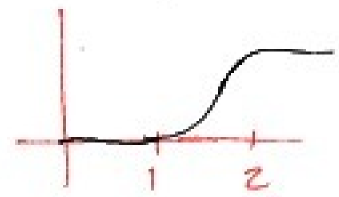
Diffeomorphism is sometimes called change of coordinates.

The C^∞ category and Partitions of Unity:

The collection of C^∞ manifolds and smooth maps form a **category**. It's a floppy category.

Lemma: There is $\phi: \mathbb{R}_+ \to \mathbb{R}$ such that

1) $\phi(x) = 0$ for $x \in [0,1]$
2) $\phi(x) = 1$ for $x \in [2, \infty)$
3) $0 \le \phi(x) \le 1$ for $x \in (1,2)$
4) $\phi \in C^\infty$



Corollary Let $V \subset U \subset M$ be compact and open, resp. There is $g \in C^\infty(M; [0,1])$ which is $\equiv 1$ on V and $\equiv 0$ outside U.

Proof: Given $p \in V$, pick a chart $(W, x^1, .., x^n)$ so that $B_2(0) \subset x(W)$.

Define $f_p: W \to \mathbb{R}$

$$q \mapsto 1 - \phi(|x(q)|)$$

Then f_p extends to 0 outside W.

The $x^{-1}(B_1(0))$ cover V, which is compact, so finitely many such $p_1, .., p_k$ suffice.

Then $f = f_{p_1} + \cdots + f_{p_K}$ is 0 outside U, ≥ 1 on V.
Now let $g(q) = \phi(1 + f(q))$. ∎

Such a g is called a bump function for the pair (V, U).

Defn An open cover $\{U_\alpha\}_{\alpha \in A}$ is locally finite if, for any $p \in X$, there is a neighborhood V of p so that $U_\alpha \cap V \neq \emptyset$ for only finitely many α.

Defn Given $\{U_\alpha\}_{\alpha \in A}$, $\{V_\beta\}_{\beta \in \Pi}$ open covers, we say $\{V_\beta\}$ refines $\{U_\alpha\}$ if for each V_β there is some U_α so that $V_\beta \subset U_\alpha$.

Defn A topological space is paracompact if every open cover has a locally finite refinement.

Theorem Every smooth manifold is paracompact.

Step 1 Construct a nice locally finite cover.

Lemma Every manifold has a countable basis of precompact coordinate balls.

Proof If M has a global chart $\varphi: M \xrightarrow{\sim} U \subset \mathbb{R}^n$, we take the cover $\{\varphi^{-1}(U \cap B_r(x)) \mid r, x \text{ rational}\}$.
If M has more charts, second-countability guarantees there is a countable atlas. Apply the above case to each chart. ∎

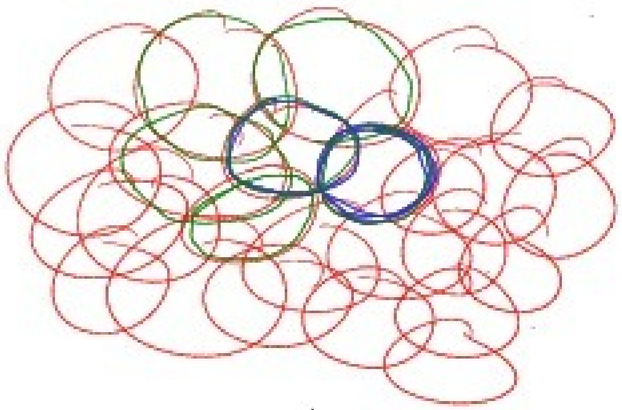
Call this covering $\{B_j\}_{j \in \mathbb{N}}$. We want to construct $\{U_j\}_{j \in \mathbb{N}}$ so that:

1) U_j is precompact
2) $\overline{U_{j-1}} \subset U_j$ (sometimes written $U_{j-1} \subset\subset U_j$)
3) $B_j \subset U_j$, so $\{B_j\}$ refines $\{U_j\}$

Set $U_1 = B_1$. Inductively, suppose $U_1, \ldots, U_K$ satisfy 1-3. $\overline{U_K}$ is compact, so $\overline{U_K} \subset B_1 \cup \cdots \cup B_{m_K}$

Set $U_{k+1} = B_1 \cup \cdots \cup B_{\max\{m_k, k+1\}}$.

({U_j} is not necessarily locally finite.)



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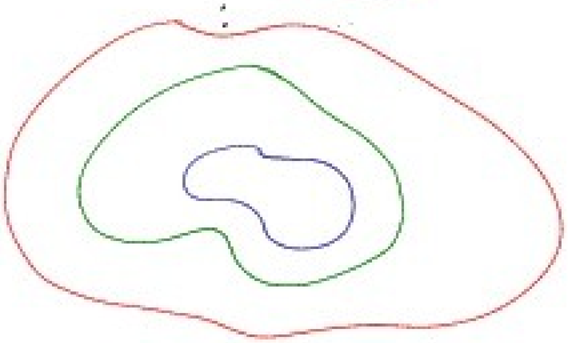


Set $V_j := U_j \setminus \overline{U_{j-2}}$. Then $\{V_j\}$ cover, and V_j intersects only V_{j+1} and V_{j-1}.

Exercise An open cover $\{V_\alpha\}_{\alpha\in A}$ so that each V_α intersects only finitely many V_β is locally finite.

So we've produced a countable, locally finite cover by precompact sets.

Step 2 The Theorem

Let $\mathcal{X}$ be any open cover, $\{V_j\}_{j\in\mathbb{N}}$ as above.

For each p, there is W_p which intersects only finitely many V_j. Set $\tilde{W}_p = W_p \cap \left(\bigcap_{\substack{j : V_j \\ \text{contains} \\ p}} V_j\right)$ open!

Then for all j, we have $\boxed{p\in V_j \Rightarrow \tilde{W}_p \subset V_j . \ (*)}$

OTOH, $p\in X \in \mathcal{X}$. Set $\tilde{\tilde{W}}_p = \tilde{W}_p \cap X$. Indeed, we can set $\tilde{\tilde{\tilde{W}}}_p = \tilde{\tilde{W}}_p \cap (\text{coord nbhd})$ and get a chart $\varphi_p : \tilde{\tilde{\tilde{W}}}_p \to B_3(0)$.

Set $U_p := \varphi_p^{-1}(B_1(0))$. By $(*)$, for each $k\in\mathbb{N}$, $\{U_p \mid p\in \overline{V}_k\}$ is an open cover of $\overline{V}_k$, hence there is a finite collection

$$\{U_k^1,\dots,U_k^{m_k}\}, \{(W_k^1,\varphi_k^1),\dots,(W_k^{m_k},\varphi_k^{m_k})\}$$

$\{W_k^i\}_{k\in\mathbb{N},\, i=1,\dots,m_k}$ is a cover by "radius-3 balls" for which the "radius-1 balls" still cover. "regular refinement" of $\mathcal{X}$.

Note This is the only real place we'll use second-countable. In fact we could have defined "manifold" to require paracompactness instead of second countability.

Def'n Given an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of a topological space, a partition of unity subordinate to $\{U_\alpha\}$ is a collection $\{\psi_\alpha : X \to \mathbb{R}\}_{\alpha\in\Lambda}$ of maps so that

1) $0 \leq \psi_\alpha \leq 1$
2) $\operatorname{supp} \psi_\alpha \subset U_\alpha$
3) $\{\operatorname{supp} \psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite.
4) $\sum_{\alpha\in\Lambda} \psi_\alpha = 1$

Note The sum in 4) is finite, so no worries about convergence.

Theorem Any open cover of a smooth manifold admits a subordinate smooth partition of unity.

Proof Let $\{W_k^i\}_{k\in\mathbb{N},\, i=1,\dots,m_k}$ be the regular refinement given by the previous theorem.

Define $f_k^i : W_k^i \to \mathbb{R}$ so that $f_k^i = 1$ on U_i, and $f_k^i = 0$ outside $(\varphi_k^i)^{-1}(B_2(0))$.

Then extend f_k^i to be $\equiv 0$ outside W_k^i.

We have
- $0 \leq f_k^i \leq 1$
- $\operatorname{supp} f_k^i \subset W_k^i$

Set $g_k^i = \dfrac{f_k^i}{\sum f_k^i}$.

Then
- $0 \leq g_k^i \leq 1$
- $\operatorname{supp} g_k^i \subset W_k^i$
- $\sum g_k^i \equiv 1$

Almost done, but not quite, since the $\{g_k^i\}$ aren't indexed by the cover $\{U_\alpha\}$.

For each pair (k,i), there is some α with

$W_K^i \subset U_\alpha$. (There could be more than one, just pick one and call it $\alpha(K,i)$.)

Set $\psi_\alpha = \sum\limits_{\substack{K,i \\ \text{such that} \\ \alpha(K,i)=\alpha}} g_K^i$. ∎

Corollary Any pair $\underset{\text{closed}}{A} \subset \underset{\text{open}}{U} \subset M$ admits a bump function.

Proof Take the open cover $\{U, M\setminus A\}$, get

ψ_1 with $\operatorname{supp}\psi_1 \subset U$

ψ_2 with $\operatorname{supp}\psi_2 \subset M\setminus A$,

$\psi_1+\psi_2=1$. $\psi_1|_A = 1-\psi_2|_A = 1$

$\psi_1|_{M\setminus U} = 0$ ∎

Manifolds-with-Boundary

Def'n A manifold-with-boundary is a Hausdorff, second countable space which is locally modeled on half-space, i.e. for each $p\in X$, there is an open set $U\ni p$ along with a homeomorphism $\varphi: U \xrightarrow{\sim} \tilde{U} \subseteq \{(x^1,\dots,x^n) \mid x^n \geq 0\}$

The boundary of X, ∂X, consists of all points which are preimages of a point with $x^n=0$ in some chart.

The interior of X is the complement of ∂X.

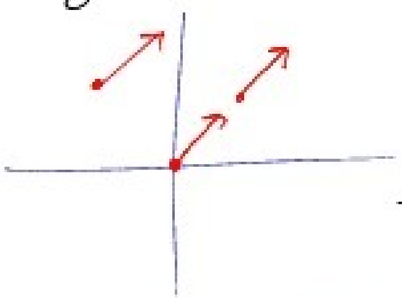
Def'n A smooth structure on a manifold-with-boundary is a maximal atlas of smoothly compatible half-space charts.

Basically everything that works for manifolds also works for manifolds-with-boundary.

The Tangent Space at a Point

One property of $\mathbb{R}^n$ we use all the time is its vector space structure. This does not immediately generalise to manifolds like S^n.

The problem is this: we usually think of a vector as being the same as its translates:



For $\mathbb{R}^n$, this isn't a problem, since we can always translate to $\vec{O} \in \mathbb{R}^n$.

Def'n A derivation at $\vec{a} \in \mathbb{R}^n$ is a map $X: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which satisfies Leibniz' rule

$$X(fg) = f(\vec{a})X(g) + g(\vec{a})X(f).$$

The space of derivations at $\vec{a}$ is $T_{\vec{a}}\mathbb{R}^n$, the tangent space at $\vec{a}$.

Proposition $T_{\vec{a}}\mathbb{R}^n$ is a vector space of dimension n.

Proof If $X, Y \in T_a\mathbb{R}^n$, then

$$\begin{aligned}(X+Y)(fg) &= X(fg) + Y(fg) \\ &= f(a)X(g) + g(a)X(f) + f(a)Y(g) + g(a)Y(f) \\ &= f(a)(X+Y)(g) + g(a)(X+Y)(f)\end{aligned}$$

$$\begin{aligned}(\alpha X)(fg) &= \alpha\big(f(a)X(g) + g(a)X(f)\big) \\ &= f(a)(\alpha X)(g) + g(a)(\alpha X)(f)\end{aligned}$$

Lemma For any derivation-at-$\vec{a}$ X

1) $X(\text{constant}) = 0$
2) If $f(a) = g(a) = 0$, $X(fg) = 0$.

Proof $X(1) = X(1 \cdot 1) = 1X(1) + 1X(1)$

$$= 2X(1)$$

I claim that $\{D_i|_a\}_{i=1,\dots,n}$ are a basis for $T_{\vec{a}}\mathbb{R}^n$.

Given $X \in T_a \mathbb{R}^n$, let $v^i = X(x^i)$. Taylor's theorem says each $f \in C^\infty(\mathbb{R})$ can be written

$$f(x) = f(a) + D_i\big|_a f\,(x^i - a^i) + \tfrac{1}{2} D^2_{ij}\big|_a f\,(x^i-a^i)(x^j-a^j) + \cdots$$

the remainder terms all vanish to high order at a, ie

$$f(x) = f(a) + D_i\big|_a f\,(x^i - a^i) + g_i(x)(x^i - a^i)$$

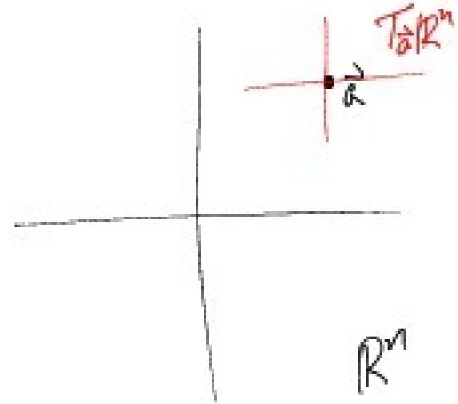
with $g_i(a) = 0$.

Then
$$X(f) = X(f(a)) + \left(D_i\big|_a f\right)\left(X(x^i)\right) + X\left(g_i(x)(x^i - a^i)\right)$$
$$= \left(D_i\big|_a f\right) v^i.$$

Thus $X = v^i D_i\big|_a$. (Why are they linearly independent?)

So we can identify
$$\mathbb{R}^n \xrightarrow{\sim} T_a \mathbb{R}^n,$$
$$v \longmapsto D_v\big|_a$$

or, alternatively, see $T_{\vec{a}}\mathbb{R}^n$ as a copy of $\mathbb{R}^n$, the vectors based at $\vec{a}$.



Very hard to draw...

Can we do this on a manifold? We have $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$.

Def'n A **smooth function element** on M^n is an open subset U, with a smooth $f: U \to \mathbb{R}$.

Given $p \in M$, we define an equivalence relation on the smooth function elements whose domains include p:

$$U \xrightarrow{f} \mathbb{R} \sim_p V \xrightarrow{g} \mathbb{R} \quad \text{exactly if}$$

there is $W \subset U \cap V$ containing p with
$$f\big|_W = g\big|_W.$$

We call the equivalence class $[U \xrightarrow{f} \mathbb{R}]$ the **germ of f at p**. The collection of germs is C^∞_p

Notice the equivalence relation $\sim_p$ means we can assume the domain of f lies in a coordinate chart.

Def'n The tangent space to M at p is the space of derivations of C_p^∞, i.e. $X: C_p^\infty \to \mathbb{R}$ so that $X[fg] = f(p)X[g] + g(p)X[f]$.

Proposition T_pM is a vector space of dimension n.

Proof If p is in a coordinate chart (U, x), we claim $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ are a basis for T_pM.

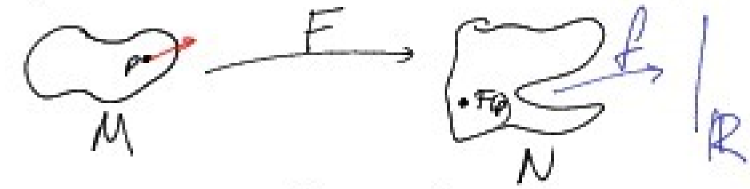
It's just the same as above! ∎

Recall the chain rule: if $\{x^i\}$ and $\{y^j\}$ are coordinates at p, $\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}$.

So if we want to change the basis of T_pM, our change-of-basis matrix is $\left(\frac{\partial x}{\partial y}\right)$.

(Re) Def'n An element $X \in T_pM$ will henceforth be called a tangent vector at p.

How do tangent spaces and smooth maps play?



Def'n Given $F: M \to N$ smooth, $X \in T_pM$, define $F_*X \in T_{F(p)}N$ by

$$(F_*X)(f) = X(f \circ F).$$

Lemma
- $F_*: T_pM \to T_{F(p)}N$ is a linear map.
- $(G \circ F)_* = G_* \circ F_*$. Just the chain rule
- $Id_* = Id$
- F diffeo $\Rightarrow$ F_* iso.

Work out the details of the proof.

$F_*: T_pM \to T_{F(p)}N$ is a linear map, so we can represent it as a matrix after we pick a basis for its domain and target.

Picking a basis for T_pM means choosing coords $(x^1, \dots, x^m)$ at p, and picking a basis for $T_{F(p)}N$ means choosing coords $(y^1, \dots, y^n)$ at $F(p)$.

Then the matrix entries for F_* are given by computing $F_* \frac{\partial}{\partial x^i}$ in terms of $\frac{\partial}{\partial y^j}$.

$$\begin{aligned}\left(F_* \frac{\partial}{\partial x^i}\right)(f) &= \frac{\partial}{\partial x^i}(f \circ F)\\ &= D_i (f \circ F \circ x^{-1})\\ &= D_i (f \circ y^{-1} \circ y \circ F \circ x^{-1})\\ &= \frac{\partial f}{\partial y^k} \cdot \frac{\partial F^k}{\partial x^i}\end{aligned}$$

That is, $\left(F_* \frac{\partial}{\partial x^i}\right) = \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial y^k}$. So $F_* = \left(\frac{\partial F^k}{\partial x^i}\right)$, and we can see the pushforward is the total derivative of F.

Defn An immersion is a smooth map whose pushforward is injective. We write $F: M \hookrightarrow N$.

A submersion is a smooth map whose pushforward is surjective.

Note: A linear map (like $F_*|_p$) being injective or surjective is a basis-independent condition. So these definitions don't depend on coordinates!

The coordinateful way to describe immersion is: $F: M^m \to N^n$ is an immersion at p if $F_*|_p$ can be represented in coordinates as a rank-m matrix.

This is one example of what differential geometry is really all about: using linear algebra on tangent spaces to understand properties of maps.

Def'n A **curve** in M is a map $\gamma: J \to M$, where J is an interval (containing one, both, or neither of its endpoints).

The **tangent vector** to a curve γ in M is
$$\gamma'_{\gamma(s)} = \gamma_*\left(\frac{d}{dt}\Big|_s\right) \quad \text{where } \frac{d}{dt}\Big|_s \in T_s J.$$

That is, $\gamma'_{\gamma(s)} \in T_{\gamma(s)}M$ is the derivation whose action on $f \in C^\infty(M)$ is given by evaluating f along the image of γ, then differentiating in the parameter.

Remarkable lemma Every $X \in T_pM$ is the tangent vector at $\gamma(s)$ to some $\gamma: J \to M$.

Proof If $(U, \varphi = x^1, \dots, x^n)$ is a coordinate at p, suppose $X = X^i \frac{\partial}{\partial x^i}\Big|_p$. Let $\gamma(t) = \varphi^{-1}(\varphi(p) + t(X^1, \dots, X^n))$.

Then
$$\gamma'_{\gamma(0)}(f) = \frac{d}{dt}\Big|_{t=0}(f \circ \gamma)$$
$$= \frac{d}{dt}\Big|_{t=0}\left(f \circ \varphi^{-1}(\varphi(p) + t(X^1, \dots, X^n))\right)$$
$$= \frac{\partial f}{\partial x^i}\Big|_p \cdot \frac{\partial}{\partial t}\Big|_{t=0}\left(\varphi(p) + t(X^1, \dots, X^n)\right)$$
$$= \frac{\partial f}{\partial x^i}\Big|_p X^i = X_p(f). \blacksquare$$

Actually it's possible to define T_pM as equivalence classes of curves passing through p. Then the pushforward F_* can be thought of as giving information about how F deforms curves. In fact we'll often use this observation to compute F_*.

The Tangent Bundle

We've seen that $F_*|_p$ carries the information about the derivatives of F at p. We want to collect this information into one object.

Def'n The *tangent bundle of M* is

$$TM = \bigsqcup_{p\in M} T_pM.$$

Proposition The tangent bundle of M^n can be equipped with the structure of a smooth $2n$-manifold.

Proof We use the smooth manifold construction lemma.

Let $\{(U_\alpha, x_\alpha)\}$ be an atlas for M.

Define $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) = \{(p,v) \mid v \in T_pM, p \in U_\alpha\}$.

Define $\tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$

$$(p,v) \mapsto (x^1,\dots,x^n,v^1,\dots,v^n)$$

Compute the transition map:

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \tilde{U}_\alpha \cap \tilde{U}_\beta \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

$$(x^1,\dots,x^n,v^1,\dots,v^n) \mapsto v^i \frac{\partial}{\partial x^i}\Big|_{\varphi_\alpha^{-1}(x)} \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \frac{\partial y}{\partial x} \cdot v\right)$$

(since $v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial y^\rho}{\partial x^i} \frac{\partial}{\partial y^\rho}$)

which is smooth since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth and $\frac{\partial y}{\partial x}\Big|_p$ is a linear map.

If $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ are a countable subcover for M, then $\{\tilde{U}_{\alpha_i}\}_{i\in\mathbb{N}}$ is a countable subcover for TM.

Given $(p,v), (q,w) \in TM$, if $p \neq q$, we can take U_α, U_β disjoint so that $p \in U_\alpha$, $q \in U_\beta$.

If $p = q$, then $(p,v), (q,w) \in \tilde{U}_\alpha$.

So the $\{(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)\}$ satisfy the requirements of the smooth manifold construction lemma. ∎

Proposition The projection $\pi : TM \to M$ is smooth.

Def'n We call $\tilde{\varphi}_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \times \mathbb{R}^n$ the *local trivialisation over U_α*.

We call $T_pM = \pi^{-1}(p)$ the *fibre over p*.

Notice that given a smooth $F: M \to N$, we get $F_*: TM \to TN$ whose representation in the above coordinates involves F and its first derivatives, hence is smooth.

That is to say, there is a functor T from the category of smooth manifolds to itself.

$$M \xrightarrow{T} TM$$

$$F: M \to N \xrightarrow{T} F_*: TM \to TN$$

Exercise: If M, N are C^k manifolds and $F: M \to N$ is a C^k map, what category does $F_*: TM \to TN$ belong to?

Note that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "F_*"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

Defn A smooth assignment, to each $p \in M$, of an element $X_p \in T_pM$, is called a vector field on M.

That is, a vector field is a smooth map $X: M \to TM$ so that the diagram commutes:

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ M \arrow[ur, "X"] \arrow[r, "id"'] & M \end{tikzcd} \qquad \pi \circ X = id$$

The space of smooth vector fields on M I'll denote $\mathfrak{X}(M)$.

With pointwise addition and scalar multiplication it's a vector space over $\mathbb{R}$, and a module over $C^\infty(M)$.

given a vector field $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have $fX: g \mapsto \{p \mapsto f(p)\, X_p(g)\}$ and this multiplication respects the strs of $\mathfrak{X}$ and C^∞.

If $X \in \mathfrak{X}(M)$, can we push it forward under $F: M \to N$? Only if F is onto. (What would $(F_*X)_q$ look like if $q \notin F(M)$?)

Only if F is one-to-one (What if $F(p)=F(q)$?)

Lemma Any $X \in T_pM$ is $\tilde{X}_p$ for some $\tilde{X} \in \mathcal{T}M$.

Proof Let (U,x) be a chart at p, and ψ a bump function which is $\equiv 1$ near p and 0 outside U.

We know $X = (x^{-1})_*(X^1, \dots, X^n)$.

For $q \in U$, set $\tilde{X}_q = \psi(q)(x^{-1})_*(X^1, \dots, X^n)$

For $q \notin U$, set $\tilde{X}_q = 0$.

Then $\tilde{X}$ is a smooth vector field with $\tilde{X}_p = X$. ■

Henceforth we'll extend vectors to vector fields as needed, sometimes without mentioning we've done so.

Defn A linear map $\bar{X}: C^\infty(M) \to C^\infty(M)$ is a derivation if $\bar{X}(fg) = f\bar{X}g + g\bar{X}f$.

We can make a derivation of C^∞ from a vector field X.

Given $g \in C^\infty(M)$, for each $p \in M$, define

$$\bar{X}(g)(p) = X_p[g] \quad \text{where } [g] = \text{germ of } g \text{ at } p$$

Check $\bar{X}(g) \in C^\infty(M)$: in coordinates (U,x),

$$\bar{X}(g) \circ x^{-1} : x(U) \to \mathbb{R}$$
$$x \longmapsto X^i \frac{\partial g}{\partial x^i}$$

(ask Lee why X^i are smooth functions)

Lemma Every derivation of $C^\infty(M)$ arises this way.

Proof Given $\bar{X}$ a derivation of $C^\infty(M)$, we want to produce a vector field. If $p \in M$, let ψ be a bump function which is $\equiv 1$ near p. Then if $[g] \in C^\infty_p$, $\psi g \in C^\infty(M)$. Set

$$X_p([g]) = \bar{X}(\psi g).$$

As an exercise, show this X is a smooth vector field. ■

We will often elide the distinction between a vector field and its associated derivation.

The Lie Bracket

The algebra of T_pM isn't that interesting — but we can try and detect algebraic structures in TM. Given $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, $Y(f) \in C^\infty(M)$.

So $X(Y(f))$ makes sense.

$$\begin{aligned} XY(fg) &= X(Y(fg)) = X(fYg + gYf) \\ &= fXYg + X(f)Y(g) + X(g)Y(f) + gXYf \\ &= fXYg + gXYf + \underbrace{XfYg + XgYf}_{\text{not a derivation!}} \end{aligned}$$

No Leibniz rule for second derivatives.

Defn Given $X, Y \in \mathfrak{X}(M)$, their Lie bracket is

$$[X,Y] : f \longmapsto XY(f) - YX(f)$$

Lemma $[X,Y] \in \mathfrak{X}(M)$.

Proof Given $f, g \in C^\infty(M)$,

$$\begin{aligned} [X,Y](fg) &= XY(fg) - YX(fg) \\ &= fXY(g) + gXY(f) + \cancel{XfYg} + \cancel{XgYf} \\ &\quad - (fYXg + gYXf + \cancel{YfXg} + \cancel{YgXf}) \\ &= f[X,Y]g + g[X,Y]f \end{aligned}$$

So $[X,Y]$ is a derivation. But we should check that $[X,Y]f$ is smooth.

In coordinates, $X = X^i \frac{\partial}{\partial x^i}$ $\quad Y = Y^i \frac{\partial}{\partial x^i}$

$$\begin{aligned} [X,Y]f &= X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial}{\partial x^j} f\right) - Y^i \frac{\partial}{\partial x^i}\left(X^j \frac{\partial}{\partial x^j} f\right) \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + \cancel{X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}} \\ &\quad - Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial f}{\partial x^j} - \cancel{Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j}} \end{aligned}$$

smooth since f, X, Y are all smooth.

Also note: $^{(*)}$ $[X,Y] = X^i \frac{\partial}{\partial x^i} Y^j \frac{\partial}{\partial x^j} - Y^i \frac{\partial}{\partial x^i} X^j \frac{\partial}{\partial x^j}$ has three implications:

- If $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, then $[X,Y]=0$.
- $[X,Y]_p$ depends on the derivatives of components of X and Y, hence on the values of X and Y in a nbhd, not just at p.
- We can use $(*)$ to prove $[X,Y]$ is a vector field: Let $\{y^1,\dots,y^n\}$, $\{x^1,\dots,x^n\}$ be two coordinate systems, with $X = X^i \frac{\partial}{\partial x^i}$ $Y = Y^i \frac{\partial}{\partial x^i}$
$X = \tilde{X}^i \frac{\partial}{\partial y^i}$ $Y = \tilde{Y}^i \frac{\partial}{\partial y^i}$

$$X^i \frac{\partial}{\partial x^i}(Y^j)\frac{\partial}{\partial x^j} - Y^i \frac{\partial}{\partial x^i}(X^j)\frac{\partial}{\partial x^j}$$

$$= X^i \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}(Y^j)\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell} - Y^i \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}(X^j)\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}$$

$$= \tilde{X}^k \frac{\partial}{\partial y^k}\left(\tilde{Y}^p \frac{\partial x^j}{\partial y^p}\right)\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell} - \tilde{Y}^k \frac{\partial}{\partial y^k}\left(\tilde{X}^p \frac{\partial x^j}{\partial y^p}\right)\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}$$

$$= \tilde{X}^k \frac{\partial}{\partial y^k}(\tilde{Y}^p)\frac{\partial x^j}{\partial y^p}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell} + \cancel{\tilde{X}^k\tilde{Y}^p \frac{\partial^2 x^j}{\partial y^k \partial y^p}}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}$$

$$-\left[\tilde{Y}^k \frac{\partial}{\partial y^k}(\tilde{X}^p)\frac{\partial x^j}{\partial y^p}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell} + \cancel{\tilde{Y}^k\tilde{X}^p \frac{\partial^2 x^j}{\partial y^k \partial y^p}}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\right]$$

$$= \tilde{X}^k \frac{\partial}{\partial y^k}(\tilde{Y}^p)\,\delta_p^\ell \frac{\partial}{\partial y^\ell} - \tilde{Y}^k \frac{\partial}{\partial y^k}(\tilde{X}^p)\,\delta_p^\ell \frac{\partial}{\partial y^\ell}$$

$$= \tilde{X}^k \frac{\partial}{\partial y^k}(\tilde{Y}^\ell)\frac{\partial}{\partial y^\ell} - \tilde{Y}^k \frac{\partial}{\partial y^k}(\tilde{X}^\ell)\frac{\partial}{\partial y^\ell}$$

same expression as in the other coords!

Proposition: If $X,Y,Z \in \mathfrak{X}(M)$, $a,b\in\mathbb{R}$, $f,g\in C^\infty(M)$,

1) $[X, aY+bZ] = a[X,Y] + b[X,Z]$
2) $[X,Y] = -[Y,X]$
3) Jacobi identity $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$
4) $[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X$

Defn A vector space with a bracket satisfying 1-3 is a Lie algebra.

Note that the Lie algebra structure lives on $\mathfrak{X}(M)$, not on T_pM!

Vector Bundles:

The tangent bundle has some nice properties we'd like to extend.

Defn A vector bundle of rank k is a tuple $(E, \pi, M, +, \cdot)$ where:

- E and M are smooth manifolds
- $\pi : E \to M$ is smooth and surjective
- For each $p \in M$, $+ : \pi^{-1}(p) \times \pi^{-1}(p) \to \pi^{-1}(p)$ and $\cdot : \mathbb{R} \times \pi^{-1}(p) \to \pi^{-1}(p)$ make $\pi^{-1}(p)$ a $\mathbb{R}$-vector space of dimension k.
- For each $p \in M$, there is $U \ni p$ open and diffeo $\Phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^k$ so that $\Phi : \pi^{-1}(q) \xrightarrow{\sim} \{q\} \times \mathbb{R}^k$ is a vector space isomorphism

and

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

E is the total space, π is projection, the Φ are local trivialisations, $\pi^{-1}(p) = E_p$ is the fibre over p. M is the base space.

E.g. $M \times \mathbb{R}^k \to M$ is the trivial bundle, since we can choose $U = M$.

E.g. The Möbius bundle:

$$\begin{tikzcd} {[0,1] \times \mathbb{R}/\sim} \arrow[d] \\ {[0,1]/\sim} = S^1 \end{tikzcd} \qquad (0,t) \sim (1,-t)$$

Notice that if U, V are two neighborhoods over which $E \xrightarrow{\pi} M$ is locally trivial, we have:

$$\begin{tikzcd} (U \cap V) \times \mathbb{R}^k \arrow[dr] & \pi^{-1}(U \cap V) \arrow[l, "\Psi"', "\sim"] \arrow[r, "\Phi", "\sim"'] \arrow[d] & (U \cap V) \times \mathbb{R}^k \arrow[dl] \\ & U \cap V & \end{tikzcd}$$

and, for each $q \in U \cap V$, $\{q\} \times \mathbb{R}^k \xrightarrow[\sim]{\Phi \circ \Psi^{-1}} \{q\} \times \mathbb{R}^k$

so we have an assignment, to each $q \in U \cap V$, of a vector space isomorphism. That is,

$$\tau_{U,V} : U \cap V \longrightarrow GL(k)$$

Moreover, since Ψ and Φ are smooth, so is τ.

If U, V, W are three such, we have the diagram

$$\begin{tikzcd}
(U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{R}^k \arrow[rr, bend left, "\tau_{\alpha\beta}"] \arrow[dr, bend right, "\tau_{\alpha\gamma}"'] & \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\chi"] & (U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{R}^k \arrow[dl, bend left, "\tau_{\beta\gamma}"] \\
& (U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{R}^k &
\end{tikzcd}$$

where
$$\Psi : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$
$$\Phi : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$$
$$\chi : \pi^{-1}(U_\gamma) \to U_\gamma \times \mathbb{R}^k$$

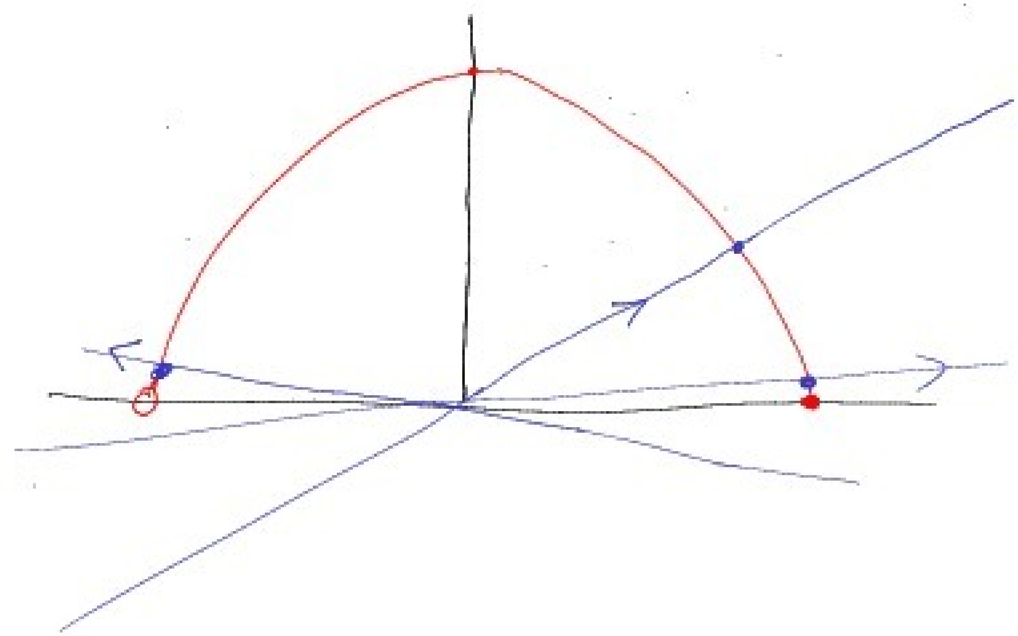
Lemma (Cocycle condition) For $q \in U_\alpha \cap U_\beta \cap U_\gamma$
$$\tau_{\alpha\beta}(q)\,\tau_{\beta\gamma}(q) = \tau_{\alpha\gamma}(q)$$

Eg: Recall $\mathbb{RP}^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$

$$\begin{tikzcd}
\Gamma_n = \{([\ell], \ell)\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1} \arrow[dr, "\pi"'] \arrow[d, "\pi"] & \\
\mathbb{RP}^n &
\end{tikzcd}$$

"tautological line bundle"

Γ_1 = Mobius bundle!



E.g. TM for any M.
The transition functions are $\left(\frac{\partial x_\alpha}{\partial x_\beta}\right)$!

Defn A section of $E \xrightarrow{\pi} M$ is a smooth map $\sigma : M \to E$ so that

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ M \arrow[ur, "\sigma"] \arrow[r, "id"] & M \end{tikzcd}$$

. The space of sections is $\Gamma(E \xrightarrow{\pi} M)$

E.g. Let $\sigma_0(p) = 0 \in E_p$ (the "zero section").
Then σ_0 is smooth (Exercise!)

In fact, $\sigma_0 : M \to E$ is an immersion which is one-to-one (an "embedding").

$\Gamma(E)$ is a vector space over $\mathbb{R}$ and a module over $C^\infty(M)$.

E.g. $C^\infty(M) = \Gamma(M \times \mathbb{R} \to M)$

Defn A bundle map $(F, f) : E_1 \xrightarrow{\pi_1} M_1 \longrightarrow E_2 \xrightarrow{\pi_2} M_2$ is a pair of smooth maps so that

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

and $F_p : (E_1)_p \to (E_2)_p$ is a linear map.

Note that if we specify F, f is determined. So we usually just say F.

A bundle isomorphism is an invertible bundle map whose inverse is also a bundle map.

So we can describe the tangent bundle construction as a covariant functor from the category of smooth n-manifolds to the category of smooth rank-n vector bundles over smooth n-manifolds.

We'll be interested in relationships between different bundles over the same base manifold

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M & \xrightarrow{id} & M \end{array}$$

Given $\sigma \in \Gamma(E_1)$, $F \circ \sigma \in \Gamma(E_2)$. Moreover, if $f \in C^\infty$, $f\sigma \in \Gamma(E_1)$, and $F \circ (f\sigma) = f(F \circ \sigma)$. We say F is linear over $C^\infty(M)$.

Lemma Suppose $\tilde{F} : \Gamma(E_1) \to \Gamma(E_2)$ is linear over C^∞, i.e. $\tilde{F}(f\sigma) = f\tilde{F}(\sigma)$ for all $\sigma \in \Gamma(E_1)$, $f \in C^\infty(M)$. Then $\tilde{F}$ is induced by a bundle map over id_M.

That is, we can specify a bundle map over id_M by specifying its action on sections.

The Cotangent Bundle

Defn Given a vector space V, the dual space V^* is $V^* = Hom(V, \mathbb{R}) = \{\text{linear } T : V \to \mathbb{R}\}$

Proposition If V has dimension n, then V^* has dimension n.

Proof Suppose $\{e_1, \dots, e_n\}$ are a basis for V. Define $e^i \in V^*$ by:

$$e^i(a^1 e_1 + \dots + a^n e_n) = a^i$$

Given $T \in V^*$, let $T_i = T(e_i)$. Claim $T = T_i e^i$.

$$T(a^1 e_1 + \dots + a^n e_n) = a^1 T_1 + \dots + a^n T_n$$

$$(T_i e^i)(a^1 e_1 + \dots + a^n e_n) = a^1 T_i e^i(e_1) + \dots + a^n T_i e^i(e_n)$$
$$= a^1 T_1 + \dots + a^n T_n. \blacksquare$$

We call $\{e^i\}_{i=1,\dots,n}$ the dual basis to $\{e_i\}_{i=1,\dots,n}$.

All vector spaces of dimension n are abstractly isomorphic. But the identification $V \to V^*$ is not unique: $\{2e^1, \dots 2e^n\}$ is also a basis for V^*.

Defn Given $A: V \to W$ a linear map, we get
$A^*: W^* \to V^*$ defined by
$(A^*\omega)(v) = \omega(Av)$ for $\omega \in W^*, v \in V$.

Note A^* is a linear map.

Since taking duals reverses arrows we say "dual" is a contravariant operation.

Proposition $(V^*)^* \cong V$ canonically.

Proof For each $v \in V$, define an element of $(V^*)^*$ by
$ev_v: \omega \mapsto \omega(v)$

Note $ev_{(av+bw)}(\omega) = \omega(av+bw) = a\omega(v) + b\omega(w)$
$= a\, ev_v(\omega) + b\, ev_w(\omega)$

So $ev: V \to (V^*)^*$ is linear.

It's also injective: Suppose $ev_v(\omega) = 0$ for all $\omega \in V^*$.
Then in particular, $\theta^1(v) = \cdots = \theta^n(v) = 0$, where $\{\theta^i\}$ are a dual basis to some $\{e_1, \ldots, e_n\}$.
Then $v = v^i e_i$, with $v^i = \theta^i(v) = 0$. So $v = 0$. ∎

We often append "co" to indicate duality, e.g. if we call elements of V "vectors", we call elements of V^* "covectors".
We can remember the above proposition as "coco = $\varnothing$".

Alfréd Rényi: A mathematician is a device for turning coffee into theorems.

Cooper: A comathematician is a device for turning cotheorems into ffee.

Defn The cotangent bundle is the bundle whose fibres are $(T_pM)^* = T_p^*M$. That is
$$T^*M = \bigsqcup_{p\in M} T_p^*M.$$
If $\{(U_\alpha, x_\alpha)\}$ are a coordinate atlas, we form the local trivialisations by
$$\begin{aligned} \tilde{U}_\alpha = \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times (\mathbb{R}^n)^* \\ (p, T) &\longmapsto (x_\alpha(p), T_1, \dots, T_n) \end{aligned}$$

Since $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ is a basis for T_pM, there is a dual basis $\{\theta_p^1, \dots, \theta_p^n\}$ for T_p^*M.

We want a more geometric interpretation of the cobasis $\{\theta_p^1, \dots, \theta_p^n\}$.

Defn Given $f \in C^\infty(M)$, define $(df)_p \in T_p^*M$ by
$$(df)_p(X_p) = X_p(f)$$

Lemma $\{dx^1|_p, \dots, dx^n|_p\}$ are a cobasis for $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$

Proof $dx^i|_p\left(\frac{\partial}{\partial x^j}\big|_p\right) = \frac{\partial}{\partial x^j}\big|_p(x^i) = \frac{\partial x^i}{\partial x^j}\big|_p = \delta_j^i$

Lemma $df \in \Gamma(T^*M)$.

Lemma $df_p = \frac{\partial f}{\partial x^i}\big|_p dx^i|_p$

Proof This is really just an application of the fact that $\{\frac{\partial}{\partial x^i}\big|_p\}$ and $\{dx^i|_p\}$ are cobases.
$$df_p(X) = X_p(f) = X_p^i \frac{\partial f}{\partial x^i}\Big|_p$$
$$\begin{aligned} \frac{\partial f}{\partial x^i}\Big|_p dx^i|_p(X) &= \frac{\partial f}{\partial x^i}\Big|_p dx^i\left(X_p^j \frac{\partial}{\partial x^j}\Big|_p\right) \\ &= \frac{\partial f}{\partial x^i}\Big|_p X_p^j \, \delta_j^i = \frac{\partial f}{\partial x^i}\Big|_p X_p^i \end{aligned}$$

We call df the differential of f.

It's a covector field which contains all of the information about the first derivatives of f.

Actually, observe that since $T_{f(p)}\mathbb{R} \cong \mathbb{R}$, we could regard $df_p : T_pM \to \mathbb{R}$ as
$$(f_*)_p : T_pM \to T_{f(p)}\mathbb{R}$$

What are the transition functions for T^*M?

Given coordinates $\{x^1,\dots,x^n\}$ and $\{y^1,\dots,y^n\}$ around $p\in M$, $\{dx^1_p,\dots,dx^n_p\}$ and $\{dy^1_p,\dots,dy^n_p\}$ are two bases for T^*_pM.

$$\delta^i_j = dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = dx^i_p\left(\frac{\partial y^k}{\partial x^j}\Big|_p \frac{\partial}{\partial y^k}\Big|_p\right) = \frac{\partial y^k}{\partial x^j}\Big|_p\, dx^i_p\left(\frac{\partial}{\partial y^k}\Big|_p\right)$$
$$= \frac{\partial y^k}{\partial x^j}\Big|_p A^i_\ell\, dy^\ell_p\left(\frac{\partial}{\partial y^k}\Big|_p\right) = \frac{\partial y^k}{\partial x^j}\Big|_p A^i_\ell\, \delta^\ell_k$$
$$= \frac{\partial y^k}{\partial x^j}\Big|_p A^i_k$$

As linear maps, $Id = \left(\frac{\partial y}{\partial x}\right)\cdot A$. So $A = \left(\frac{\partial y}{\partial x}\right)^{-1} = \left(\frac{\partial x}{\partial y}\right)$.

Chain Rule (covector version)

$$dx^i_p = \frac{\partial x^i}{\partial y^k}\Big|_p dy^k\big|_p$$

This also allows us to show that the differential could have been defined in coordinates:

$$\frac{\partial f}{\partial x^i}\Big|_p dx^i_p = \frac{\partial y^k}{\partial x^i}\Big|_p \frac{\partial f}{\partial y^k}\Big|_p \frac{\partial x^i}{\partial y^\ell}\Big|_p dy^\ell_p$$
$$= \frac{\partial y^k}{\partial x^i}\Big|_p \frac{\partial x^i}{\partial y^\ell}\Big|_p \frac{\partial f}{\partial y^k}\Big|_p dy^\ell_p$$
$$= \delta^k_\ell \frac{\partial f}{\partial y^k}\Big|_p dy^\ell_p = \frac{\partial f}{\partial y^k}\Big|_p dy^k_p.$$

One property that covector fields can have is closedness.

Defn A covector field is called closed if, in coordinates, we have $\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$ for all $i,j = 1,\dots,n$, where $\omega = \omega_i dx^i$.

Lemma The closedness property is well-defined.

Proof Suppose $\omega = \omega_i dx^i = \tilde\omega_j d\tilde x^j$, so that

$$\omega_i dx^i = \omega_i \frac{\partial x^i}{\partial \tilde x^j} d\tilde x^j = \tilde\omega_j d\tilde x^j$$

$\tilde\omega_j = \omega_i \frac{\partial x^i}{\partial \tilde x^j}$. Suppose $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.

$$\frac{\partial \tilde\omega_i}{\partial \tilde x^j} - \frac{\partial \tilde\omega_j}{\partial \tilde x^i} = \frac{\partial}{\partial \tilde x^j}\left(\omega_k \frac{\partial x^k}{\partial \tilde x^i}\right) - \frac{\partial}{\partial \tilde x^i}\left(\omega_\ell \frac{\partial x^\ell}{\partial \tilde x^j}\right)$$
$$= \frac{\partial x^p}{\partial \tilde x^j}\frac{\partial}{\partial x^p}\left(\omega_k \frac{\partial x^k}{\partial \tilde x^i}\right) - \frac{\partial x^q}{\partial \tilde x^i}\frac{\partial}{\partial x^q}\left(\omega_\ell \frac{\partial x^\ell}{\partial \tilde x^j}\right)$$
$$= \frac{\partial x^p}{\partial \tilde x^j}\frac{\partial x^k}{\partial \tilde x^i}\left(\frac{\partial \omega_k}{\partial x^p}\right) - \frac{\partial x^q}{\partial \tilde x^i}\frac{\partial x^\ell}{\partial \tilde x^j}\left(\frac{\partial \omega_\ell}{\partial x^q}\right)$$
$$+ \omega_k \frac{\partial x^p}{\partial \tilde x^j}\frac{\partial^2 x^k}{\partial x^p \partial \tilde x^i} - \omega_\ell \frac{\partial x^q}{\partial \tilde x^i}\frac{\partial^2 x^\ell}{\partial x^q \partial \tilde x^j}$$
$$= \frac{\partial x^p}{\partial \tilde x^j}\frac{\partial x^k}{\partial \tilde x^i}\left(\frac{\partial \omega_k}{\partial x^p}\right) - \frac{\partial x^q}{\partial \tilde x^i}\frac{\partial x^\ell}{\partial \tilde x^j}\left(\frac{\partial \omega_q}{\partial x^\ell}\right)$$
$$+ \omega_k \frac{\partial x^p}{\partial \tilde x^j}\left(\frac{\partial^2 x^k}{\partial \tilde x^i \partial x^p}\right) - \omega_\ell \frac{\partial x^q}{\partial \tilde x^i}\frac{\partial^2 x^\ell}{\partial \tilde x^j \partial x^q}$$

$$= 0 + \omega_k \frac{\partial x^p}{\partial x^j}\frac{\partial}{\partial x^i}(\delta_p^k) - \omega_\ell \frac{\partial x^q}{\partial x^i}\frac{\partial}{\partial x^j}(\delta_q^\ell)$$

$$= 0. \quad \blacksquare$$

Note, however, that the quantity $\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}$ is not well-defined under a change of coordinates! We'll correct this later, with some more technology.

Lemma Any differential is closed.

The Pullback

The real reason to use T^*M is actually contravariance.

Def'n Given $F: M \to N$, we have

$(F_*)_p : T_pM \to T_{F(p)}N$, so taking duals:

$((F_*)_p)^* : T^*_{F(p)}N \to T^*_pM$

We write $(F^*)_p : T^*_{F(p)}N \to T^*_pM$ and call it the pullback.

What does the pullback do?

Let $\omega \in T^*_{F(p)}N$, $X \in T_pM$

$$(F_p^*\omega)(X) = \omega(F_* X)$$

Not all that enlightening. Let's try if $\{x^1,\dots,x^n\}, \{y^1,\dots,y^m\}$ are coords near p and $F(p)$:

$$(F_p^* dy^i)\left(\frac{\partial}{\partial x^j}\right) = dy^i\left(F_*\frac{\partial}{\partial x^j}\right) = dy^i\left(\frac{\partial F^p}{\partial x^j}\frac{\partial}{\partial y^p}\right)$$

$$= \frac{\partial F^p}{\partial x^j} dy^i\left(\frac{\partial}{\partial y^p}\right) = \frac{\partial F^p}{\partial x^j}\delta_p^i = \frac{\partial F^i}{\partial x^j}$$

So the matrix representing F^* in coordinates is $\left(\frac{\partial F}{\partial x}\right)$.

Why should we use F^* instead of F_*?

Consider what happens with a vector field. $X \in \mathfrak{X}(M)$.
If $F: M \to N$ is not onto, then how do we define $(F_* X)_q$ for $q \notin F(M)$?
If $F: M \to N$ is not injective, and $F(p_1) = F(p_2)$, which F_* do we use?

OTOH, if $\omega \in \Gamma(T^*N)$ is a covector field, then we can define $(F^*\omega)_p = (F^*)_p \, \omega_{F(p)}$ without ambiguity.

We say covector fields pull back.

There is $F^*: \Gamma(N) \to \Gamma(M)$

However, there is no bundle map $T^*N \to T^*M$.

Prop'n
- $id^* = id$
- Given $M \xrightarrow{F} N \xrightarrow{G} P$,
  $(G \circ F)^*_p = F^*_p \cdot G^*_{F(p)}$

The covector fields on M, $\mathfrak{X}^*(M) = \Gamma(T^*M)$, form a module over $C^\infty(M)$, since $(f\omega)_p(X) = f(p)\, \omega_p(X_p)$ makes perfect sense.

Lemma Given $F: M \to N$, F^* is a linear map from $\mathfrak{X}^*(N)$ to $\mathfrak{X}^*(M)$. That is, given $\omega, \mu \in \mathfrak{X}^*(N)$ $f, g \in C^\infty(N)$, we have

$$F^*(f\omega + g\mu) = (f \circ F) F^*\omega + (g \circ F) F^*\mu$$

Pf Prove at a point $p \in M$, $X \in T_pM$.

$$\begin{aligned}
[F^*(f\omega + g\mu)]_p(X) &= F^*([f\omega + g\mu]_{F(p)})(X) && \text{defn.} \\
&= [f\omega + g\mu]_{F(p)}(F_* X) && \text{defn.} \\
&= f(F(p))\, \omega_{F(p)}(F_* X) + g(F(p))\, \mu_{F(p)}(F_* X) \\
&= [(f \circ F)(F^*\omega)_p + (g \circ F)(F^*\mu)_p](X) && \text{defn.}
\end{aligned}$$

∎

We describe this by saying "F^* is linear over C^∞ functions"

Integration along curves

The fact that covector fields pull back allows us to integrate them.

First note that the rank of $T^*\mathbb{R}$ is 1. So any element of $\mathfrak{X}^*(\mathbb{R})$ looks like

$$f(t)\,dt$$

for some function $f \in C^\infty(\mathbb{R})$.

Now given a curve $\gamma : J \to M$ and $\omega \in \mathfrak{X}^*(M)$, we can define

$$\int_\gamma \omega := \int_J \gamma^*\omega := \int_J f(t)\,dt$$

Integral of f in the usual sense

where $f(t)$ = coefficient of $\gamma^*_{\gamma(t)}\omega$

Proposition If $\gamma : J \to M$ is a smooth curve, ϕ is a positive reparametrisation of J, and $\omega \in \mathfrak{X}^*(M)$, then

$$\int_\gamma \omega = \int_{\gamma\circ\phi} \omega$$

Tensors

We want a unified way of dealing with objects like vector fields, covector fields, etc., so as to make sense of notions like "closed".

Almost every operation we want to perform is <u>linear</u>.

Defn A map $T : \underbrace{V_1 \times \cdots \times V_p}_{\text{vector spaces}} \to \underbrace{W}_{\text{vector space}}$ is *multilinear* if it is linear in each argument, i.e.

$$T(A_1, \dots, \alpha A + \beta B, \dots, A_p) = \alpha T(A_1, \dots, A, \dots, A_p) + \beta T(A_1, \dots, B, \dots, A_p)$$

Theorem For any vector spaces $V_1, \dots, V_p$, there is a vector space $V_1 \otimes \cdots \otimes V_p$ and a map $V_1 \times \cdots \times V_p \xrightarrow{\otimes} V_1 \otimes \cdots \otimes V_p$ so that for any multilinear $T : V_1 \times \cdots \times V_p \to W$, there is a unique linear $\tilde{T} : V_1 \otimes \cdots \otimes V_p \to W$ so that

$$\begin{tikzcd}
V_1 \times \cdots \times V_p \arrow[r, "T"] \arrow[d, "\otimes"'] & W \\
V_1 \otimes \cdots \otimes V_p \arrow[ur, "\tilde{T}"'] &
\end{tikzcd}$$

Moreover, $V_1 \otimes \cdots \otimes V_p$ is unique up to vector space isomorphism.

Proof We'll construct $V_1 \otimes V_2$.

Let $\mathbb{R}\langle V_1 \times V_2\rangle$ be the free vector space on $V_1 \times V_2$, that is, the space of formal sums $\sum a_{(v_1,v_2)}(v_1,v_2)$ over elements $v_1 \in V_1$, $v_2 \in V_2$, $a_{(v_1,v_2)} \in \mathbb{R}$.

Let $\mathcal{R}$ be the subspace generated by elements of the form
$$a(v_1,v_2) - (av_1,v_2)$$
$$a(v_1,v_2) - (v_1,av_2)$$
$$(v_1+w_1,v_2) - (v_1,v_2) - (w_1,v_2)$$
$$(v_1,v_2+w_2) - (v_1,v_2) - (v_1,w_2)$$
and $V_1 \otimes V_2 = \mathbb{R}\langle V_1 \times V_2\rangle / \mathcal{R}$.

Note that $V_1 \times V_2 \hookrightarrow \mathbb{R}\langle V_1 \times V_2\rangle$, so we define
$V_1 \times V_2 \longrightarrow V_1 \otimes V_2$ by $(v_1,v_2) \longmapsto [(v_1,v_2)]$

Given $T: V_1 \times V_2 \to W$ multilinear, T induces a (linear) map
$$\bar{T}: \mathbb{R}\langle V_1 \times V_2\rangle \longrightarrow W$$
$$\sum a_{v_1,v_2}(v_1,v_2) \longmapsto \sum a_{(v_1,v_2)} T(v_1,v_2)$$

$\operatorname{Ker} \bar{T} \supset \mathcal{R} \Leftrightarrow T$ is multilinear!

So $\bar{T}$ descends to a linear map $\tilde{T}: V_1 \otimes V_2 \to W$.

Uniqueness of $\tilde{T}$ and of the pair $(V_1 \otimes V_2, \otimes)$ is to be pondered.

Prop'n If we write $v \otimes w$ for $\otimes(v,w) = [(v,w)]$, then the map $\otimes$ has the following properties:

1) $\otimes$ is multilinear: $(av) \otimes w = a\, v \otimes w = v \otimes aw$
$$(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1+w_2) = v \otimes w_1 + v \otimes w_2$$

2) $0 \otimes w = v \otimes 0 = 0$

Prop'n

$\oplus, \otimes$ give a ring str.

2) $(V \oplus W) \otimes U \cong (V \otimes U) \oplus (W \otimes U)$ canonically

3) $V \otimes W \cong W \otimes V$ canonically

4) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ canonically

5) $V \otimes \mathbb{R} \cong V$ canonically.

6) $(V \otimes W)^* \cong V^* \otimes W^*$

Proposition $V^*\otimes W^*$ is canonically isomorphic to the space $B(V,W)$ of bilinear maps $V\times W\to\mathbb{R}$.

Proof. To define the isomorphism $V^*\otimes W^*\to B(V,W)$ first define $\Phi: V^*\times W^*\to B(V,W)$ by

$$(\Phi(\omega,\eta))(v,w) = \omega(v)\eta(w)$$

just multiplication of $\mathbb{R}$.

Since Φ is bilinear, it descends to some linear
$\tilde{\Phi}: V^*\otimes W^*\to B(V,W)$.

Show that $\tilde{\Phi}$ is an isomorphism by making an inverse.

Let $\{\omega^1,\dots,\omega^n\}$ be a basis for V^*, dual to $\{e_1,\dots,e_n\}$
$\{\eta^1,\dots,\eta^m\}$ a basis for W^*, dual to $\{f_1,\dots,f_m\}$.

Then $\{\omega^i\otimes\eta^j\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$ span $V^*\otimes W^*$. Define

$$\Psi: B(V,W)\to V^*\otimes W^*$$
$$b\mapsto b(e_i,f_j)\,\omega^i\otimes\eta^j$$

If $\tau=\tau_{ij}\,\omega^i\otimes\eta^j$ is an arbitrary element of $V^*\otimes W^*$,
then
$$\Psi\circ\tilde{\Phi}(\tau) = \tilde{\Phi}(\tau)(e_i,f_j)\,\omega^i\otimes\eta^j$$
$$= \tilde{\Phi}(\tau_{kl}\,\omega^k\otimes\eta^l)(e_i,f_j)\,\omega^i\otimes\eta^j$$
$$= (\tau_{kl}\tilde{\Phi}(\omega^k\otimes\eta^l))(e_i,f_j)\,\omega^i\otimes\eta^j$$
$$= \tau_{kl}\Phi(\omega^k,\eta^l)(e_i,f_j)\,\omega^i\otimes\eta^j$$
$$= \tau_{kl}\,\omega^k(e_i)\eta^l(f_j)\,\omega^i\otimes\eta^j = \tau_{kl}\delta_i^k\delta_j^l\,\omega^i\otimes\eta^j$$
$$= \tau_{ij}\,\omega^i\otimes\eta^j=\tau$$

So $\Psi\circ\tilde{\Phi}=\mathrm{id}_{V^*\otimes W^*}$.

If $b\in B(V,W)$,
$$\tilde{\Phi}\circ\Psi(b)(v,w) = \tilde{\Phi}(b(e_i,f_j)\,\omega^i\otimes\eta^j)(v,w)$$
$$= b(e_i,f_j)\,\tilde{\Phi}(\omega^i\otimes\eta^j)(v,w)$$
$$= b(e_i,f_j)\,\omega^i(v)\eta^j(w)$$
$$= b(e_i,f_j)\,v^i w^j = b(v,w)$$

So $\tilde{\Phi}\circ\Psi=\mathrm{id}_{B(V,W)}$. ∎

Corollary 1) $\dim(V^*\otimes W^*) = \dim V^*\dim W^* = \dim V\dim W$

2) $\{\omega^i\otimes\eta^j\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$ are a basis!

Proposition $V_1^*\otimes\cdots\otimes V_k^*$ is isomorphic to the space of k-linear maps $V_1\times\cdots\times V_k\to\mathbb{R}$.

Def'n $V^{\otimes k} = V \otimes \overset{k}{\cdots} \otimes V$

$V^{\otimes 0} = \mathbb{R}, \quad V^{\otimes 1} = V$

Def'n We call an element of $(V^*)^{\otimes k}$ a covariant k-tensor and an element of $(V)^{\otimes k}$ a contravariant k-tensor.

An element of $(V^*)^{\otimes k} \otimes V^{\otimes l}$ is a tensor of type (k,l) or $\binom{k}{l}$

$T^k(V) = (V^*)^{\otimes k} \quad T^k_l(V) = T^k(V) \otimes T_l(V)$

$T_l(V) = V^{\otimes k}$

Def'n Given $S \in T^k(V)$, $T \in T^l(V)$, define $S \otimes T \in T^{k+l}(V)$ by

$(S \otimes T)(v_1, \dots, v_{k+l}) = S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+l})$

Note that $S \otimes T$ and $T \otimes S$ are distinct elements of $T^{k+l}(V)$!

Tensor Bundles

Def'n Given a smooth manifold M,

$$T^k_l(M) = \bigsqcup_{p \in M} T^k_l(T_pM)$$

Exercise $T^k_l(M)$ is a vector bundle of rank __?

$T^1(M) = T^*M \qquad T^0M = T_0M = M \times \mathbb{R}$

$T_1(M) = TM$

By our construction,

T^k_pM has basis $\{dx^{i_1}_p \otimes \cdots \otimes dx^{i_k}_p\}$ $i_\ell = 1, \dots, n$

$(T_p)_\ell M$ has basis $\left\{\frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\ell}}\Big|_p\right\}$ $i_\ell = 1, \dots, n$

What are the transition functions for T^kM?

Let $\{x^1, \dots, x^n\}$ and $\{y^1, \dots, y^n\}$ be coordinates at $p \in M$.

Then $dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \left(\frac{\partial x^{i_1}}{\partial y^{p_1}} dy^{p_1}\right) \otimes \left(\frac{\partial x^{i_2}}{\partial y^{p_2}} dy^{p_2}\right) \otimes \cdots \otimes \left(\frac{\partial x^{i_k}}{\partial y^{p_k}} dy^{p_k}\right)$

$= \frac{\partial x^{i_1}}{\partial y^{p_1}} \frac{\partial x^{i_2}}{\partial y^{p_2}} \cdots \frac{\partial x^{i_k}}{\partial y^{p_k}} \left(dy^{p_1} \otimes \cdots \otimes dy^{p_k}\right)$

The linear map cor. to $\frac{\partial x}{\partial y} \otimes \cdots \otimes \frac{\partial x}{\partial y}$.

Def'n A tensor field of type $\binom{k}{\ell}$ is a section of $T^k_\ell M$. We write $\mathcal{T}^k_\ell(M) = \Gamma(T^k_\ell M)$.

Lemma If $A \in \mathcal{T}^k_\ell(M)$, and $\{x^1, \dots, x^n\}, \{\tilde{x}^1, \dots, \tilde{x}^n\}$ are coordinates near $p \in M$, with

$$A = A^{j_1 \dots j_\ell}_{i_1 \dots i_k}\, dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_\ell}}$$
$$= \tilde{A}^{r_1 \dots r_\ell}_{s_1 \dots s_k}\, d\tilde{x}^{s_1} \otimes \dots \otimes d\tilde{x}^{s_k} \otimes \frac{\partial}{\partial \tilde{x}^{r_1}} \otimes \dots \otimes \frac{\partial}{\partial \tilde{x}^{r_\ell}}$$

Then $\tilde{A}^{r_1 \dots r_\ell}_{s_1 \dots s_k} = A^{j_1 \dots j_\ell}_{i_1 \dots i_k} \frac{\partial \tilde{x}^{r_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{r_\ell}}{\partial x^{j_\ell}} \frac{\partial x^{i_1}}{\partial \tilde{x}^{s_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{s_k}}$.

The way to remember this is —
Upper indices are "covariant": the tilde goes with them
Lower indices are "contravariant": the tilde goes against them.

Lemma The following are equivalent for a "rough" section σ:

1) $\sigma \in \mathcal{T}^k_\ell(M)$
2) The (local) components $\sigma^{j_1 \dots j_\ell}_{i_1 \dots i_k}$ are smooth functions
3) If $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\omega^1, \dots, \omega^\ell \in \mathfrak{X}^*(M)$, then the function $\sigma(p) = \sigma_p(X_{1p}, \dots, X_{kp}, \omega^1_p, \dots, \omega^\ell_p)$ is smooth.

We can describe a tensor field either invariantly or in coordinates.

E.g. Define a $\binom{1}{1}$ tensor field by: $\sigma^i_j = \delta^i_j$

Check this is well-defined. If $\{x^1, \dots, x^n\}, \{y^1, \dots, y^n\}$ are coordinates then

$$\underbrace{\delta^k_\ell}_{\text{"in } x\text{"}} \frac{\partial x^i}{\partial y^k} \frac{\partial y^\ell}{\partial x^j} = \frac{\partial x^i}{\partial y^\ell} \frac{\partial y^\ell}{\partial x^j} = \underbrace{\delta^i_j}_{\text{"in } y\text{"}}$$

What is the meaning of this tensor?

$$\sigma = \delta^i_j\, dx^j \otimes \frac{\partial}{\partial x^i}$$

Let $X \in \mathfrak{X}(M)$, $\omega \in \mathfrak{X}^*(M)$. Then $X = X^k \frac{\partial}{\partial x^k}$, $\omega = \omega_\ell\, dx^\ell$

$$\sigma(X, \omega) = \left(\delta^i_j\, dx^j \otimes \frac{\partial}{\partial x^i}\right)\left(X^k \frac{\partial}{\partial x^k}, \omega_\ell\, dx^\ell\right)$$
$$= \delta^i_j\, dx^j\left(X^k \frac{\partial}{\partial x^k}\right) \frac{\partial}{\partial x^i}(\omega_\ell\, dx^\ell)$$
$$= \delta^i_j X^k \delta^j_k \omega_\ell \delta^\ell_i = X^k \omega_k = \omega(X)$$

So σ is just "evaluate my second slot on my first slot".

E.g. $\delta_{ij}\, dx^i \otimes dx^j$ Sim. $\delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$

$$\text{"in } x\text{"} \to \delta_{k\ell} \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j} \neq \underbrace{\delta_{ij}}_{\text{"in } y\text{"}} \; !$$

is not a tensor!

Proposition: Let M be a smooth manifold.

1) Given a $\sigma \in \mathcal{T}^{*}_{\ell}(M)$, define

$$\bar{\sigma} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \times \mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M) \longrightarrow C^\infty(M)$$
$$(X_1, \ldots, X_k, \omega_1, \ldots, \omega_\ell) \mapsto \sigma(X_1, \ldots, X_k, \omega_1, \ldots, \omega_\ell)$$

Then $\bar{\sigma}$ is multilinear over $C^\infty(M)$, e.g. for $f, g \in C^\infty(M)$,

$$\bar{\sigma}(X_1, \ldots, fX + gY, \omega_1, \ldots, \omega_\ell) = f\sigma(X_1, \ldots, X, \omega_1, \ldots, \omega_\ell) + g\sigma(X_1, \ldots, Y, \omega_1, \ldots, \omega_\ell)$$

2) Any multilinear map

$$\tau : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \times \mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M) \longrightarrow C^\infty(M)$$

is induced by some $\tau \in \mathcal{T}^{k}_{\ell}(M)$.

Eg. $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is not tensorial:

$$[fX + gY, Z] = f[X,Z] + g[Y,Z] - Z(f)X - Z(g)Y$$

Zero is a tensor, so the fact that $[\cdot,\cdot]$ vanishes on coordinate vectors should have led us to expect this.

Proposition Covariant tensor fields pull back, i.e. given $F : M \to N$, $\sigma \in \mathcal{T}^k(N)$, we can define $F^*\sigma \in \mathcal{T}^k(M)$ by $(F^*\sigma)_p(X_1, \ldots, X_k) = \sigma_{F(p)}(F_* X_1, \ldots, F_* X_k)$

Proposition Pushforwards of contravariant tensor bundles are bundle maps! $F_* : T_k(M) \to T_k(N)$

$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \mapsto \left(F_* \frac{\partial}{\partial x^{i_1}}\right) \otimes \cdots \otimes \left(F_* \frac{\partial}{\partial x^{i_k}}\right)$$

Defn Given a tensor of type $\binom{k}{\ell}$ T and a vector field X, we define a $\binom{k-1}{\ell}$ tensor $X \lrcorner T$ by:

$$(X \lrcorner T)(Y_1, \ldots, Y_{k-1}, \omega_1, \ldots, \omega_\ell) = T(X, Y_1, \ldots, Y_{k-1}, \omega_1, \ldots, \omega_\ell)$$

called the contraction of T by X or the interior product of X with T.

In coordinates, $X = X^p \frac{\partial}{\partial x^p}$, $T = T^{i_1 \cdots i_\ell}_{j_1 \cdots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \frac{\partial}{\partial x^{i_\ell}}$

$$(X \lrcorner T)^{i_1 \cdots i_\ell}_{j_1 \cdots j_{k-1}} = X^p T^{i_1 \cdots i_\ell}_{p j_1 \cdots j_{k-1}}$$

Similarly, we can contract a covector field into a $\binom{k}{\ell}$ tensor to get a $\binom{k}{\ell-1}$ tensor.

We can also form a $\binom{k-1}{\ell-1}$ tensor (the trace of T) by:

$$(\operatorname{tr} T)^{i_1 \cdots i_{\ell-1}}_{j_1 \cdots j_{k-1}} = T^{s i_1 \cdots i_{\ell-1}}_{s j_1 \cdots j_{k-1}}$$

Alternating Tensors

Note that the contraction and tracing operations can be done on any slot. This ambiguity leads us to cower and retreat, and only consider tensors with some kind of symmetry.

Defn $T \in \mathcal{T}^k(V)$ is **totally symmetric** if, for any choice $X_1, \dots, X_k \in V$, and any $1 \le i < j \le k$,

$$T(X_1, \dots, X_i, \dots, X_j, \dots, X_k) = T(X_1, \dots, X_j, \dots, X_i, \dots, X_k)$$

Lemma T is totally symmetric iff its coordinate representation is totally symmetric, i.e. $T_{s_1 \dots s_i \dots s_j \dots s_k} = T_{s_1 \dots s_j \dots s_i \dots s_k}$

Defn $T \in T^k(V)$ is **alternating** if, for any $Y, X_1, \dots, X_{k-2} \in V$,

$$T(X_1, \dots, Y, \dots, Y, \dots, X_{k-2}) = 0.$$

Lemma A k-tensor T is alternating iff

$$T(X_1, \dots, X_i, X_{i+1}, \dots, X_k) = -T(X_1, \dots, X_{i+1}, X_i, \dots, X_k).$$

(unless we are working over a field of characteristic 2.)

The space of alternating k-tensors is a vector subspace of $T^k(V)$. Call it $\Lambda^k(V)$.

What is the projection $T^k(V) \to \Lambda^k(V)$?

Defn Given $T \in T^k(V)$, the **alternation of T** is

$$(\mathrm{Alt}(T))(V_1, \dots, V_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \, T(V_{\sigma(1)}, \dots, V_{\sigma(k)})$$

Eg. $(\mathrm{Alt}\, T)(X, Y) = \frac{1}{2}[T(X,Y) - T(Y,X)]$

$$(\mathrm{Alt}\, T)(X,Y,Z) = \frac{1}{6}\big[T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) - T(Y,X,Z) - T(Z,Y,X) - T(X,Z,Y)\big]$$

Lemma
1) $\mathrm{Alt}(T) \in \Lambda^k(V)$
2) $\mathrm{Alt}(T) = T$ if $T \in \Lambda^k(V)$
3) $\mathrm{Alt} : T^k(V) \to \Lambda^k(V)$ is linear.

(in particular, $\mathrm{Alt}^2 = \mathrm{Alt}$ and Alt is onto.)

Just as we have $\otimes : T^k(V) \times T^l(V) \to T^{k+l}(V)$, there is

Defn The **exterior product** or **wedge** is defined by:

$$\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$$

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta)$$

Exercise Alt is associative!

Proposition Given a cobasis pair $\{\omega^1,\dots,\omega^n\}$, $\{e_1,\dots,e_n\}$, $\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_k} \mid 1\le i_1<\cdots<i_k\le n\}$ is a basis for $\Lambda^k(V)$.

Proof What's important here is that the indices are increasing!

We know that $\{\omega^{i_1}\otimes\cdots\otimes\omega^{i_k} \mid i_1,\dots,i_k\in\{1,\dots,n\}\}$ are a basis of $T^k(V)$, so since Alt is onto, $\{\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_k}) \mid i_1,\dots,i_k\in\{1,\dots,n\}\}$ must span $\Lambda^k(V)$.

Claim. If any two of $i_1,\dots,i_k$ are equal, then $\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_k})=0$.

Proof. $(k=3)$ $\mathrm{Alt}(\omega^1\otimes\omega^2\otimes\omega^2)(X,Y,Z)$

$$= \tfrac{1}{6}\big[\omega^1\otimes\omega^2\otimes\omega^2(X,Y,Z)+\omega^1\otimes\omega^2\otimes\omega^2(Y,Z,X)+\omega^1\otimes\omega^2\otimes\omega^2(Z,X,Y) - \omega^1\otimes\omega^2\otimes\omega^2(Y,X,Z)-\omega^1\otimes\omega^2\otimes\omega^2(Z,Y,X)-\omega^1\otimes\omega^2\otimes\omega^2(X,Z,Y)\big]$$

$$= \tfrac{1}{6}\big[X^1Y^2Z^2+Y^1Z^2X^2+Z^1X^2Y^2-Y^1X^2Z^2-Z^1Y^2X^2-X^1Z^2Y^2\big] = 0.$$

Exercise: Write this down for arbitrary k.

Claim. $\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_j}\otimes\omega^{i_{j+1}}\otimes\cdots\otimes\omega^{i_k})$
$= -\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_{j+1}}\otimes\omega^{i_j}\otimes\cdots\otimes\omega^{i_k})$

Proof $(k=3,\ j=2)$

$$\mathrm{Alt}(\omega^1\otimes\omega^2\otimes\omega^3)(X,Y,Z) = \tfrac{1}{6}\big[X^1Y^2Z^3+Y^1Z^2X^3+Z^1X^2Y^3-Y^1X^2Z^3-Z^1Y^2X^3-X^1Z^2Y^3\big]$$

$$\mathrm{Alt}(\omega^1\otimes\omega^3\otimes\omega^2)(X,Y,Z) = \tfrac{1}{6}\big[X^1Y^3Z^2+Y^1Z^3X^2+Z^1X^3Y^2-Y^1X^3Z^2-Z^1Y^3X^2-X^1Z^3Y^2\big]$$

Exercise: Write this for arbitrary k and j.

So from our spanning set $\{\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_k})\}$ we can omit anything with repeated indices, and the second claim allows us to choose just one representative from each choice of k distinct indices, namely the <u>increasing</u> choice.

Claim. The $\{\mathrm{Alt}(\omega^{i_1}\otimes\cdots\otimes\omega^{i_k}) \mid 1\le i_1<\cdots<i_k\le n\}$ are linearly independent.

Exercise: Prove this (Hint: Similar to previous proofs of linear independence.)

Vastly Important Corollary: If $\dim V = n$, then
$$\dim(\Lambda^k V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In particular:

- $\dim(\Lambda^k V) = \dim(\Lambda^{n-k} V)$ (so they are abstractly isom.)
- $\dim(\Lambda^n V) = 1$
- $\dim(\Lambda^k V) = 0$ if $k > n$

$$\mathrm{Alt}(\omega^1 \otimes \cdots \otimes \omega^n)(e_1, \ldots, e_n)$$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \omega^1 \otimes \cdots \otimes \omega^n (e_{\sigma(1)}, \ldots, e_{\sigma(n)})$$
Only the identity permutation survives!
$$= \frac{1}{n!}$$
← Why we put the combinatorial coefficient in the definition of $\wedge$.

$$\omega^1 \wedge \cdots \wedge \omega^n (e_1, \ldots, e_n) = 1.$$

Proposition
1) $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is bilinear
2) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

Another name for $\omega^1 \wedge \cdots \wedge \omega^n$

Given $X_1, \ldots, X_n \in V$, we can choose a basis $\{e_1, \ldots, e_n\}$, write $X_1, \ldots, X_n$ in terms of that basis, put them into a matrix, and take its determinant.

So $\det_{\{e_1,\ldots,e_n\}} \in \Lambda^n(V)$, hence $\det = c\, \omega^1 \wedge \cdots \wedge \omega^n$

OTOH, $\det_{\{e_1,\ldots,e_n\}}(e_1, \ldots, e_n) = 1$.
$$\omega^1 \wedge \cdots \wedge \omega^n (e_1, \ldots, e_n) = 1$$
So $c = 1$, ie $\omega^1 \wedge \cdots \wedge \omega^n$ is the determinant!

Proposition If $\omega^1, \ldots, \omega^k \in V^*$, $X_1, \ldots, X_k \in V$, then
$$\omega^1 \wedge \cdots \wedge \omega^k (X_1, \ldots, X_k) = \det(\omega^i(X_j)).$$

Exercise $\wedge$ is the unique bilinear, associative, skew map which satisfies the proposition.

Defn The *alternating algebra* on V is
$$\Lambda^*(V) = \bigoplus_k \Lambda^k(V) \qquad \dim \Lambda^*(V) = 2^n$$
It is an anticommutative associative graded algebra.

Differential Forms

Defn Given a smooth manifold M of dimension n,

$$\Lambda^k(M) = \coprod_{p \in M} \Lambda^k(T_pM)$$ alternating bundle of rank k

wedge bundle

The sections of $\Lambda^k(M)$ are called differential forms.

$$\mathcal{A}^k(M) = \Gamma(\Lambda^k(M))$$

At $p \in M$, any k-form ω can be expressed in coordinates as

$$\omega_p = \omega_{i_1 \cdots i_k}(p)\, dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k} \quad \text{where } 1 \le i_1 < \cdots < i_k \le n$$

$$= \omega_I(p)\, dx_p^I \quad \text{where } I \text{ means any } k\text{-tuple of indices.}$$

If $J = \{j_1, \ldots, j_k\}$ is a k-index

$$\omega\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \omega_I\, dx^I\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right)$$

$$= \omega_I \det\left(dx^i\left(\frac{\partial}{\partial x^j}\right)\right) = \omega_I\, \delta_J^I$$

where $\delta_J^I = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{otherwise} \end{cases}$ as sets!

So $\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$

Change-of-coordinates formula for forms:

If $\omega = \omega_I\, dx^I = \tilde{\omega}_J\, d\tilde{x}^J$

$$\tilde{\omega}_J = \omega\left(\frac{\partial}{\partial \tilde{x}^{j_1}}, \ldots, \frac{\partial}{\partial \tilde{x}^{j_k}}\right)$$

$$= \omega\left(\frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}}\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}}\frac{\partial}{\partial x^{i_k}}\right)$$

$$= \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}}\, \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$$

Need to swap to ensure the i's are increasing!

$$= \det\left(k \times k \text{ minor of } \left(\frac{\partial x}{\partial \tilde{x}}\right) \text{ cor. to } \begin{matrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{matrix}\right) \omega_I$$

E.g. Change of coordinates formula from multivariate Calculus: Given (x^1, x^2), (r, θ) coordinates for $\mathbb{R}^2$,

$$\omega = dx^1 \wedge dx^2$$

Then $\omega_{12} = 1$, $x^1 = r\cos\theta$ $x^2 = r\sin\theta$

$$\frac{\partial x^1}{\partial r} = \cos\theta \quad \frac{\partial x^2}{\partial r} = \sin\theta \qquad \det\left(\frac{\partial x}{\partial \tilde{x}}\right) = r$$

$$\frac{\partial x^1}{\partial \theta} = -r\sin\theta \quad \frac{\partial x^2}{\partial \theta} = r\cos\theta \qquad \tilde{\omega}_{12} = r$$

So $\omega = dx^1 \wedge dx^2 = r\, dr \wedge d\theta$ which is just what we expect but with wedges.

Prop'n Λ^k and $\wedge$ are natural.

That is, if $F: M \to N$, $\omega \in \Lambda^k(N)$, $\eta \in \Lambda^l(M)$, then $F^*: \Lambda^k(N) \to \Lambda^k(M)$ has

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

Moreover, in coordinates

$$(\dagger) \quad F^*(\omega_I \, dy^I) = (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)$$

Proof That $(F^*\omega)_p(X_1, \dots, X_k) = \omega_{F(p)}(F_* X_1, \dots, F_* X_k)$ is alternating is clear.

Actually $(\dagger)$ should prove naturality of $\wedge$. (Exercise!)

$$(F^*\omega)_p(X_1, \dots, X_k) = \omega_{F(p)}(F_* X_1, \dots, F_* X_k)$$
$$= \omega_I(F(p))\, dy^I(F_* X_1, \dots, F_* X_k)$$

So we just need to check that

$$(dy^{i_1} \wedge \cdots \wedge dy^{i_k})(F_* X_1, \dots, F_* X_k)$$

and $\left(d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)\right)(X_1, \dots, X_k)$ are the same.

$$dy^i(F_* X) = dy^i\left(\frac{\partial F^r}{\partial x^j} X^j \frac{\partial}{\partial y^r}\right) = \frac{\partial F^r}{\partial x^j} X^j \delta^i_r = X^j \frac{\partial F^i}{\partial x^j}$$

$$d(y^i \circ F)(X) = X(y^i \circ F) = X^j \frac{\partial}{\partial x^j}(y^i \circ F) = X^j \frac{\partial F^i}{\partial x^j}$$

Use the above to compute change-of-coordinates:

$$(\mathrm{Id}^*)(dx \wedge dy) = d(r\cos\theta) \wedge d(r\sin\theta)$$
$$= \left(\frac{\partial}{\partial r}(r\cos\theta)\,dr + \frac{\partial}{\partial \theta}(r\cos\theta)\,d\theta\right) \wedge \left(\frac{\partial}{\partial r}(r\sin\theta)\,dr + \frac{\partial}{\partial \theta}(r\sin\theta)\,d\theta\right)$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= \cos\theta\sin\theta\, dr \wedge dr + r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr - r^2\sin\theta\cos\theta\, d\theta \wedge d\theta$$
$$= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$
$$= r\cos^2\theta\, dr \wedge d\theta + r\sin^2\theta\, dr \wedge d\theta$$
$$= r\, dr \wedge d\theta$$

E.g. Given a one-form (i.e. covector field) ω, set $d\omega = \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) dx^i \wedge dx^j$

$$\left(\frac{\partial \tilde\omega_i}{\partial \tilde x^j} - \frac{\partial \tilde\omega_j}{\partial \tilde x^i}\right) d\tilde x^i \wedge d\tilde x^j$$

$$= \left(\frac{\partial x^k}{\partial \tilde x^j}\frac{\partial \tilde\omega_i}{\partial x^k} - \frac{\partial x^l}{\partial \tilde x^i}\frac{\partial \tilde\omega_j}{\partial x^l}\right)\frac{\partial \tilde x^i}{\partial x^p}\frac{\partial \tilde x^j}{\partial x^q}\, dx^p \wedge dx^q$$

$$= \left(\frac{\partial x^k}{\partial \tilde x^j}\frac{\partial}{\partial x^k}\left(\frac{\partial x^s}{\partial \tilde x^i}\omega_s\right) - \frac{\partial x^l}{\partial \tilde x^i}\frac{\partial}{\partial x^l}\left(\frac{\partial x^r}{\partial \tilde x^j}\omega_r\right)\right)\frac{\partial \tilde x^i}{\partial x^p}\frac{\partial \tilde x^j}{\partial x^q}\, dx^p \wedge dx^q$$

$$= \left(\frac{\partial x^k}{\partial \tilde x^j}\frac{\partial x^s}{\partial \tilde x^i}\frac{\partial}{\partial x^k}\omega_s - \frac{\partial x^l}{\partial \tilde x^i}\frac{\partial x^r}{\partial \tilde x^j}\frac{\partial}{\partial x^l}\omega_r\right)\frac{\partial \tilde x^i}{\partial x^p}\frac{\partial \tilde x^j}{\partial x^q}\, dx^p \wedge dx^q$$

$$= \left(\delta^k_q \delta^s_p \frac{\partial}{\partial x^k}\omega_s - \delta^l_p \delta^r_q \frac{\partial}{\partial x^l}\omega_r\right) dx^p \wedge dx^q$$

$$= \left(\frac{\partial \omega_p}{\partial x^q} - \frac{\partial}{\partial x^p}\omega_q\right) dx^p \wedge dx^q$$

Note that we need $\wedge$ to be alternating in order to cancel the second derivatives.

Defn Given $\omega \in \mathcal{A}^k(M)$, define $d\omega \in \mathcal{A}^{k+1}(M)$ by:

$$d(\omega_I\, dx^I) = d\omega_I \wedge dx^I = \frac{\partial}{\partial x^j}\omega_I\, dx^j \wedge dx^I \qquad *$$

Exercise Show $d\omega$ is well-defined.
(Not!)

Proposition d as defined by $*$ has the properties:

1) d is linear over $\mathbb{R}$.
2) If $\omega \in \mathcal{A}^k(M)$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3) $d(d\omega) = 0$.

Proof 1) is since $\frac{\partial}{\partial x^j}$ and $dx^j \wedge$ are linear.

2) By linearity, suffices to prove in the case $\omega = f\, dx^I$, $\eta = g\, dx^J$.

$$\omega \wedge \eta = fg\, dx^I \wedge dx^J$$

$$d(\omega \wedge \eta) = d(fg) \wedge dx^I \wedge dx^J$$

$$= (f\, dg) \wedge dx^I \wedge dx^J + g\, df \wedge dx^I \wedge dx^J$$

$$= (-1)^k f\, dx^I \wedge dg \wedge dx^J + g\, d\omega \wedge dx^J$$

$$= (-1)^k \omega \wedge d\eta + d\omega \wedge \eta$$

3) Again, wlog assume $\omega = f dx^I$.

Then $d\omega = \frac{\partial f}{\partial x^j} dx^j \wedge dx^I$

$$d(d\omega) = \frac{\partial^2 f}{\partial x^l \partial x^j} dx^l \wedge dx^j \wedge dx^I$$

4 kinds of terms:
- l or $j \in I$ $\quad = 0$
- $l = j$ $\quad = 0$
- $l < j$
- $j < l$

These cancel one another.

Unique Differential Theorem

There is a unique linear map $d : \Lambda^*(M) \to \Lambda^*(M)$ with

1) d has degree 1, i.e. $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$
2) If $\omega \in \Lambda^k(M)$, then (d obeys Leibniz' rule)
$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$$
3) $df(X) = X(f)$ (d is the differential on functions)
4) $d^2 = 0$

Proof: Case: M has a global chart.

Show $d(f dx^I) = \bar{d}(f\, dx^I)$

$$\underset{df \wedge dx^I}{\overset{\shortparallel}{d(f dx^I)}} \qquad \underset{\bar{d}f \wedge dx^I + f \bar{d}(dx^I)}{\overset{\shortparallel}{\bar{d}(f\, dx^I)}}$$

So we want to show $\bar{d}(dx^I) = 0$.

Now $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$
$$= \bar{d}x^{i_1} \wedge \dots \wedge \bar{d}x^{i_k}$$

So $\bar{d}(dx^I) = \bar{d}(\bar{d}x^{i_1}) \wedge dx^{i_2} \wedge \dots \wedge \bar{d}x^{i_k}$
$$- \bar{d}x^{i_1} \wedge \bar{d}(\bar{d}x^{i_2} \wedge \dots \wedge \bar{d}x^{i_k})$$
$$= \bar{d}^2 x^{i_1} \wedge \bar{d}x^{i_2} \wedge \dots \wedge \bar{d}x^{i_k}$$
$$- \bar{d}x^{i_1} \wedge \bar{d}^2 x^{i_2} \wedge \bar{d}x^{i_3} \wedge \dots \wedge \bar{d}x^{i_k}$$
$$+ \bar{d}x^{i_1} \wedge \bar{d}x^{i_2} \wedge \bar{d}(\bar{d}x^{i_3} \wedge \dots \wedge \bar{d}x^{i_k})$$
$$= \sum_j (-1)^{j+1} \bar{d}x^{i_1} \wedge \dots \wedge \bar{d}^2 x^{i_j} \wedge \dots \wedge \bar{d}x^{i_k}$$
$$= 0.$$

Case: M is a general manifold.

It suffices to show $(d\omega)_p = (\bar{d}\omega)_p$, so we can just apply the above on a chart near p.

Corollary d is well-defined, that is, $(d\omega_I)\wedge dx^I = (d\tilde{\omega}_J)\wedge d\tilde{x}^J$ for any choices of coordinates $\{x^1,\dots,x^n\}$ $\{\tilde{x}^1,\dots,\tilde{x}^n\}$

Theorem (invariant defn of d)

If $\omega\in\Lambda^k(M)$, define $\tilde{d}:\Lambda^k(M)\to\mathcal{T}^{k+1}(M)$

$$(\tilde{d}\omega)(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1}))$$
$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1})$$

Then $\tilde{d}\omega\in\Lambda^{k+1}(M)$, and $\tilde{d}$ satisfies the unique differential theorem.

Theorem d is natural, that is, $F^*(d\omega)=d(F^*\omega)$

Proof Locally if $\omega = f\,dx^{i_1}\wedge\dots\wedge dx^{i_k}$,

$$d(F^*\omega) = d(f\circ F\, d(x^{i_1}\circ F)\wedge\dots\wedge d(x^{i_k}\circ F))$$
$$= d(f\circ F)\wedge d(x^{i_1}\circ F)\wedge\dots\wedge d(x^{i_k}\circ F)$$

OTOH, $F^*(d\omega) = F^*(df\wedge dx^{i_1}\wedge\dots\wedge dx^{i_k})$
$$= d(f\circ F)\wedge d(x^{i_1}\circ F)\wedge\dots\wedge d(x^{i_k}\circ F)$$

Defn A manifold M is smoothly contractible if there is a C^∞ map

$$H: M\times[0,1]\to M$$

such that $H(\cdot,1)=\mathrm{id}_M$

$H(\cdot,0)=p_0$ for some $p_0\in M$.

Poincaré Lemma

If M is contractible to a point, then every closed form is an exact differential.

Proof We'll focus our attention on $M\times[0,1]$ as a manifold.

Define $i_t: M\to M\times[0,1]$

$p\mapsto(p,t)$

Claim If $\omega\in\Lambda^k(M\times[0,1])$ is closed, then $i_1^*\omega - i_0^*\omega = d\eta$ for some $\eta\in\Lambda^{k-1}(M)$.

We'll do the case $k=1$

Any $\omega \in \Lambda^1(M\times[0,1])$ can be written as
$\omega = \omega_i dx^i + f dt$ where $\{x^1,\dots,x^n\}$ are coordinates on M.

Then $(i_t^*\omega) = i_t^*(\omega_i dx^i) + i_t^*(f dt)$
$= (\omega_i \circ i_t)\, dx^i + f\, i_t^* dt$

Now $(i_t^* dt)(X) = dt(i_{t*}X) = ((i_t)_* X)(t) = 0$

So $d\omega = d\omega_i \wedge dx^i + df \wedge dt$
$$= \frac{\partial}{\partial x^j}\omega_i\, dx^j\wedge dx^i + \frac{\partial \omega_i}{\partial t}\, dt\wedge dx^i + \frac{\partial f}{\partial x^j}\, dx^j \wedge dt + \frac{\partial f}{\partial t}\cancel{dt\wedge dt}^{\,0}$$

$d\omega = 0$, so
$$0 = \frac{\partial}{\partial t}\lrcorner\, d\omega = \frac{\partial \omega_i}{\partial t} dx^i - \frac{\partial f}{\partial x^j} dx^j$$

So $\frac{\partial \omega_i}{\partial t} = \frac{\partial f}{\partial x^i}$.

Then $\omega_i(p,1) - \omega_i(p,0) = \int_0^1 \frac{\partial \omega_i}{\partial t}(p,t)\, dt$
$$= \int_0^1 \frac{\partial f}{\partial x^i}(p,t)\, dt$$

$$\omega(p,1) - \omega(p,0) = \left(\int_0^1 \frac{\partial f}{\partial x^i}(p,t)\,dt\right) dx^i + (f(p,1) - f(p,0))\, dt$$

So $i_1^*\omega - i_0^*\omega = \left(\int_0^1 \frac{\partial f}{\partial x^i}(p,t)\,dt\right) dx^i$
$$= \frac{\partial}{\partial x^i}\left(\int_0^1 f(p,t)\,dt\right) dx^i$$
$$= d\left(\int_0^1 f(p,t)\,dt\right).$$

Returning to our smooth contraction H, given a closed form $\alpha \in \Lambda^k(M)$,
$$\alpha = (H\circ i_1)^*\alpha = i_1^*(H^*\alpha) \quad \text{and } d(H^*\alpha) = H^* d\alpha = 0.$$
$$0 = (H\circ i_0)^*\alpha = i_0^*(H^*\alpha)$$

So by the claim
$$\alpha - 0 = i_1^*(H^*\alpha) - i_0^*(H^*\alpha) = d\eta$$

Hence $\alpha = d\eta$ is exact. ■

Eg. $d\theta \in A^1(\mathbb{R}^2 \setminus \{0\})$

$$d\theta = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}dx + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}dy = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$d\theta(X) = X\left(\arctan\left(\frac{y}{x}\right)\right)$$

$d(d\theta) = 0$. So $d\theta$ is closed. But if $f: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ had $df = d\theta$, then

$$\frac{\partial f}{\partial x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} \qquad \frac{\partial f}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}$$

So $f(x,y) = \arctan\left(\frac{y}{x}\right)$. $\rightarrow\leftarrow$

Thus $d\theta$ is closed but not exact!

Cor: $\mathbb{R}^2 \setminus \{0\}$ is not smoothly contractible.

Orientations

Defn. A *local frame* for the vector bundle $E \xrightarrow{\pi} M$ is a locally trivial nbhd $U \subset M$ together with local sections $E_1, \dots, E_k : U \to E$ which are pointwise linearly independent.

Defn Given a f.d. vector space V, two bases $\{e_1, \dots, e_n\}$, $\{f_1, \dots, f_n\}$ are *cooriented* if the change-of-basis matrix from $\{e_1, \dots, e_n\}$ to $\{f_1, \dots, f_n\}$ has positive determinant.

Exercise Coorientation is an equivalence relation on the set of bases. There are exactly two equivalence classes, called *orientations*.

Defn An *orientation* on a manifold M is a continuous choice of orientation for each tangent space of M.

Continuous means there is a cover of M by domains of local frames which agree with the orientation.

A manifold is orientable if it admits an orientation.

Prop'n M is orientable iff there is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ M such that for all α, β,

$$\det\left(D(\varphi_\beta \circ \varphi_\alpha^{-1})\right) > 0 \quad \text{on} \quad \varphi_\alpha(U_\alpha \cap U_\beta)$$

Prop'n 1) If M is a connected n-manifold, then a nowhere-vanishing $\Omega \in \Lambda^n(M)$ determines an orientation on M.

2) Conversely, an orientation on M gives rise to a nowhere-vanishing n-form on M.

Proof 1) Locally, $\Omega = f\,dx^1 \wedge \dots \wedge dx^n$ for some $f \neq 0$. If the chart U is connected, $f \neq 0$ implies that either $f > 0$ or $f < 0$. If $f < 0$ replace x^1 by $-x^1$. Having chosen such coordinates,

Define an orientation on M by: $\{E_1, \dots, E_n\}$ is positive if $\Omega(E_1, \dots, E_n) > 0$.

2) Given an orientation, call $\Omega_p \in \Lambda^n(T_pM)$ "positive" if $\Omega(E_1, \dots, E_n) > 0$ for any positive basis.

The space $\Lambda_+ \subset \Lambda^n(T_pM)$ of positive n-covectors is convex, i.e. if $\Omega_1, \Omega_2 \in \Lambda_+$, so is

$$t\Omega_1 + (1-t)\Omega_2 \quad \text{for} \quad t \in [0,1].$$

Proposition If $E \xrightarrow{\pi} M$ is a vector bundle and $V \subset E$ is an open set so that

$$V_p = V \cap E_p \text{ is convex, nonempty in } E_p,$$

then there is a section $\sigma \in \Gamma(E)$ with $\sigma(p) \in V_p$ for all $p \in M$.

Proof of Prop'n

Cover M by local trivialisations $\{U_\alpha\}_{\alpha \in A}$.

For each α, choose some local section $\sigma_\alpha : U_\alpha \to E$

with $\sigma_\alpha(p) \in V_p$.
Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$.
Set $\sigma = \sum \psi_\alpha \sigma_\alpha$.
Then σ is a section of E, which at a point q is a convex combination of elements of V_q. ■

So the proposition gives a section of $\Lambda^n_+(M)$ ■

Proposition. Suppose TM is trivial. Then M is orientable.

Proof TM has a global frame $\{E_1, .., E_n\}$, and T^*M has a dual global frame $\{\omega^1, .., \omega^n\}$. So

$\Omega = \omega^1 \wedge .. \wedge \omega^n$ is a nowhere-vanishing n-form.

Eg. $\mathbb{RP}^n$ is orientable exactly when n is odd.

$\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$

Proposition The antipodal map $\alpha: S^n \to S^n$, $x \mapsto -x$

is orientation preserving exactly when n is odd.

(Defn A diffeomorphism F is orientation preserving if F_* takes positive bases to positive bases and negative bases to negative bases.)

Proof Since S^n is so symmetric, we'll work at $S = (0, .., 0, 1)$.
$\alpha(S) = N = (0, .., 0, 1)$. $\quad \sigma: S^n \setminus \{N\} \to \mathbb{R}^n$

In stereographic coordinates, $\tilde{\sigma} = -\sigma(-): S^n \setminus \{S\} \to \mathbb{R}^n$

$$\tilde{\sigma} \circ \alpha \circ \sigma^{-1}(u^1, .., u^n) = \tilde{\sigma}\left(-\frac{(2u, |u|^2 - 1)}{|u|^2+1}\right)$$
$$= -\sigma\left(\frac{(2u, |u|^2-1)}{|u|^2+1}\right) = -(u^1, .., u^n)$$

So really it's a question about the antipodal map on $\mathbb{R}^n$.

$\alpha: \mathbb{R}^n \to \mathbb{R}^n$
$x \mapsto -x$

If $E \in T_p\mathbb{R}^n$,

$$(\alpha_* E)(f) = E(f \circ \alpha) = E(f(-\cdot)) = -Ef$$

So $\alpha_*: \{E_1, .., E_n\} \mapsto \{-E_1, .., -E_n\}$.
The change of basis matrix is

$$\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$$

and its determinant is $(-1)^n$. ∎

Back to $\mathbb{RP}^n$:

If $\alpha: S^n \to S^n$ is orientation-preserving then given a point $[p] \in \mathbb{RP}^n$, we can lift to $p, -p \in S^n$ and use the orientation on either p or $-p$.

The Orientation Cover

Def'n A **covering space** is a triple $\tilde{X} \xrightarrow{\pi} X$ of two top. spaces and a continuous map such that: for every $p \in X$, there is a nbhd U of p, such that $\pi^{-1}(U)$ is a disjoint collection of open sets $\{\tilde{U}_\alpha\}$ in $\tilde{X}$ such that $\pi: \tilde{U}_\alpha \to U$ is a homeomorphism.

If $\tilde{X}$ and X are smooth manifolds and $\pi: \tilde{U}_\alpha \to U$ is a diffeomorphism, we call $\tilde{X} \xrightarrow{\pi} X$ a **smooth covering space**.

If for each $p \in M$, the collection of $\{U_\alpha\}$ is finite, we call that number the **degree** of the covering.

E.g. $S^1 \to S^1$
$z \mapsto z^k$

Covering space of degree k.

$(\cos\theta, \sin\theta) \mapsto (\cos k\theta, \sin k\theta)$

Theorem Given any smooth manifold M^n, there is a $\hat{M}^n$ which satisfies:

- $\hat{M} \to M$ is a covering of degree 2
- $\hat{M}$ is orientable
- M is orientable iff there is a global section of $\hat{M} \to M$
- If M is not orientable, $\hat{M}$ is connected.
- If M is orientable $\hat{M}$ is two disjoint copies of M.

Proof Idea: The points of $\hat{M}$ will be $(p, \pm)$.

Let $\Lambda^n_* M = \{(p, \Omega) \mid \Omega \in \Lambda^n(T_pM) \text{ is nonzero}\} \subset \Lambda^n M$.

If we consider $\mathbb{R}_+ = (0, \infty)$ as a group (with multiplication), there is an action of $\mathbb{R}_+$ on Λ^n_* by:

$$s \cdot (p, \Omega) = (p, s\Omega).$$

This action is smooth, free, and proper.

Smooth: The coordinates for $\Lambda^n M$ are given by

$$f\,dx^1 \wedge \dots \wedge dx^n \longmapsto (x^1, \dots, x^n, f)$$

So the action $s \cdot (p, \Omega)$ in coordinates is

$$(x^1, \dots, x^n, f) \longmapsto (x^1, \dots, x^n, sf)$$

which is smooth

free: (an action is free if the only group element which has a fixed point is the identity element)

proper: (an action is proper if for any compact $K \subset X$ has $G_K := \{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is compact in G.)

Given a compact $K \subset \Lambda^n_* M$, K is covered by finitely many preimages of local trivializations call them $\pi^{-1}(U_1), \dots, \pi^{-1}(U_k)$. In fact, K is covered by some $U_1 \times K_1, \dots, U_k \times K_k$ where $K_i \subset \mathbb{R} \setminus \{0\}$ are compact. Let $A = \min\{|x| \mid x \in K_i\}$, $B = \max\{|x| \mid x \in K_i\}$.

Then $\left(\frac{B}{A} + \varepsilon\right) \cdot K \cap K = \emptyset$ for any ε and $\left(\frac{A}{B} - \varepsilon\right) \cdot K \cap K = \emptyset$. With some choice of K_is, we can make sure $G_K = \left[\frac{A}{B}, \frac{B}{A}\right]$

Theorem. If $G \curvearrowright X$ is a smooth, free, proper action on a smooth manifold X, then X/G is a smooth manifold.

So $\Lambda^n_* M / \mathbb{R}_+$ is a smooth manifold of dimension n.

Moreover, the map $\Lambda^n_* M \xrightarrow{\pi} M$ descends to

$$\hat{M} = \Lambda^n_* M / \mathbb{R} \xrightarrow{\hat{\pi}} M$$

If U is a local trivial nbhd, then

$\hat{\pi}^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, \pm)$, so

$\hat{\pi}^{-1}(U) = U \times \{+, -\}$, hence $\hat{\pi}$ is a covering map

M is orientable iff $\hat{M}$ has a global section. But then $\hat{\pi}$ is a local diffeomorphism which is injective, hence a diffeomorphism.

Embeddings and Immersions and the IFT

Recall

Def'n An **immersion** is a smooth map $F: M \to N$ for which $(F_*)_p : T_pM \to T_{F(p)}N$ is injective for all $p \in M$.

* An **embedding** is an immersion which is globally one-to-one.

A **submersion** is a smooth map whose pushforward is surjective at each $p \in M$.

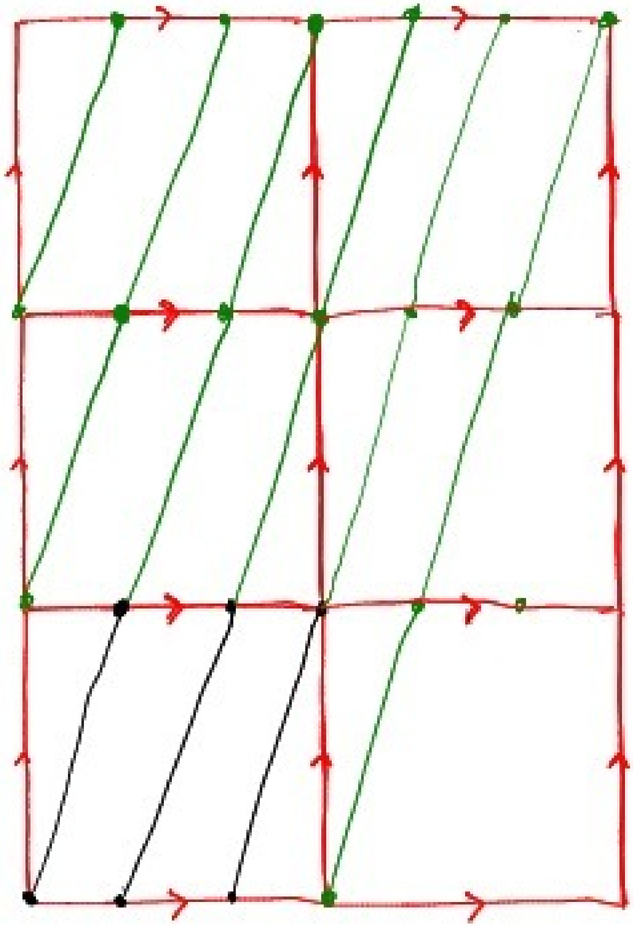
Eg. A curve $\gamma: J \to M$ is an immersion iff $\gamma'(t) = \gamma_*\left(\frac{d}{dt}\right) \neq 0$ for any $t \in J$.

So consider $\gamma: \mathbb{R} \to S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C} \times \mathbb{C}$

given by $t \mapsto (e^{2\pi i t}, e^{2\pi i 3t}) = (\cos(2\pi t), \sin(2\pi t), \cos(6\pi t), \sin(6\pi t))$

$$\gamma'(t) = -2\pi \sin(2\pi t)\frac{\partial}{\partial x^1} + 2\pi\cos(2\pi t)\frac{\partial}{\partial y^1} - 6\pi\sin(6\pi t)\frac{\partial}{\partial x^2} + 6\pi \sin(2\pi t)\frac{\partial}{\partial y^2}$$

which is never the zero vector.



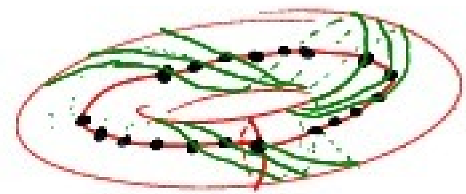
So this curve is an immersion, but it's periodic, hence <u>not</u> an embedding.

What if we consider the curve

$$\gamma_c(t) : \mathbb{R} \to S^1 \times S^1$$
$$t \mapsto (e^{2\pi i t}, e^{2\pi i c t})$$

for c irrational?

It's possible to show that, in fact, such an irrational curve's image is <u>dense</u> in the torus $S^1 \times S^1$!



Moreover, we can show that $\gamma : \mathbb{R} \to \gamma(\mathbb{R}) \subset S^1 \times S^1$ isn't even a homeomorphism onto its image!

So embedding as we've defined it above isn't all you might desire.

<u>New Defn</u> An injective immersion which is a homeomorphism onto its image ("topological embedding") is a (smooth) embedding.

Def: A map $F: M \to N$ is **proper** if, for any compact $K \subset N$, $F^{-1}(K) \subset M$ is compact.

Proposition An injective immersion $F: M \to N$ is a smooth embedding if F is proper, or if M is compact.

How do we get submanifolds?

Inverse Function Theorem (Euclidean)

Suppose $U, V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ and $F: U \to V$ has $DF(p)$ invertible. Then there are connected nbhds $U_0 \subset U$, $V_0 \subset V$ $p \in U_0$, $F(p) \in V_0$, and $G: V_0 \to U_0$ so that $G \circ F = \mathrm{Id}_{U_0}$ $F \circ G = \mathrm{Id}_{V_0}$ (G is as smooth as F was)

"invertibility of the derivative implies invertibility of the map, at least locally"

Inverse Function Theorem (manifolds)

Suppose M, N are smooth manifolds, $F: M \to N$ is a smooth map, and $p \in M$ has $(F_*)_p : T_pM \to T_{F(p)}N$ invertible. Then F is a local diffeo near p.

Cor If M and N have the same dimension and $F: M \to N$ is either a submersion or an immersion, then F is a local diffeo.

Implicit Function Theorem (Euclidean)

Suppose $U \underset{\text{open}}{\subseteq} \overset{x}{\mathbb{R}^n} \times \overset{y}{\mathbb{R}^k}$ and $\Phi: U \to \mathbb{R}^k$ has $\left(\frac{\partial \Phi}{\partial y}\right)$ nonsingular at $p = (a,b) \in U$. Then there are nbhds $\underset{a}{V_0} \subset \mathbb{R}^n$, $\underset{b}{W_0} \subset \mathbb{R}^k$, a smooth $\Psi: V_0 \to W_0$ so that $\Phi^{-1}(\Phi(p)) \cap (V_0 \times W_0)$ is the graph of Ψ

Proof idea: Let $F: U \to \mathbb{R}^n \times \mathbb{R}^k$ be
$(x,y) \mapsto (x, \Phi(x,y))$

Then $DF = \begin{pmatrix} \mathrm{Id} & 0 \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \end{pmatrix}$ Apply IFT to F.

Rank Theorem (Euclidean)

Suppose $U \underset{open}{\subseteq} \mathbb{R}^n$, $V \underset{open}{\subseteq} \mathbb{R}^m$, $F: U \to V$ smooth so that rank DF is constant (say $= k$) on U.
For any $p \in U$, there are charts $(\underset{p\in}{U_0}, \varphi)$, $(\underset{F(p)\in}{V_0}, \psi)$ so that $\varphi(p) = 0$, $\psi(F(p)) = 0$,

$$\psi \circ F \circ \varphi^{-1} : (x^1, \dots, x^k, x^{k+1}, \dots, x^n) \longmapsto (x^1, \dots, x^k, 0, \dots, 0)$$

(That is, in the new coordinates $F_* = \begin{pmatrix} Id_k & 0 \\ 0 & 0 \end{pmatrix}$.
So this is the smooth version of the fact from linear algebra: If $T: V \to W$ has rank k, then there are bases of V and W so that T written in those bases is $\begin{pmatrix} Id_k & 0 \\ 0 & 0 \end{pmatrix}$)

Rank Theorem (manifolds)

Same statement, but we replace $U \underset{open}{\subseteq} \mathbb{R}^n$ by M, $V \underset{open}{\subseteq} \mathbb{R}^m$ by N.

Defn If $U \underset{open}{\subseteq} \mathbb{R}^n$, a k-slice of U is a subset $S \subset U$ of the form $S = \{x \in U \mid x^{k+1} = c^{k+1}, x^{k+2} = c^{k+2}, \dots, x^n = c^n\}$
(Allowing for reordering, we can set any choice of $n-k$ coordinates to constants.)
Note that a k-slice is homeo to an open subset of $\mathbb{R}^k$.

Defn An embedded k-submanifold of M^n is a subset $S \subset M$ s.t. for every point $p \in S$, there is a chart (U, φ) of M for which $\varphi(S)$ is a k-slice of $\varphi(U)$.
The codimension of S is $n-k$

Lemma (Being an embedded submanifold is local in M)
Let M be a smooth manifold, $S \subset M$. If each $p \in S$ has a nbhd $_{p\in}U \subset M$ so that $S \cap U$ is an embedded k-submanifold of U, then S is an embedded k-submanifold of M.

Theorem (embedded submanifolds are images of embeddings)

Let $S \subset M$ be an embedded k-submanifold. Then there is a unique smooth structure for S so that the inclusion $\iota: S \hookrightarrow M$ is an embedding.

Proof. ① S is a topological k-manifold.

Automatically Hausdorff & second-countable.

Given slice coordinates (U, φ), define $V = U \cap S$,

$$\psi(p) = (\varphi^1(p), \dots, \varphi^k(p)).$$

Then verify (V, ψ) give a k-chart for S.

$$\psi^{-1}(x^1, \dots, x^k) = \varphi^{-1}(x^1, \dots, x^k, c^{k+1}, \dots, c^n)$$

② Check these coords are smoothly compatible.

③ Check $\iota: S \hookrightarrow M$ is smooth

(in coords above $\iota: (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, c^{k+1}, \dots, c^n)$)

④ Uniqueness

Exercise!

Theorem The image of an embedding is an embedded submanifold.

Proof This is just the Rank Theorem.

Embedded Submanifolds Are Precisely Images of Smooth Embeddings!

If $S \subset M$ is a k-submanifold, then for any $p \in S$, $X_p \in T_pS$, we can consider $\iota_{*p} X_p \in T_pM$.

In fact, this is a case where a vector field $X \in \mathfrak{X}(S)$ pushes forward along S to give a section of

$$\begin{array}{c} TM|_S = \{(p,v) \in TM \mid p \in S\} \\ \downarrow \\ S \end{array}$$

In coordinates $\iota_*: T_pS \to T_pM$ is

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$$

Prop^n $T_pS \subset T_pM$ is characterised by:

$$T_pS = \{ X_p \in T_pM \mid Xf = 0 \text{ for all } f \text{ with } f|_S \equiv 0 \}$$

Proof $(\subset)$ The condition $f|_S \equiv 0$ is $f \circ i \equiv 0$, so $(i_* X_p) f = X_p(f \circ i) = 0$

$(\supset)$

Integral Curves, Vector Fields and Flows

Defn Given a vector field $X \in \mathfrak{X}(M)$, we call $\gamma : J \to M$ an *integral curve* of X if, for each $t \in J$,

$$\gamma'(t) = V_{\gamma(t)}$$

E.g. Let $M = \mathbb{R}^2$, $X = x\frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$.

Then for $\gamma(t) = (x(t), y(t))$ to be an integral curve of X, we'd need

$$\gamma'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y} = X_{\gamma(t)} = x(t)\frac{\partial}{\partial x} + y(t)\frac{\partial}{\partial y}$$

So that $\frac{dx}{dt} = x \qquad \frac{dy}{dt} = y$

$$x(t) = x(0)e^t$$

$$y(t) = y(0)e^t$$

i.e. $\gamma(t) = e^t \gamma(0)$.

The integral curves of X are rays emanating from the origin

Integral curves are so called because we start with $\gamma' = X$ and try to find γ.

Finding integral curves amounts to solving an ODE given by the coordinate expression of X:

If $\{x^1, ..., x^n\}$ are coordinates we want to solve

$$\gamma'(t_0) = \gamma_* \frac{d}{dt}\Big|_{t_0} = \frac{d\gamma^i}{dt}\Big|_{t_0} \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)} = X^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)}$$

i.e. $\frac{d\gamma^i}{dt} = X^i(\gamma^1, ..., \gamma^n)$

Lemma Let X be a vector field, $\gamma : (\alpha, \omega) \to M$ an integral curve of X. Then for any $a \in \mathbb{R}$,

$$\delta : (\alpha + a, \omega + a) \to M$$
$$\delta(t) = \gamma(t - a)$$

is an integral curve of X.

Proof Consider the map $T_a : (\alpha, \omega) \to (\alpha + a, \omega + a)$
$t \longmapsto t + a$.

Then T_a is a diffeo, and

$$T_{a*} \frac{d}{dt} = \frac{d}{dt}.$$ ∎

So we can always assume $0 \in J$ when convenient.

We'd like to understand the family of all integral curves for a given vector field $X \in \mathfrak{X}(M)$.

Suppose, for the moment, that the vector field X has:

(*) For each $p \in M$, there is a unique integral curve $\theta^p : \mathbb{R} \to M$ with $\theta^p(0) = p$.

Then we could define $\theta_t : M \to M$ by $\theta_t(p) = \theta^p(t)$ so that $\theta_0 = id_M$.

By the Lemma, the map $\eta_s : t \mapsto \theta^p(t+s)$ is an integral curve for X. But $\eta_s(0) = \theta^p(s)$, and we've assumed uniqueness. So $\eta_s(t) = \theta^{\theta^p(s)}(t)$.

$$\theta_{t+s}(p) = \theta^p(t+s) = \theta^{\theta^p(s)}(t) = \theta_t(\theta^p(s))$$
$$= \theta_t(\theta_s(p))$$

i.e. $\boxed{\theta_{t+s} = \theta_t \circ \theta_s}$

Def'n A global flow is a group action of $(\mathbb{R}, +)$ on M:
$\theta : \mathbb{R} \times M \to M$

Given a global flow Θ, we can define curves

$$\Theta^p : \mathbb{R} \to M$$
$$t \mapsto \Theta(t,p)$$

That is, the image $\Theta^p(\mathbb{R}) \subset M$ is the orbit of p under the action Θ.

General Fact About Group Actions

The orbits of a group action are disjoint and cover.

Defn The infinitesimal generator of a global flow Θ is $X_\Theta \in \mathfrak{X}(M)$ given by

$$(X_\Theta)_p = (\Theta^p)'(0)$$

Prop'n The infintesimal generator of the global flow Θ is a smooth vector field. The orbits of Θ are integral curves of X_Θ.

Proof $(X_\Theta)_p f = \frac{d}{dt}\Big|_{t=0}\big(f(\Theta(t,p))\big) = \frac{\partial f}{\partial x^i}\frac{d\Theta^i}{dt}\Big|_{t=0}$

To show Θ^p is an integral curve for X_Θ, we need to show, for any $f \in C^\infty(M)$, $t_0 \in \mathbb{R}$

$$\frac{d}{dt}\Big|_{t=t_0}\big(f(\Theta^p(t))\big) \stackrel{?}{=} (X_\Theta)_{\Theta^p(t_0)} f = (X_\Theta)_q f$$

where $q = \Theta^p(t_0)$.

For any t, $\Theta^q(t) = \Theta^p(t_0+t)$. So

$$\frac{d}{dt}\Big|_{t=0}\big(f(\Theta^q(t))\big) = \frac{d}{dt}\Big|_{t=0}\big(f(\Theta^p(t_0+t))\big) = \frac{d}{dt}\Big|_{t=t_0}\big(f(\Theta^p(t))\big)$$

$\overset{\shortparallel}{(X_\Theta)_q f}$ by def'n. ∎

So a flow gives rise to a vector field, and a decomposition of M into the (disjoint) integral curves of that vector field. What about the converse?

Eg (Anything nonlinear.)

Let $X = x^3 \frac{\partial}{\partial x}$. Then the integral curve (in $\mathbb{R}^2$) of X has

$$\frac{dx}{dt} = x^3$$
$$\int \frac{dx}{x^3} = \int dt$$
$$-\tfrac{1}{2}x^{-2} = t + C$$
$$x(t) = \sqrt{C - 2t}$$

which stalls out at $x=0$ as $t \to \frac{1}{2}C$.

Defn A flow domain on M is an open subset $\mathcal{D} \subset \mathbb{R} \times M$ so that for each $p \in M$, $\mathcal{D}^p = \mathcal{D} \cap (\mathbb{R} \times \{p\}) = (a_p, b_p)$ for some $a_p < 0 < b_p$.

A flow on M with domain $\mathcal{D}$ is a map $\theta : \mathcal{D} \to M$ so that, if $s \in \mathcal{D}^p$, $t \in \mathcal{D}^{\theta(s,p)}$, and $s+t \in \mathcal{D}^p$, then
$$\theta(t, \theta(s,p)) = \theta(t+s, p)$$
and $\theta(0,p) = p$.

"(local) flow on M" "local one-parameter action"

<u>Proposition</u> The infinitesimal generator of a local flow is a vector field. Each $\theta^p : \mathcal{D}^p \to M$ is an integral curve.

Same proof, noting that derivatives are only local.

<u>Fundamental Theorem of ODE</u>

Let $U \underset{\text{open}}{\subset} \mathbb{R}^n$, $V : U \to \mathbb{R}^n$ smooth. For $t_0 \in \mathbb{R}$, $x \in U$, consider the IVP
$$(\dagger) \begin{cases} (\gamma^i)'(t) = V^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \gamma^i(t_0) = x^i \end{cases}$$

a) for each $t_0 \in \mathbb{R}$, $x_0 \in U$, there are an interval $J_0 \ni t_0$ and an open $x_0 \in U_0 \subset U$ so that for each $x \in U_0$, $(\dagger)$ has a solution $\gamma : J_0 \to U$

b) Any two smooth solutions to $(\dagger)$ agree on their common domain.

c) Define $\theta : J_0 \times U_0 \to U$ by
$$(t, x) \mapsto \gamma_x(t).$$
Then θ is smooth.

Think about how to leverage this to give a converse to the local-flows-generate-vector-fields prop'n.

Theorem (Fundamental Theorem for Flows)

Let X be a vector field on M. Then there is a unique maximal flow domain $\mathcal{D} \subset \mathbb{R} \times M$ and a unique local flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is X.

Defn We call θ the *flow generated by X.*

Proof Given X, $p \in M$, and $\{x^1, .., x^n\}$ coordinates around p, the integral curve equation in these coordinates is:

$$(\dagger) \begin{cases} (\tilde{\gamma}^i)'(t) = X^i(\gamma^1(t), .., \gamma^n(t)) \\ \tilde{\gamma}^i(0) = x^i(p) \end{cases}$$

So that $\tilde{\gamma} : J \to \mathbb{R}^n$ has $x \circ \gamma = \tilde{\gamma}$ $\quad \gamma = x^{-1} \circ \tilde{\gamma}$

The FTODE says we can solve this system, to get an integral curve $\gamma_p : J_p \to M$.

Now suppose $\gamma, \bar{\gamma} : J \to M$ are two integral curves for X which intersect at some $t_0 \in J$, ie

$$\gamma(t_0) = \bar{\gamma}(t_0) = p$$

In coordinates at p, $x \circ \gamma$ and $x \circ \bar{\gamma}$ are sol'ns to $(\dagger)$, hence by FTODE must agree in that chart.

"Method of Continuity"

Let $S = \{t \in J \mid \gamma(t) = \bar{\gamma}(t)\}$.

Then $t_0 \in S$, so S is nonempty.

We just showed S is open.

Since $\gamma, \bar{\gamma}$ are continuous, S is closed.

So $S = J$.

For $p \in M$, define $\mathcal{D}^p = \bigcup \{J \mid 0 \in J, \exists \gamma_p : J \to M \text{ s.t. } \gamma(0) = p, \gamma_p \text{ integral curve}\}$

and define $\theta^p : \mathcal{D}^p \to M$
$$t \mapsto \gamma_p(t)$$

Define $\mathcal{D} = \bigcup_p \mathcal{D}^p = \{(t,p) \mid t \in \mathcal{D}^p\}$,

$$\theta : \mathcal{D} \to M$$
$$(t,p) \mapsto \theta^p(t)$$

Then it's more or less clear that $\mathcal{D}, \theta$ satisfy all the properties we need, except $\mathcal{D}$ is open.

But again, this follows from a careful reading of FTODE. ∎

Theorem

1) If $s \in \mathcal{D}^p$, then $\mathcal{D}^{\theta(s)} = \mathcal{D}^p - s$

2) For each $t \in \mathbb{R}$, the set $M_t = \{p \in M \mid t \in \mathcal{D}^p\}$ is open in M. $\theta_t : M_t \to M_{-t}$ is diffeo with inverse θ_{-t}.

3) X is invariant under θ_t, i.e.
$$(\theta_{t*} X)_p = X_{\theta_t(p)}$$

Proof (of 3)

$$(\theta_{t_0 *} X)_p f = X_p(f \circ \theta_{t_0}) = \frac{d}{dt}\Big|_{t=0}(f \circ \theta_{t_0} \circ \theta^p(t))$$
$$= \frac{d}{dt}\Big|_{t=0}(f \circ \theta^p_{t_0+t})$$
$$= (\theta^p)'(t_0) f = X_{\theta_{t_0}(p)} f$$

Escape Lemma

Let X be a v.f. If γ is an integral curve whose domain is not all of $\mathbb{R}$, then the image of γ is not contained in any compact set.

Proof Suppose γ is defined on (a,b), $b<\infty$, and the image of γ lies in some compact K.

Let $t_i \nearrow b$. Then $\{\gamma(t_i)\}_{i\in\mathbb{N}}$ is a sequence in K, hence by compactness after passing to a subsequence, $\gamma(t_i) \to p \in K$

By the FT of flows, there are $\varepsilon, U \ni p$ so that $\theta : (-\varepsilon,\varepsilon) \times U \to M$ is a flow.

For each i, set
$$\sigma(t) = \begin{cases} \gamma(t), & a<t<b \\ \theta_{t-t_i} \circ \theta_{t_i}(p) & t_i - \varepsilon < t < t_i + \varepsilon \end{cases}$$

This σ is well-defined by uniqueness of integral curves. So γ can be extended to $t_i + \varepsilon > b$, hence b is not maximal.

We call an integral curve Complete if its domain is $\mathbb{R}$.

Cor All of a compact manifold's integral curves are complete.

Theorem Let $X \in \mathfrak{X}(M)$. If $X_p = 0$, then $\mathcal{D}^p = \mathbb{R}$ and the integral curve through p is constant.

If $X_p \neq 0$ then the integral curve $\theta^p : \mathcal{D}^p \to M$ is an immersion.

Proof $X_p = (\theta^p)'_0 \neq 0$. But we need to show $(\theta^p)'_t \neq 0$ for any $t \in \mathcal{D}^p$.

By the FT of flows $(\theta^p)'_t = X_{\theta^p(t)} = (\theta_t)_* X_p$

θ_t is a diffeo, so θ_t^* is iso. ∎

Canonical Form Theorem

If $X \in \mathfrak{X}(M)$ and $X_p \neq 0$, then there is a coordinate nbhd for p so that $X = \frac{\partial}{\partial x^1}$ and $\theta_t(x^1, \dots, x^n) = (x^1 + t, x^2, \dots, x^n)$

Lie Derivatives

Vector fields are defined as differential operators on functions.

But what if we want to differentiate something else, e.g. a vector field?

In $\mathbb{R}^n$, it makes sense to write

$$(D_X Y)_p = \langle X_p(Y^1), \dots, X_p(Y^n) \rangle$$
$$= \frac{d}{dt}\Big|_{t=0} \left(Y_{p+tX_p} \right) = \lim_{t \to 0} \frac{1}{t} \left[Y_{p+tX_p} - Y_p \right]$$

Notes:

- $(D_X Y)_p$ depends only on the value of X at p
- We can subtract $Y_{p+tX_p} - Y_p$ since we identify all tangent spaces of $\mathbb{R}^n$ with $\mathbb{R}^n = T_0 \mathbb{R}^n$.
- $p + tX_p \in \mathbb{R}^n$ again under the identification of $\mathbb{R}^n$ with its tangent spaces.

What role does $p+tX_p$ play?

It's a path from p in the direction X_p (and since calculus works, up to first order any such path should yield the same result.)

<u>Defn</u> The Lie derivative of $Y \in \mathfrak{X}(M)$ wrt to $X \in \mathfrak{X}(M)$ is
$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{1}{t}\left[(\Theta_{-t})_* Y_{\Theta_t(p)} - Y_p\right]$$
where Θ_t is the flow of X.

That is we do precisely what we did in $\mathbb{R}^n$, in the coordinates for which $X = \frac{\partial}{\partial x^1}$.

Since T_pM is a vector space, $(\mathcal{L}_X Y)_p$ should be a derivation (Exercise: prove this directly.)

In fact $p \mapsto (\mathcal{L}_X Y)_p$ gives a <u>smooth</u> vectorfield, since in coordinates $\{x^1, .., x^n\}$:

$$(\mathcal{L}_X Y)_p(x^i) = \lim_{t\to 0} \frac{1}{t}\left[(\Theta_{-t})_* Y_{\Theta_t(p)}(x^i) - Y_p x^i\right]$$
$$= \lim_{t\to 0} \frac{1}{t}\left[\frac{\partial \Theta^k_{-t}}{\partial x^j} Y^j_{\Theta_t(p)} \frac{\partial}{\partial x^k}(x^i) - Y^i_p\right]$$
$$= \lim_{t\to 0} \frac{1}{t}\left[\frac{\partial \Theta^i_{-t}}{\partial x^j} Y^j_{\Theta_t(p)} - Y^i_p\right]$$
Smooth if Θ_t, Y are.

<u>Theorem</u> $\mathcal{L}_X Y = [X,Y]$

<u>Proof</u> <u>Case:</u> $X_p \neq 0$.

By continuity, $X_q \neq 0$ in some nbhd of p.

We can pick coordinates $\{x^1, .., x^n\}$ so that
$$\Theta_t(x) = (x^1+t, x^2, .., x^n)$$
In these coordinates, $\Theta_{t*} = Id$

So
$$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0}\left[(Id)\, Y(x^1+t, .., x^n) - Y(x^1, .., x^n)\right]$$
$$= \frac{\partial Y}{\partial x^1} = \frac{\partial Y^i}{\partial x^1}\frac{\partial}{\partial x^i}$$

In these coordinates OTOH, $X = \frac{\partial}{\partial x^1}$. So the coordinate formula
$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$$

becomes
$$[X,Y] = \left[X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial}{\partial x^i}(\delta^{ij})\right]\frac{\partial}{\partial x^j}$$
$$= \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j}$$

Case: $p \in \operatorname{supp} X$
This follows from case 1 and continuity.

Case: p lies in an open set where $X \equiv 0$.
Then $[X,Y]=0$ by the coord. formula.
Also $\theta_t = \mathrm{id}$, so $\mathcal{L}_X Y = 0$. ∎

Corollary
1) $\mathcal{L}_X Y = -\mathcal{L}_Y X$
2) $\mathcal{L}_{[X,Y]} Z + \mathcal{L}_{[Z,X]} Y + \mathcal{L}_{[Y,Z]} X = 0$
3) $\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z$
4) $\mathcal{L}_X(fY) = f\mathcal{L}_X Y + X(f)Y$
5) $F_*(\mathcal{L}_X Y) = \mathcal{L}_{F_*X}(F_*Y)$ for a diffeo F.

Theorem The following are equivalent:
1) $[X,Y]=0$
2) $\mathcal{L}_X Y = 0$
3) $\mathcal{L}_Y X = 0$
4) Y is invariant under the flow of X
5) X is invariant under the flow of Y
6) The flows of X and Y commute, ie.
$\theta_t \circ \eta_s = \eta_s \circ \theta_t$ if s,t are times at which one side is defined.

Proof 1-3 follows from the previous thm.
4) $\Leftrightarrow$ $\frac{d}{dt}\left((\theta_{-t})_* Y_{\theta_t(p)}\right) = 0$
3) shows $= 0$ when $t=0$, need for all t.
So use the group law to translate time:
$$\left.\frac{d}{dt}\right|_{t=t_0}\left((\theta_{-t})_* Y_{\theta_t(p)}\right) = \left.\frac{d}{ds}\right|_{s=0}\left((\theta_{-t_0-s})_* Y_{\theta_{s+t_0}(p)}\right)$$
$$= \left.\frac{d}{ds}\right|_{s=0}(\theta_{-t_0})_* \theta_{-s*} Y_{\theta_s(\theta_{t_0}(p))}$$

$$= (\Theta_{-t_0})_* \frac{d}{ds}\Big|_{s=0} \Theta_{-s*} Y_{\Theta_s(\Theta_{t_0}(p))}$$

$$= (\Theta_{-t_0})_* (\mathcal{L}_X Y)_{\Theta_{t_0}(p)} = 0.$$

$4) \Leftrightarrow 5) \Leftrightarrow 6)$ is an exercise. ∎

Theorem Suppose $X_1, \dots, X_k$ are linearly independent vector fields on M, such that $[X_\alpha, X_\beta] \equiv 0$. Then for each $p \in M$, there are coordinates $\{x^1, \dots, x^n\}$ so that $X_\alpha = \frac{\partial}{\partial x^\alpha}$ in a nbhd of p.

Proof. The statement is local, so we may assume $M = \mathbb{R}^n$, $p = 0$.

Then the $\{X_\alpha\}$ being indep. allows us to arrange $X_\alpha|_0 = \frac{\partial}{\partial x^\alpha}\Big|_0$

Define χ by $\chi(t^1, \dots, t^n) = \Theta^1_{t^1} \circ \Theta^2_{t^2} \cdots \circ \Theta^k_{t^k}(0, \dots, 0, t^{k+1}, \dots, t^n)$

Then $\chi_*\left(\frac{\partial}{\partial t^\alpha}\Big|_0\right) = \begin{cases} X_\alpha|_0 & \alpha = 1, \dots, k \\ \frac{\partial}{\partial x^\alpha}\Big|_0 & \alpha = k+1, \dots, n \end{cases}$,

i.e. $\chi_*|_0 = \mathrm{id}$. So by IFT, χ gives a coord. system $x = \chi^{-1}$ near 0

Also $\frac{\partial}{\partial x^1} = X_1$.

But since $[X_\alpha, X_\beta] = 0$, the flows also commute. So we can reorder and get $\frac{\partial}{\partial x^\lambda} = X^\lambda$. ∎

Defn If $X \in \mathcal{X}(M)$, $\omega \in \Lambda^k(M)$,

$$(\mathcal{L}_X \omega)_p = \frac{d}{dt}\Big|_{t=0} \Theta_t^*(\omega_{\Theta_t(p)}) = \frac{d}{dt}\Big|_{t=0} (\Theta_t^* \omega)_p$$

Proposition $\mathcal{L}_X$ obeys every conceivable Leibniz rule.

In particular, if $\sigma \in \Lambda^k(M)$ and $Y_1, \dots, Y_k \in \mathcal{X}(M)$,

$$\mathcal{L}_X(\sigma(Y_1, \dots, Y_k)) = (\mathcal{L}_X \sigma)(Y_1, \dots, Y_k) + \sum_{j=1}^k \sigma(Y_1, \dots \mathcal{L}_X Y_j, \dots, Y_k)$$

Equiv,

$$(\mathcal{L}_X \sigma)(Y_1, \dots, Y_k) = X(\sigma(Y_1, \dots, Y_k)) - \sum_{j=1}^k \sigma(Y_1, \dots, [X, Y_j], \dots, Y_k)$$

Cor If $g \in C^\infty$, $X \in \mathcal{X}$, then $d(\mathcal{L}_X g) = \mathcal{L}_X(dg)$

Proof $(\mathcal{L}_X dg)(Y) = X(dg(Y)) - dg([X, Y])$

$$= X(Y(g)) - [X, Y](g) = YX(g)$$

$$(d\mathcal{L}_X g)(Y) = d(X(g))(Y) = YX(g) \;∎$$

Cartan's Formula

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega)$$

Proof. Case: $\omega \in \Lambda^0(M) = C^\infty(M)$.

$$\mathcal{L}_X \omega = X(\omega) \qquad X \lrcorner d\omega = X(\omega) \qquad X \lrcorner \omega = 0$$

Case: $\omega \in \Lambda^1(M)$

Since both sides are linear over $\mathbb{R}$ in ω suffices to prove for ω of the form $f\,dg$.

$$\mathcal{L}_X(f\,dg) = X(f)\,dg + f\,\mathcal{L}_X(dg)$$

$$X \lrcorner (d(f\,dg)) + d(X \lrcorner f\,dg)$$

$$= X \lrcorner (df \wedge dg) + d(f\,X(g))$$

$$= X(f)\,dg - \cancel{X(g)\,df} + \cancel{df\,X(g)} + f\,d(X(g))$$

$$= X(f)\,dg + f\,d(\mathcal{L}_X g)$$

Case: $\omega \in \Lambda^k$, $k > 1$.

Again by linearity prove for $\omega = \alpha \wedge \beta$ for $\alpha \in \Lambda^1$, $\beta \in \Lambda^{k-1}$.

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$$

$$= (X \lrcorner d\alpha + d(X \lrcorner \alpha)) \wedge \beta + \alpha \wedge (X \lrcorner d\beta + d(X \lrcorner \beta))$$

$$= (X \lrcorner d\alpha) \wedge \beta \;\textcircled{1} + (d(X \lrcorner \alpha)) \wedge \beta \;\textcircled{2}$$

$$+ \alpha \wedge (X \lrcorner d\beta) \;\textcircled{3} + \alpha \wedge (d(X \lrcorner \beta)) \;\textcircled{4}$$

$$X \lrcorner d(\alpha \wedge \beta) + d(X \lrcorner (\alpha \wedge \beta))$$

$$= X \lrcorner (d\alpha \wedge \beta - \alpha \wedge d\beta) + d((X \lrcorner \alpha) \wedge \beta - \alpha \wedge (X \lrcorner \beta))$$

$$= X \lrcorner (d\alpha \wedge \beta) - X \lrcorner (\alpha \wedge d\beta)$$

$$+ (d(X \lrcorner \alpha)) \wedge \beta + (X \lrcorner \alpha) \wedge d\beta - d\alpha \wedge (X \lrcorner \beta)$$

$$+ \alpha \wedge d(X \lrcorner \beta)$$

$$= (X \lrcorner d\alpha) \wedge \beta \;\textcircled{1} + \cancel{d\alpha \wedge (X \lrcorner \beta)} - \cancel{(X \lrcorner \alpha) \wedge d\beta}$$

$$+ \alpha \wedge (X \lrcorner d\beta) \;\textcircled{3}$$

$$+ (d(X \lrcorner \alpha)) \wedge \beta \;\textcircled{2} + \cancel{(X \lrcorner \alpha) \wedge d\beta} - \cancel{d\alpha \wedge (X \lrcorner \beta)}$$

$$+ \alpha \wedge d(X \lrcorner \beta) \;\textcircled{4} \qquad \blacksquare$$

Cor $d\mathcal{L}_X\omega = \mathcal{L}_X d\omega$

Proof. $\mathcal{L}_X d\omega = X \lrcorner (d^2\omega) + d(X\lrcorner d\omega) = d(X\lrcorner d\omega)$

$$d\mathcal{L}_X\omega = d(X\lrcorner d\omega + d(X\lrcorner\omega))$$
$$= d(X\lrcorner d\omega)$$

Exercise: Simpler, more conceptual proof?

Integral Manifolds

Defn A *k-distribution* (or *k-plane field*) D is an assignment, to each $p\in M$, of a k-dimensional subspace $D_p \subset T_pM$.

A k-distribution is *smooth* if ...?

$$G_k(TM) = \{(p,V) \mid V\subset T_pM \text{ k-dim'l subspace}\}$$

$$\begin{tikzcd} G_k(TM) \arrow[d] \\ M \arrow[u, dashed, bend left, "D"] \end{tikzcd}$$

"Grassmann bundle"

Lemma D is smooth if for every $p\in M$ there are a nbhd U and smooth local vector fields on U $Y_1,\dots,Y_k$ so that for each $q\in U$, $\{Y_1|_q,\dots,Y_k|_q\}$ span D_q.

The $\{Y_1,\dots,Y_k\}$ are a *local frame* for D.

E.g. Consider $\Sigma^k \times N$.
Then $T_p\Sigma \xhookrightarrow{i_*} T_{(p,q)}(\Sigma\times N)$ gives a k-subspace for each $(p,q)\in\Sigma\times N$.

E.g. $M=\mathbb{R}^n\setminus\{0\}$, $D_p=\{v_p\in T_p\mathbb{R}^n \mid v_p\cdot(p^1,\dots,p^n)=0\}$

Defn An immersed submanifold Σ is an *integral submanifold* for D if $i_*T_p\Sigma = D_p$ for all $p\in\Sigma$

E.g. $\Sigma\times\{q\}\subset\Sigma\times N$
$S^{n-1}(|p|)\subset\mathbb{R}^n\setminus\{0\}$

But! Not all distributions have integral submanifolds

E.g. $\mathbb{R}^3$, $D_p=\ker(dz-x\,dy)$
$$=\{V \mid dz(V)=x\,dy(V)\}$$
$$=\{V^1\tfrac{\partial}{\partial x}+V^2\tfrac{\partial}{\partial y}+V^3\tfrac{\partial}{\partial z} \mid V^3(x,y,z)=x\,V^2(x,y,z)\}$$
D_p is clearly 2-dimensional, but has no integral submanifolds!

Defn D is *integrable at p* if D has an integral submanifold through p.
D is *integrable* if it is integrable at each p.

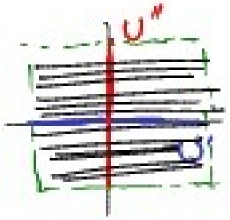
Defn D is *involutive* if, for any $X,Y\in D$, $[X,Y]\in D$.

Prop'n Suppose D is integrable. Then D is involutive.

Proof Let $X,Y\in D$, $p\in M$, Σ the integral submanifold through p.
Then $X=i_*\tilde X$, $Y=i_*\tilde Y$ for $\tilde X,\tilde Y\in\mathcal{X}(\Sigma)$.
So $[X,Y]_p=[i_*\tilde X,i_*\tilde Y]_p=i_*[\tilde X,\tilde Y]\in i_*T_p\Sigma=D_p$

Lemma D is involutive if it is involutive on local frames.
That is, if for every $p\in M$ there is a local frame $\{X_1,\dots,X_k\}$ for D with $[X_i,X_j]\in D$, then D is involutive.
So only need to check $\binom{k}{2}$, not ∞ly many.

Defn A distribution is **completely integrable** if it is integrable and for each p, there are coordinates (U, φ) so that $\varphi(U)$ is a product $U' \times U'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $D = \text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\}$ and the integral manifolds of D given by:



Frobenius Theorem

Involutive $\Rightarrow$ completely integrable.

Proof The Canonical Form Theorem shows that the distribution spanned by a frame of commuting vector fields is completely integrable. So we'll show

D involutive $\Rightarrow \exists$ local commuting frame for D

The conclusion is local (i.e. about the existence of particular coordinates) so work at $\vec{0} \in \mathbb{R}^n$.

$D_0 \subset T_0\mathbb{R}^n$ can, by linear algebra, be written as

$D_0 = \text{span}\{\frac{\partial}{\partial t^1}\big|_0, \dots, \frac{\partial}{\partial t^k}\big|_0\}$ for some coords $(t^1, \dots, t^n)$.

Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ to be projection on the first k.

Then $\pi_*|_0 \, D_0 \to T_0\mathbb{R}^k$ is an isomorphism.

By continuity of π_*, $\pi_*|_q : D_q \to T_{\pi(q)}\mathbb{R}^k$ is one-to-one for q near 0.

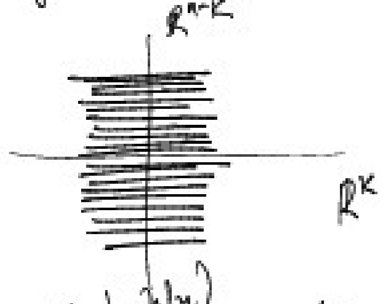
So there are $X_i|_q$ with $\pi_*|_q X_i|_q = \frac{\partial}{\partial t^i}\big|_{\pi(q)}$

So $\pi_*|_q [X_i, X_j]_q = [\pi_* X_i, \pi_* X_j]_{\pi(q)} = [\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}]\big|_{\pi(q)} = 0.$

So $[X_i, X_j]_q = 0.$

Thus these $\{X_1, \dots, X_k\}$ are a commuting local frame. ∎

Defn A k-submanifold Σ of M is a *foliation* of M if for each $p \in M$, there are coordinates (U, x) so that the components of $\Sigma \cap U$ are the images of:



Σ is (usually highly) disconnected, since it has dimension k and fills out M. A connected component of Σ is called a *leaf* or *folium*.

Frobenius+ Suppose D is an involutive distribution on M. Then M is foliated by an integral submanifold of D.

Proof By Frobenius, cover M by countably many charts $\{(U_\alpha, x_\alpha)\}$ so that $D|_{U_\alpha} = \text{span}\{\frac{\partial}{\partial x^1_\alpha}, \dots, \frac{\partial}{\partial x^k_\alpha}\}$.

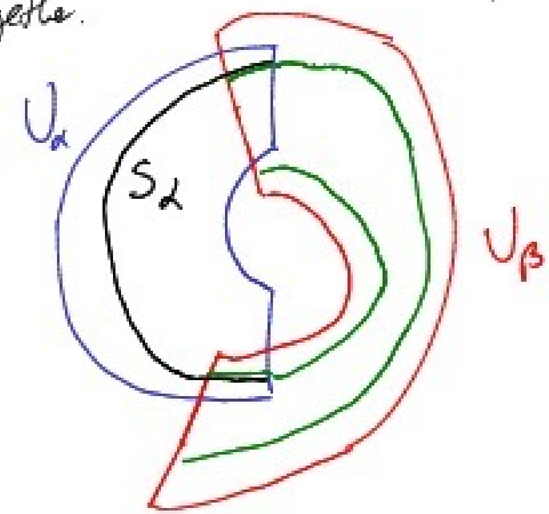
If Σ is any integral submanifold for D, then in these coordinates, and $i: \Sigma \hookrightarrow M$, then in these coordinates

$$d(x^l \circ i)(X_k) = X_k(x^l \circ i) = (i_* X_k)(x^l) = \frac{\partial}{\partial x^k}(x^l) = \delta^l_k$$

$= 0$ if $k \neq l$.

In particular, $x^l \circ i \equiv$ constant on Σ. So any integral submanifold of D is a horizontal slice in these coordinates.

Suppose S_α is a slice of U_α. Need to patch such S_α together.



S_α could intersect more than one slice of U_β.
But $S_\alpha \cap U_\beta$ is a manifold so it has only countably many components.

Since there are countably many U_α, if we start at p, and just start tacking on $S \cap U_\beta$, there will only be countably many tackings-on.

Exercise: A countable union of connected manifolds which overlap in open sets is a connected manifold.

So we can extend S^p to a maximal integral submanifold.

Take $\Sigma = \bigcup_{p \in M} S^p$ as the foliation.

Integration

Warning: This is not how Lee does it!

Def'n A singular k-cell σ on M^n is a smooth map $\sigma : [0,1]^k \to M$

Singular 0-cells are points.

Singular 1-cells are curves.

Def'n If $\omega \in \Omega^k_0(\mathbb{R}^k)$ is a k-form on $\mathbb{R}^k$ with compact support, we define

$$\int_{\mathbb{R}^k} \omega = \int_{\operatorname{supp}\omega} f(x^1,\dots,x^k)\,dx^1\cdots dx^k$$

ordinary multi-variable integral

where $\omega = f\,dx^1\wedge\dots\wedge dx^k$

Lemma If $\{y^1,\dots,y^k\}$ are other coordinates for $\mathbb{R}^k$, and $\det \frac{\partial x}{\partial y} > 0$, and $\omega = \tilde{f}\,dy^1\wedge\dots\wedge dy^k$, then

$$\int_{\operatorname{supp}\omega} f(x^1,\dots,x^k)\,dx^1\cdots dx^k = \int_{\operatorname{supp}\omega} \tilde{f}(y^1,\dots,y^k)\,dy^1\cdots dy^k$$

This is just the change-of-variables formula for multivariable integration!

Def'n If $\omega \in \Lambda^k(M)$ and $\sigma: [0,1]^k \to M$ is a singular k-cell, define

$$\int_\sigma \omega = \int_{[0,1]^k} \sigma^* \omega$$

Lemma If $p: [0,1]^k \to [0,1]^k$ is an orientation-preserving diffeo, $\sigma: [0,1]^k \to M$, and $\omega \in \Lambda^k(M)$, then

$$\int_\sigma \omega = \int_{\sigma \circ p} \omega$$

If p is orientation-reversing, then

$$\int_\sigma \omega = -\int_{\sigma \circ p} \omega$$

Def'n A singular k-chain on M is a (finite) linear combination of singular k-cells on M.

If $\sigma = \sum_{i=1}^N a_i \sigma_i$ is a singular k-chain, $\omega \in \Lambda^k(M)$

$$\int_\sigma \omega = \sum_{i=1}^N a_i \int_{\sigma_i} \omega$$

$[0,1]^k$ has $2k$ faces, each of which is $[0,1]^{k-1}$

$I_{1,0} = \{(0, x^2, \ldots, x^k)\}$ $\quad I_{1,1} = \{(1, x^2, \ldots, x^k)\}$

$I_{2,0} = \{(x^1, 0, x^3, \ldots, x^k)\}$ $\quad I_{2,1} = \{(x^1, 1, x^3, \ldots, x^k)\}$

$\vdots$

$I_{k,0}$ $\quad I_{k,1}$

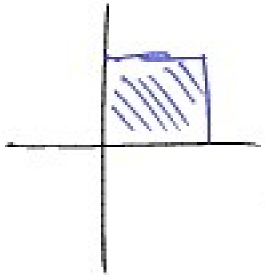
Def'n If $\sigma: [0,1]^k \to M$ is a k-cell, we define the $(k-1)$-cells $\sigma_{i,0}$, $\sigma_{i,1}$ by:

$$\sigma_{i,0} = \sigma\big|_{I_{i,0}} : [0,1]^{k-1} \to M$$

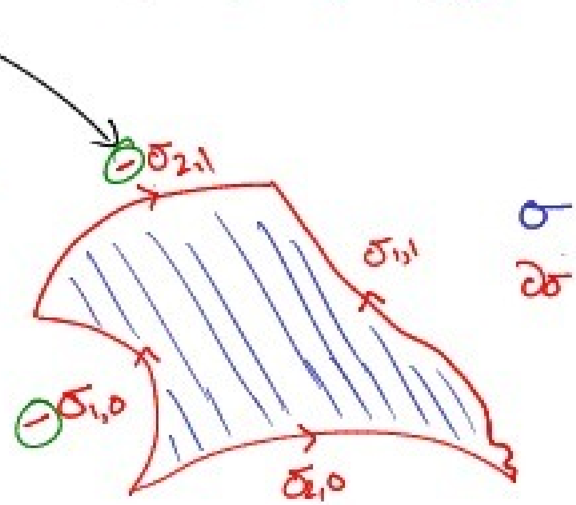
$$\sigma_{i,1} = \sigma\big|_{I_{i,1}} : [0,1]^{k-1} \to M$$

The boundary of σ, $\partial\sigma$, is the $k-1$ cell

$$\partial\sigma = \sum_{i=1}^k (-1)^{i+1} [\sigma_{i,1} - \sigma_{i,0}]$$



$$\partial\sigma = \sigma_{1,1} - \sigma_{1,0} - (\sigma_{2,1} - \sigma_{2,0})$$
$$= \sigma_{1,1} - \sigma_{1,0} - \sigma_{2,1} + \sigma_{2,0}$$

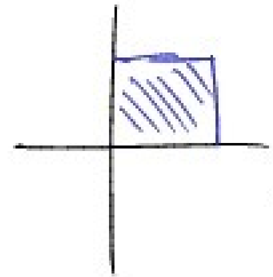


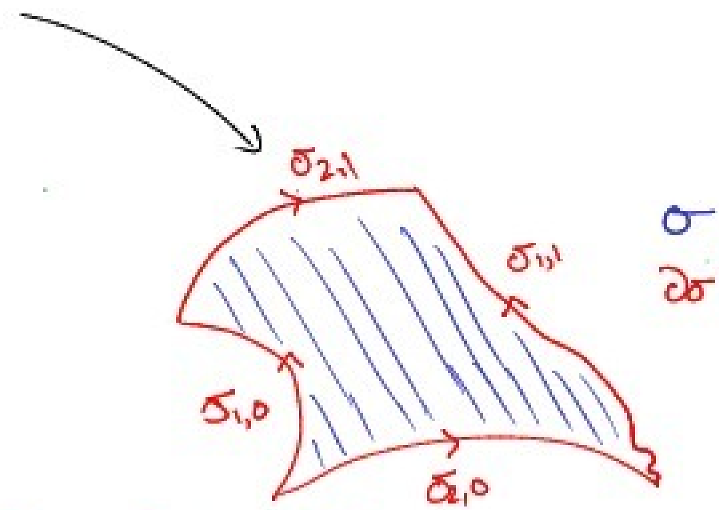
Our choice of signs for the boundary cells is consistent with expectations.

The boundary operator ∂ extends by linearity:

$$\partial\left(\sum_{i=1}^{N} a_i \sigma_i\right) = \sum_{i=1}^{N} a_i \partial\sigma_i$$

We define $\partial(\text{any } 0\text{-chain}) = 1$.





$$\partial^2\sigma =$$
$$= \partial\sigma_{1,1} - \partial\sigma_{1,0} - \partial\sigma_{2,1} + \partial\sigma_{2,0}$$
$$= (\sigma_{1,1})_{1,1} - (\sigma_{1,1})_{1,0} - \left((\sigma_{1,0})_{1,1} - (\sigma_{1,0})_{1,0}\right)$$
$$- \left((\sigma_{2,1})_{1,1} - (\sigma_{2,1})_{1,0}\right) + (\sigma_{2,0})_{1,1} - (\sigma_{2,0})_{1,0}$$
$$= \sigma(1,1) - \sigma(1,0) - \sigma(0,1) + \sigma(0,0)$$
$$- \sigma(1,1) + \sigma(0,1) + \sigma(1,0) - \sigma(0,0)$$
$$= 0$$

Proposition $\partial^2 = 0$

Proof Just bookkeeping, really.

It suffices to show that

$$(\sigma_{i,\alpha})_{j,\beta} = (\sigma_{j+1,\beta})_{i,\alpha} \quad \text{where } 1 \le i,j \le k,\ \alpha,\beta \in \{0,1\}$$

Then in $\partial^2\sigma = \partial\left(\sum_{i=1}^{k}(-1)^{i+1}(\sigma_{i,1} - \sigma_{i,0})\right)$

$$= \sum_{i=1}^{k}(-1)^{i+1}(\partial\sigma_{i,1} - \partial\sigma_{i,0})$$

$$= \sum_{i=1}^{k}(-1)^{i+1}\sum_{j=1}^{k-1}(-1)^{j+1}\left[(\sigma_{i,1})_{j,1} - (\sigma_{i,1})_{j,0}\right]$$

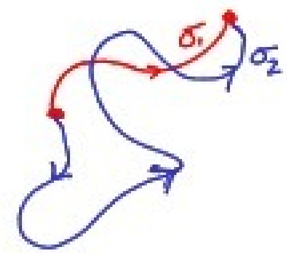
$$- \sum_{i=1}^{k}(-1)^{i+1}\sum_{j=1}^{k-1}(-1)^{j+1}\left[(\sigma_{i,0})_{j,1} - (\sigma_{i,0})_{j,0}\right]$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k-1}(-1)^{i+j+\alpha+\beta}(\sigma_{i,\alpha})_{j,\beta}$$

Swapping i,j gives a (-1), so terms cancel in pairs.

Defn If $\partial\sigma = 0$, we call σ closed.

$\sigma = \sigma_1 - \sigma_2$ is closed.



If $\sigma = \partial\tau$, we call τ an exact boundary (or just a boundary)

Stokes Theorem for k-chains

integral of a $(k+1)$-form over a $(k+1)$-chain

$$\int_{\sigma} d\omega = \int_{\partial\sigma}\omega$$

integral of a k-form over a k-chain

Proof

Case: σ is the standard singular k-cell in $\mathbb{R}^k$

Then $\omega \in \Lambda^{k-1}(\mathbb{R}^k)$, so $\omega = f\, dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^k$

Then $d\omega = df \wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^k$

$(-1)^{i-1}\frac{\partial f}{\partial x^i}\, dx^1\wedge\cdots\wedge dx^i\wedge\cdots\wedge dx^k$

$$\int_\sigma d\omega = \int_{[0,1]^k} (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^i \wedge \dots \wedge dx^k$$

$$= \int_{[0,1]^k} \frac{\partial f}{\partial x^i} \, dx^1 \dots dx^k$$

$$= \left[\int_{[0,1]^{k-1}} f(x^1, \dots, \underset{i\text{th}}{1}, \dots, x^{k-1}) dx^1 \dots dx^{k-1} - \int_{[0,1]^{k-1}} f(x^1, \dots, \underset{i\text{th}}{0}, \dots x^{k-1}) \right]$$

$$\int_{\partial\sigma} \omega = \sum_{j=1}^{k} (-1)^{j+1} \left(\int_{\sigma_{j,1}} \omega - \int_{\sigma_{j,0}} \omega \right)$$

If we consider $dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$ along $\sigma_{j,\alpha}$ for $i \neq j$, the fact that $x^j \equiv \alpha$ along $\sigma_{j,\alpha}$ means $dx^j = 0$.
So $dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k = 0$. Only $\sigma_{i,\alpha}$ contributes:

$$\int_{\partial\sigma} f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k = (-1)^{i+1} \left[\int f(x^1, \dots, 1, \dots, x^k) - \int f(x^1, \dots, 0, \dots, x^k) \right]$$

If σ is some other k-cell $[0,1]^k \to M$, then

$$\int_\sigma d\omega = \int_{[0,1]^k} \sigma^* d\omega = \int_{[0,1]^k} d(\sigma^* \omega) = \int_{\partial[0,1]^k} \sigma^* \omega = \int_{\partial\sigma} \omega$$

Theorem Suppose σ_1, σ_2 are n-cells which can be extended to diffeos on a nbhd, $\omega \in \mathcal{A}^n(M)$ has $\mathrm{supp}(\omega) \subset \sigma_1([0,1]^n) \cap \sigma_2([0,1]^n)$.

Suppose also M is oriented and σ_1, σ_2 are orientation-preserving.

Then $\int_{\sigma_1} \omega = \int_{\sigma_2} \omega$

This follows from invariance under reparametrisation

If M is oriented and $\mathrm{supp}(\omega) \subset \sigma([0,1]^n)$, then $\int_\sigma \omega$ is indep. of σ. So we write $\int_M \omega$.

Def'n Given an oriented M^n there is a cover of M by open sets $\{U_\alpha\}_{\alpha\in A}$ with $U_\alpha \subset \sigma_\alpha([0,1]^n)$ for some one-to-one n-cell $\sigma_\alpha : [0,1]^n \to M$ which is orientation-pres.
Let $\{\phi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to $\{U_\alpha\}_\alpha$.
For $\omega \in \Lambda_0^n(M)$, define

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \cdot \omega$$

finite sum since ω is compactly supported

Prop'n This definition is independent of the cover $\{U_\alpha\}$.

Proof. If $\{V_\beta\}$, $\{\psi_\beta\}$ are another cover/partition of unity, then

$$\sum_\alpha \int_M \phi_\alpha \omega = \sum_\alpha \int \phi_\alpha \sum_\beta \psi_\beta \omega$$

finite sums can be interchanged!

$$= \sum_\alpha \sum_\beta \int_M \phi_\alpha \psi_\beta \omega = \sum_\beta \sum_\alpha \int \phi_\alpha \psi_\beta \omega$$

$$= \sum_\beta \int \psi_\beta \sum_\alpha \phi_\alpha \omega = \sum_\beta \int \psi_\beta \omega$$

Notice this means $\int_M \omega$ depends on the orientation for M.

Def'n If Σ is an immersed k-submanifold of M^n and $\omega \in \Lambda_0^k(M)$, we define

$$\int_\Sigma \omega = \int_\Sigma i^*\omega$$

So a k-submanifold gives a map

$$\int_\Sigma : \Lambda_0^k(M) \longrightarrow \mathbb{R}$$

$$\omega \longmapsto \int_\Sigma \omega$$

So we can include $\{k\text{-submanifolds}\} \subset \left(\Lambda_0^k(M)\right)^*$
But clearly not every element of $\Lambda_0^k(M)$ is a smooth k-submanifold. Here starts geometric measure theory.

Stokes' Theorem for Manifolds

Suppose M^n is an oriented manifold-with-boundary, $\omega \in \Lambda_0^{n-1}(M)$.
Then

$$\int_M d\omega = \int_{\partial M} \omega$$

Any manifold is a manifold-with-boundary ($\partial M = \emptyset$)

So

Cor. If M^n is a manifold and $\omega \in A_0^{n-1}(M)$, then
$$\int_M d\omega = 0.$$

In order to define $\int_{\partial M} \omega$, ∂M must be given an orientation.

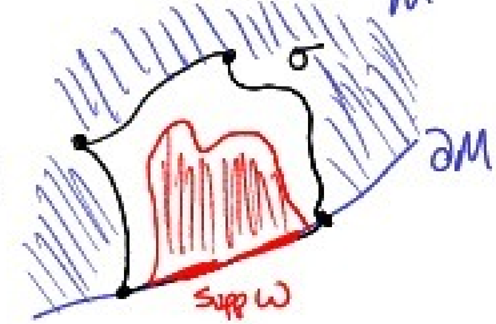
Defn If M^n is a manifold-with-boundary with orientation μ, we define an orientation $\partial\mu$ on ∂M by $\{v_1|_p, \dots, v_{n-1}|_p\} \in (\partial\mu)_p$ if $\{\nu_p, v_1|_p, \dots, v_{n-1}|_p\} \in \mu_p$ where $\nu|_p = -\frac{\partial}{\partial x^n}\big|_p$ for the boundary chart.

Exercise Show this is a well-defined orientation.

Proof of Stokes' Theorem

Case supp ω is contained in the interior of a bijective n-cell σ which is disjoint from ∂M. Then $\operatorname{supp} d\omega \subset \operatorname{supp}\omega \subset \operatorname{int} \sigma([0,1]^n)$, so we have
$$\int_M d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = 0 = \int_{\partial M} \omega$$

Case. supp ω is contained in the interior of a bijective n-cell σ which intersects ∂M only along the face $\sigma_{n,0}$.



In $\partial\sigma$, the coefficient of $\sigma_{n,0}$ is $(-1)^n$.

So by Stokes for cells,
$$\int_M d\omega = \int_\sigma d\omega = (-1)^n \int_{\sigma_{n,0}} \omega$$

With our choice of orientation $\partial\mu$ on ∂M, the map $\sigma_{n,0} : [0,1]^{n-1} \to \partial M$ is orientation-preserving if n is even, and orientation-reversing if n is odd. (**Exercise** Sort this out.)

So $\int_{\partial M} \omega = (-1)^n \int_{\sigma_{n,0}} \omega$.

General Case

Cover M with a cover $\{U_\alpha\}$ and a subordinate partition of unity $\{\phi_\alpha\}$ so that each $\phi_\alpha \omega$ is one of the above cases.

Note that $d\left(\sum \phi_\alpha\right) = d(1) = 0$

So $\sum_\alpha (d\phi_\alpha \wedge \omega) = \left(\sum_\alpha d\phi_\alpha\right) \wedge \omega = 0$

Thus

$$\int_M d\omega = \sum_\alpha \int_M \phi_\alpha \, d\omega = \sum_\alpha \int_M d\phi_\alpha \wedge \omega + \phi_\alpha d\omega$$

$$= \sum_\alpha \int_M d(\phi_\alpha \omega) = \sum_\alpha \int_{\partial M} \phi_\alpha \omega = \int_{\partial M} \omega$$ ∎

Stokes' Theorem tells that the duality between forms and submanifolds includes their boundary operations.

Cor. A compact oriented manifold (without boundary) is not contractible to a point.

Proof. Since M is oriented, $\exists$ some positive n-form ω. $d\omega = 0$ since $\omega \in \Lambda^n(M)$.

So ω is <u>closed</u>. If M were contractible, then ω would be exact, i.e. $\omega = d\eta$ for some η. So

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0$$

OTOH, ω is positive, so $\int_M \omega > 0$. $\rightarrow\leftarrow$. ∎

Stokes' Theorem and its sidekick, the Poincaré Lemma allows us to understand the <u>shape</u> of a manifold in terms of its forms.

de Rham Cohomology

Def's On a manifold M, define $Z^k(M) = \{\omega \in \Lambda^k \mid d\omega = 0\}$

$B^k(M) = \{d\eta \mid \eta \in \Lambda^{k-1}\}$ and

$$H^k(M) = Z^k(M) / B^k(M)$$

quotient vector space

$[\omega] = \{\omega + d\eta \mid \eta \in \Lambda^{k-1}\}$

k^{th} de Rham Cohomology group

$$\Lambda^0(M) \xrightarrow{d^0} \Lambda^1(M) \xrightarrow{d^1} \Lambda^2(M) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-2}} \Lambda^{n-1}(M) \xrightarrow{d^{n-1}} \Lambda^n(M)$$

$$H^k(M) = \ker d^k / \operatorname{im} d^{k-1}$$

Z^k and B^k are really big. The exception:

Lemma $H^0(M) = \mathbb{R}^{\#\text{ of components of } M}$

Proof. $B^0(M) = d(\Lambda^{-1}(M)) = 0$

$Z^0(M) = \{f \in C^\infty(M) \mid df = 0\}$

$= \{\text{constant functions}\}$

Restatement of Poincaré Lemma

$H^k(M) = 0$ if M is contractible and $k \geq 1$.

Proof. $H^k(M) = 0$ means

$\underset{\text{closed } k\text{-forms}}{Z^k(M)} = B^k(M) =$ exact k-forms

Proposition If M^n is a connected compact oriented manifold, then $H^n(M) = \mathbb{R}$.

Proof By orientability, there is some positive form $\omega \in \Lambda^n(M)$. As before, $\omega \notin d(\Lambda^{n-1}(M))$, so $\dim H^n(M) \geq 1$.

On the other hand, $\dim H^n(M) \leq \dim Z^n(M) \leq \dim \Lambda^n(M) = \infty$

Hmm... we need to fix this

Defn The de Rham cohomology groups with compact support are $H^k_c(M) = Z^k_c(M) / B^k_c(M)$

where $Z^k_c(M) = \{\omega \in \Lambda^k_0(M) \mid d\omega = 0\}$

$B^k_c(M) = \{d\eta \mid \eta \in \Lambda^{k-1}_0(M)\}$

ie the homology of the cochain complex

$$\to \Lambda^0_0 \xrightarrow{d} \Lambda^1_0 \xrightarrow{d} \Lambda^2_0 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n_0 \to .$$

$B^k_c(M)$ is not the space of exact forms with compact support.

Proposition. $H^1_c(\mathbb{R}) = \mathbb{R}$

Proof By the argument above, $\dim H^1_c(\mathbb{R}) \geq 1$

We'll show that $\int_{\mathbb{R}} : H^1_c(\mathbb{R}) \to \mathbb{R}$, $[\omega] \mapsto \int_{\mathbb{R}} \omega$ is injective.

Suppose $\int_{\mathbb{R}} \omega = 0$. Since $\mathbb{R}$ is contractible, the Poincaré Lemma says $\omega = df$ for some $f \in C^\infty(\mathbb{R})$.

Then $\int_{\mathbb{R}} \omega = \int_{\mathbb{R}} df = f(\infty) - f(-\infty)$.

Since ω has compact support, df is constant outside some $[-N, N]$, and $f(+\infty) = f(N)$, $f(-\infty) = f(-N)$.

So $f(N) = f(-N)$. Then $\omega = df = d(f - f(N))$ and $f - f(N) \in C^\infty_c$. ∎

Theorem If M^n is connected & oriented then $H^n_c(M) = \mathbb{R}$

Proof. We've shown this for $\mathbb{R}$.

Lemma If $H^n_c(\mathbb{R}^n) = \mathbb{R}$, then $H^n_c(M) = \mathbb{R}$ for any connected oriented n-manifold.

Proof Let $\omega \in \Lambda^n_c(M)$ be a form with support contained in some open set diffeo. to $\mathbb{R}^n$, with $\int_M \omega > 0$. (E.g. $\phi\, dx^1 \wedge \dots \wedge dx^n$, where $x^1, \dots, x^n$ are coords and ϕ is a bump fn for their chart.)

Given any other $\tilde{\omega} \in \Lambda^n_c(M)$, we want to show $\tilde{\omega} = c\omega + d\eta$ for some $\eta \in \Lambda^{n-1}_c(M)$.

Since $\operatorname{supp} \tilde{\omega}$ is compact, we can write $\tilde{\omega} = \phi_1 \tilde{\omega} + \dots + \phi_k \tilde{\omega}$ where $\{\phi_\alpha\}$ are a partition of unity subordinate to a cover by diffeomorphic copies of $\mathbb{R}^n$.

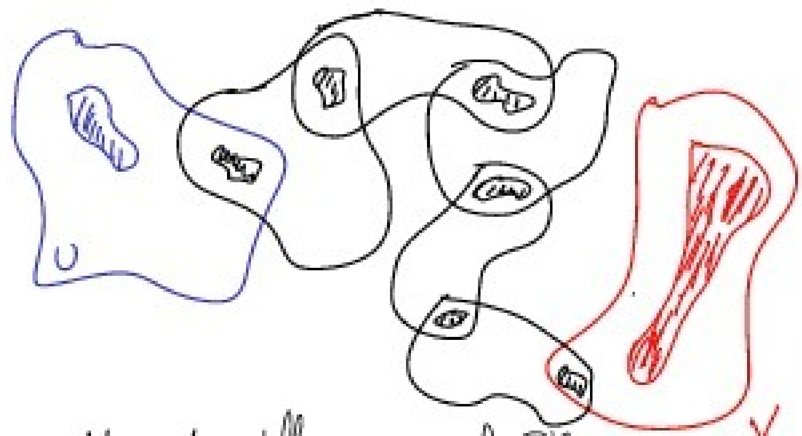
So each $\phi_i\tilde{\omega}$, like ω, is supported in $\mathbb{R}^n$.

If $\phi_i\tilde{\omega} = c_i\omega + d\eta_i$, we can write

$$\omega = \sum \phi_i\tilde{\omega} = \left(\sum c_i\right)\omega + d\left(\sum \eta_i\right)$$

and we'll be done.

So WLOG suppose $\tilde{\omega}$ has support contained in a diffeo copy of $\mathbb{R}^n$, say V.



Choose $V_1,\dots,V_r$ diffeo. copies of $\mathbb{R}^n$ with $V_i \cap V_j \neq \emptyset$, $V_1 = U$, $V_r = V$.

Let ω_i be a form supported in $V_i \cap V_{i+1}$, $\int_M \omega_i \neq 0$

Then $\omega_1 = c_1\omega + d\eta_1$

$$\omega_2 = c_2\omega_1 + d\eta_2$$
$$\vdots$$
$$\omega_{r-1} = c_{r-1}\omega_{r-2} + d\eta_{r-1}$$
$$\tilde{\omega} = c_r\omega_{r-1} + d\eta_r$$

So $\tilde{\omega} = \left(\prod_{i=1}^{r} c_i\right)\omega + d\left(c_r\cdots c_2\eta_1 + c_r\cdots c_3\eta_2 + \dots + \eta_r\right)$

Hence $[\tilde{\omega}] = \left(\prod_{i=1}^{r} c_i\right)[\omega]$ as required. ∎

Since $H^1_c(\mathbb{R}) = \mathbb{R}$, we've shown also $H^1_c(S^1) = \mathbb{R}$.

Strategy: If $H^n_c(S^n) = \mathbb{R}$, then $H^{n+1}_c(\mathbb{R}^{n+1}) = \mathbb{R}$.

To relate forms on S^n to forms on $\mathbb{R}^{n+1}$, we'll use polar coordinates.

Consider the form on $\mathbb{R}^{n+1}$: $\sigma = \sum_{i=1}^{n+1}(-1)^{i-1}x^i dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^{n+1}$

($n=1$ $\quad x\,dy - y\,dx = r^2 d\theta$)

There is also a map $r: \mathbb{R}^{n+1}\setminus\{0\} \to S^n$, $v \mapsto \frac{v}{|v|}$ with the property that $r|_{S^n} = \mathrm{id}_{S^n}$

(ie. r is a retraction).

Consider $r^*(\sigma|_{S^n}) = \sigma' \in \Omega^n(\mathbb{R}^{n+1}\setminus\{0\})$.

($n=1$ $\quad \sigma' = \frac{1}{r^2}\sigma = d\theta$)

Lemma For $p \in \mathbb{R}^{n+1} \setminus \{0\}$, $(r^* \sigma')(p) = \frac{\sigma(p)}{|p|^{n+1}}$

Proof Let $v_1, \ldots, v_n$ be vectors at p.

$$\frac{\sigma(p)}{|p|^{n+1}}(v_1, \ldots, v_n) = \frac{\det(p, v_1, \ldots, v_n)}{|p|^{n+1}}$$

OTOH, $(r^* \sigma')(p)(v_1, \ldots, v_n) = \sigma'\left(\frac{p}{|p|}\right)\left(r_* |_p v^1, \ldots, r_* |_p v^n\right)$

We'll prove equality on a spanning set.

If any of the v^i is parallel to the position vector p, then LHS $= 0$. Also, $r_*|_p p = 0$, so RHS $= 0$ in this case.

If all v^i are $\perp$ to the position vector, then they are tangent to the sphere of radius $|p|$. But r_* shrinks this sphere to have radius 1, so $r_*|_p v = \frac{1}{|p|} v_{r(p)}$ for such a vector.

Total of $(n+1)$ factors of $|p|$, so

$$(r^* \sigma')(p)(v_1, \ldots, v_n) = \frac{1}{|p|^{n+1}} \det(p, v^1, \ldots, v^n)$$

Lemma (Spherical integration)

Suppose $f : B^{n+1}(1) \to \mathbb{R}$, and define $g : S^n \to \mathbb{R}$ by

$$g(p) = \int_0^1 r^n f(rp)\, dr$$

Then $\int_B f = \int_{S^n} \int_0^1 r^n f(rp)\, dr\, \sigma' = \int_{S^n} g \sigma'$.

(Proof as Exercise)

Now we'll show $\int_{\mathbb{R}^{n+1}} : H_c^{n+1}(\mathbb{R}^{n+1}) \to \mathbb{R}$ is injective.

Suppose $\omega = f\, dx^1 \wedge \ldots \wedge dx^{n+1}$ is a compactly supported $n+1$-form. WLOG assume $\operatorname{supp} \omega \subset B^{n+1}(1)$.

If $0 = \int_{\mathbb{R}^{n+1}} \omega = \int_{B(1)} f = \int_{S^n} g\sigma'$, then $g\sigma' \in \ker \int_{S^n}$

so by inductive hypothesis, $g\sigma' = d\lambda$ for some $\lambda \in A^{n-1}(S^n)$.

Now define a n-form on $\mathbb{R}^{n+1}$ by

$$\eta(p) = \sum_{i=1}^{n+1} (-1)^{i-1} \left(\int_0^1 r^n f(rp)\, dr \right) x^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^{n+1}$$

We know that $\omega = d\eta$ for some η by Poincaré Lemma, but it is helpful to have this η.

$$d\eta\big|_p = \sum_{i=1}^{n+1} \left(\frac{\partial}{\partial x^i}\left(\int_0^1 r^n f(r\cdot p)\,dr\; x^i\right)\right) dx^1\wedge\cdots\wedge dx^{n+1}$$

$$= \sum_{i=1}^{n+1}\left[\int_0^1 r^{n+1}\frac{\partial f}{\partial x^i}(r\cdot p)\,\frac{\partial}{\partial x^i}p\,dr\; x^i + \int_0^1 r^n f(r\cdot p)\,dr\right]dx^1\wedge\cdots\wedge dx^{n+1}$$

$$= \left[(n+1)\int_0^1 r^n f(r\cdot p)\,dr + \int_0^1 r^{n+1}\,\nabla f(rp)\cdot p\,dr\right]dx^1\wedge\cdots\wedge dx^{n+1}$$

IBP

$$= \left[r^{n+1} f(r\cdot p)\Big|_{r=0}^{r=1}\right] dx^1\wedge\cdots\wedge dx^{n+1}$$

$$= f(p)\,dx^1\wedge\cdots\wedge dx^{n+1} = \omega\big|_p.$$

Claim: $r^*(g\sigma') = \eta$

So $\eta = r^*(g\sigma') = r^*(d\lambda) = d(r^*\lambda)$

Now let ϕ be a bump function for the complement of $B^{n+1}(1)$.

$\omega = d\eta = d(\eta - d(\phi r^*\lambda))$.

Claim: $\eta - d(\phi r^*\lambda) \in \Omega_c(\mathbb{R}^{n+1})$

On the complement of $B^{n+1}(1)$, $\eta = d(r^*\lambda) = d(\phi r^*\lambda)$ ∎

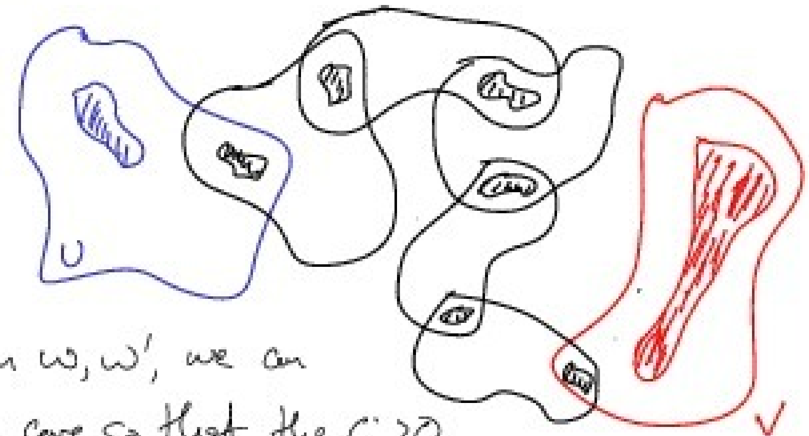
Thus $\omega \in d(\Omega_c^n(\mathbb{R}^{n+1}))$, so $[\omega] = 0$ in $H_c^{n+1}(\mathbb{R}^{n+1})$

Then, inductively, we are done. ∎

Cor. If M^n is a compact oriented manifold, $H^n(M) = \mathbb{R}$

Theorem If M^n is a connected nonorientable manifold, $H^n_c(M) = 0$.

Proof Go back to the argument



Given ω, ω', we can take care so that the $c_i > 0$, so that $[\omega'] = c[\omega]$ for some $c > 0$. Since M is not orientable, there is some sequence of charts $U = V_1, \dots, V_r = U$ for which the orientation gets reversed somewhere along the line. This allows us to relate ω and $-\omega$. So $[-\omega] = c[\omega]$ for some $c > 0$. That is, $-[\omega] = c[\omega]$, $c > 0 \Rightarrow [\omega] = 0$.

Homotopy Invariance of de Rham Cohomology

If $F: M \to N$ and $G: M \to N$ are homotopic smooth maps, then $F^* = G^*: H^k(N) \to H^k(M)$.

Def'n

$$\begin{tikzcd}
{} \arrow[r, "d"] & \mathcal{A}^{p-1}(M) \arrow[r, "d"] & \mathcal{A}^{p}(M) \arrow[r] & \mathcal{A}^{p+1}(M) \arrow[r] & {} \\
{} \arrow[r] \arrow[u, "h"'] & \mathcal{A}^{p-1}(N) \arrow[r, "d"] \arrow[u, "F^*-G^*"'] \arrow[ul, "h"] & \mathcal{A}^{p}(N) \arrow[r] \arrow[u, "F^*-G^*"'] \arrow[ul, "h"] & \mathcal{A}^{p+1}(N) \arrow[r] \arrow[u, "F^*-G^*"'] \arrow[ul, "h"] & {} \arrow[ul, "h"]
\end{tikzcd}$$

Given two cochain homotopy maps $F^*, G^*: \mathcal{A}(N) \to \mathcal{A}(M)$, we call a map $h: \mathcal{A}^p(N) \to \mathcal{A}^{p-1}(M)$ a cochain homotopy between F^* and G^* if the diagram above parallelograms commute, i.e.

$$F^* - G^* = h \circ d + d \circ h$$

Lemma If F^* and G^* are cochain-homotopic, then they induce the same map $H^k(N) \to H^k(M)$.

Proof Suppose $[\omega] \in H^k(N)$, so $d\omega = 0$. Then

$$F^*[\omega] = [F^*\omega] = [G^*\omega + h(\cancelto{0}{d\omega}) + d(h\omega)]$$
$$= [G^*\omega] = G^*[\omega] \quad \blacksquare$$

Proposition Any smooth homotopy induces a cochain homotopy on forms.

Proof (We've seen this before.)

Case. Consider a manifold M and $M \times I$. Then $i_0: M \to M \times \{0\}$ and $i_1: M \to M \times \{1\}$ are homotopic via $id_{M \times I}$.

We want $h: \mathcal{A}^p(M \times I) \to \mathcal{A}^{p-1}(M)$ which is a cochain homotopy.

$$h\omega = \int_0^1 \left(\frac{\partial}{\partial t} \lrcorner\, \omega\right) dt$$

In coordinates if $\omega = f(x,t)\, dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}} + g(x,t)\, dx^{j_1} \wedge \dots \wedge dx^{j_p}$,

then $h\omega = \int_0^1 f(x,t)\, dt\; dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$

so $d(h\omega) = \int_0^1 \frac{\partial f}{\partial x^i}(x,t)\, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$

$$d\omega = \frac{\partial f}{\partial x^i}(x,t)\, dx^i \wedge dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}} + \frac{\partial g}{\partial t}(x,t)\, dt \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p} + \frac{\partial g}{\partial x^i}(x,t)\, dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

$$h(d\omega) = \int_0^1 (-1)\frac{\partial f}{\partial x^i}\, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$
$$+ \int_0^1 \frac{\partial g}{\partial t}(x,t)\, dt\; dx^{i_1} \wedge \dots \wedge dx^{i_p}$$
$$= -d(h\omega) + g(x,1)\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$
$$- g(x,0)\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$
$$= -d(h\omega) + i_1^*\omega - i_0^*\omega$$

So $h(d\omega) + d(h\omega) = (i_1^* - i_0^*)\omega$ and h is a cochain homotopy from i_0^* to i_1^*.

<u>General case</u>

If $F, G: M \to N$ are smoothly homotopic then

$F: M \xrightarrow{i_0} M \times I \xrightarrow{H} N$

$G: M \xrightarrow{i_1} M \times I \xrightarrow{H} N$

Define $\tilde{h}: A^p(N) \to A^{p-1}(M)$ by
$\tilde{h} = h \circ H^*$

Then $\tilde{h}(d\omega) + d(\tilde{h}\omega) = h\, H^* d\omega + d\, h H^*\omega$
$= h\, d(H^*\omega) + d\, h(H^*\omega)$
$= i_1^* H^*\omega - i_0^* H^*\omega = G^*\omega - F^*\omega$ ∎

<u>Theorem</u> de Rham cohomology is a homotopy invariant.
That is, homotopic maps induce the same map
homotopy-equivalent manifolds have the same cohomology.

(X, Y are homotopy equivalent if $\exists$ $F: X \to Y$, $G: Y \to X$
so that $F \circ G \simeq id_Y$
$G \circ F \simeq id_X$)

<u>Proof</u> First we need the following:

<u>Whitney Approximation Theorem</u>

Any continuous map H between smooth manifolds is homotopic to a smooth map.
If H is smooth on a closed set, we can take the homotopy to fix that closed set.

Our homotopy equivalence has $F \circ G \simeq id_Y$,
so there is some $H: Y \times I \to Y$ with $H(0) = id_Y$, $H(1) = F \circ G$

We apply Whitney to the homotopy $H: Y\times I\to Y$ to get a smooth map $\tilde{H}: Y\times I\to Y$ which agrees with H on the closed sets $Y\times\{0\}$ and $Y\times\{1\}$.

This $\tilde{H}$ is a smooth homotopy between id_Y and $F\circ G$. Then $(F\circ G)^*: H^k(Y)\to H^k(Y)$ is $id_Y^* = id$.

Sim. for $G\circ F$.

So $H^k(X)\cong H^k(Y)$. ▨

Defn A topological space X is **simply connected** if every continuous map $S^1\to X$ is homotopic to a constant map.

Prop. If M is a compact simply connected smooth manifold, $H^1(M)=0$.

Proof 1 Let $[\omega]\in H^1(M)$. Given any loop $\gamma: S^1\to M$, (by Whitney WLOG take everything smooth) there is a homotopy to $\gamma_0: S^1\to\{x_0\}\subset M$.

Clearly $\gamma_0^*: H^1(M)\to H^1(S^1)=\mathbb{R}$ is the zero map.

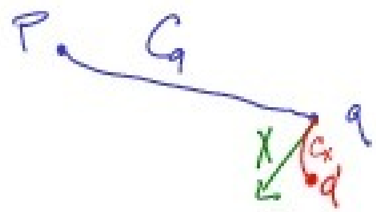
So γ^* is the zero map.

This means that $\int_\gamma \omega = 0$.

(Recall from calculus)

Lemma If ω is a 1-form so that $\int_\gamma \omega = 0$ for any loop γ, then $\omega = df$ for some $f\in C^\infty(M)$.

Proof Pick any $p\in M$ and define $f(q)=\int_{C_q}\omega$, where C_q is a curve starting at p and ending at q.



$$df_q(X) = X_q(f) = \lim_{t\to 0}\frac{f(q')-f(q)}{t} = \lim_{t\to 0}\frac{1}{t}\int_{C_X}\omega = \omega_q(X)$$ ▨

Proof 2 If γ is a contractible loop, the homotopy is a map from $S^1\times[0,1]$ which crushes $S^1\times\{1\}$.

So we get a map $\sigma: I^2\to M$, ie a 2-cell.

The boundary of this 2-cell is (a chainified) γ.

By Stokes', $\int_\gamma\omega = \int_{\partial\sigma}\omega = \int_\sigma d\omega$.

But $d\omega$ is closed by assumption. So $\int_\gamma \omega = 0$.

Then apply the lemma as in Proof 1. ▨

The Mayer-Vietoris Sequence

Def'n A sequence of linear maps

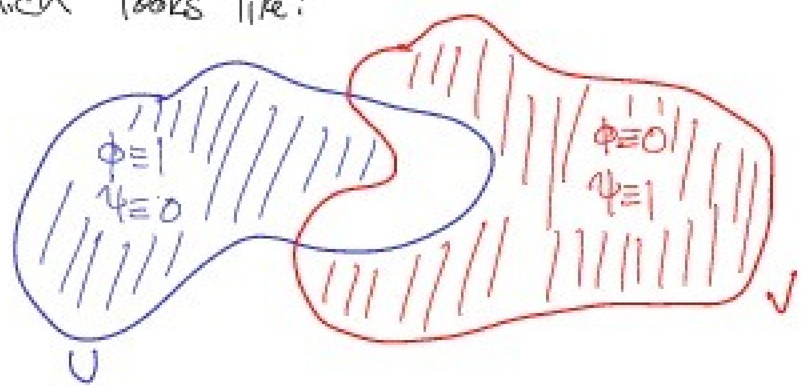
$$\cdots \to V^{p-1} \xrightarrow[F_{p-1}]{} V^{p} \xrightarrow[F_{p}]{} V^{p+1} \to \cdots$$

is exact if $\ker F_p = \operatorname{im} F_{p-1}$

E.g. $0 \xrightarrow{0} V^1 \xrightarrow{A} V^2 \xrightarrow{B} V^3 \xrightarrow{0} 0$ exact means:

1. $\ker A = \operatorname{im} 0 = 0$ so A is injective.
2. $\operatorname{im} B = \ker 0 = V^3$ so B is surjective
3. $\ker B = \operatorname{im} A$

Now let's suppose we have a manifold M which looks like:



Pick a partition of unity ϕ, ψ as above.

We have the diagram of inclusions (which commutes by def'n of "inclusion")

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

Define the sequence

$$* \quad 0 \to \Lambda^p(M) \xrightarrow{A} \Lambda^p(U) \oplus \Lambda^p(V) \xrightarrow{B} \Lambda^p(U\cap V) \to 0$$

$$\sigma \longmapsto (k^*\sigma, l^*\sigma)$$

$$(\alpha, \beta) \longmapsto i^*\alpha - j^*\beta$$

Lemma (*) is exact.

Proof The maps k^*, l^*, i^*, j^* are just restrictions.

- If $A\sigma = 0$, then $k^*\sigma = \sigma|_U = 0$
  $l^*\sigma = \sigma|_V = 0$
  So $\sigma = 0$. Thus A is injective.
- $B \circ A = 0$, so $\operatorname{im} A \subset \ker B$
  Suppose $(\alpha, \beta) \in \ker B$. Then $\alpha|_{U\cap V} = \beta|_{U\cap V}$.
  So define $\sigma(p) = \begin{cases} \alpha(p) & p \in U \\ \beta(p) & p \in V \end{cases}$.
  Then $A\sigma = (\alpha, \beta)$. So $\ker B \subset \operatorname{im} A$.

- Let $\omega \in A^p(U\cap V)$. Define $\alpha = \begin{cases} \psi\omega, & U\cap V \\ 0, & U\setminus V \end{cases}$

and $\beta = \begin{cases} -\phi\omega, & U\cap V \\ 0 & V\setminus U \end{cases}$

Then $i^*\alpha - j^*\beta = \psi\omega - -\phi\omega = (\psi+\phi)\omega = \omega$.

Thus B is surjective. $\blacksquare$

Now this holds for each p, and the maps A & B commute with d. So we have a "short exact sequence of cochain complexes."

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & A^{p-1}(M) \arrow[r] \arrow[d, "A"] & A^p(M) \arrow[r] \arrow[d, "A"] & A^{p+1}(M) \arrow[r] \arrow[d, "A"] & {} \\
\arrow[r] & A^{p-1}(U)\oplus A^{p-1}(V) \arrow[r] \arrow[d, "B"] & A^p(U)\oplus A^p(V) \arrow[r] \arrow[d, "B"] & A^{p+1}(U)\oplus A^{p+1}(V) \arrow[r] \arrow[d, "B"] & {} \\
\arrow[r] & A^{p-1}(U\cap V) \arrow[r] \arrow[d] & A^p(U\cap V) \arrow[r] \arrow[d] & A^{p+1}(U\cap V) \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

<u>Algebraic Fact</u> Any short exact sequence of complexes induces a "long" exact sequence in cohomology.

<u>Mayer-Vietoris Theorem</u> If M is a smooth manifold with $M = U\cup V$ where U and V are open subsets. Then there is a long exact sequence:

$$\xrightarrow{B} H^{k-1}(U\cap V) \xrightarrow{\delta} H^k(M) \xrightarrow{A} H^k(U)\oplus H^k(V) \xrightarrow{B} H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(M) \xrightarrow{A}$$

The map δ is called the "connecting map", and it's where all the juice is!

But what does δ mean geometrically? It turns out we can define δ on the level of forms nicely.

Given $\omega \in A^{p-1}(U\cap V)$, we want to define $\sigma = \delta\omega \in A^p(M)$.

Only need this for closed ω for cohomological purposes.

Given ω with $d\omega = 0$, let α, β be as above, so that $i^*\alpha - j^*\beta = \omega$. Then $(d\alpha)|_{U\cap V} = i^*d\alpha = d\,i^*\alpha$

$$= d(j^*\beta + \omega) = j^*d\beta = (d\beta)|_{U\cap V}$$

So $d\alpha \in \Lambda^p(U)$ and $d\beta \in \Lambda^p(V)$ agree on $U \cap V$.

$\operatorname{supp} d\alpha \subset \operatorname{supp} \alpha \subset U$

$\operatorname{supp} d\beta \subset \operatorname{supp} \beta \subset V$

Thus $\operatorname{supp} d\alpha \subset U \cap V$, so we can extend $d\alpha$ to M by 0 outside $U \cap V$. Call this extension σ, and set $\delta\omega = \sigma$

Check exactness: $(k^* \oplus \ell^*)(\delta\omega) = (d\alpha, d\beta) \in \Lambda^p(U) \oplus \Lambda^p(V)$

So $\operatorname{im} \delta \subset \ker (k^* \oplus \ell^*)$

On the other hand, if $[\sigma] \in \ker(k^* \oplus \ell^*)$, then $\sigma|_U = d\eta$, $\sigma|_V = d\nu$, and on $U \cap V$, $d\eta|_{U\cap V} = \sigma|_{U \cap V} = d\nu|_{U\cap V}$ so $\sigma|_{U\cap V}$ has a primitive $\omega = i^*\eta$. But by our definition, $\delta[\omega] = [\sigma]$. So $\ker(k^* \oplus \ell^*) \subset \operatorname{im} \delta$.

Let's use the Mayer-Vietoris sequence to compute some cohomology.

<u>Proposition</u> The de Rham cohomology of the spheres is

$$H^k(S^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & 1 \le k < n \\ \mathbb{R} & k = 0 \end{cases}$$

<u>Proof</u> We have $k=0$ and $k=n$ already, so the proposition holds for S^1.

Let's try the Mayer-Vietoris sequence for S^n given by $U = S^n \setminus \{N\}$, $V = S^n \setminus \{S\}$. Then by stereographic projection, $U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^n$. Moreover, $U \cap V \cong \mathbb{R}^n \setminus \{0\}$

So $\begin{cases} H^k(U) = 0 \text{ for } k > 1 \\ H^k(V) = 0 \text{ for } k > 1 \end{cases}$ and $U \cap V$ is homotopy-equivalent to S^{n-1}.

Thus M-V says:

$$0 \to H^0(S^n) \to H^0(U) \oplus H^0(V) \to H^0(S^{n-1}) \xrightarrow{\delta} H^1(S^n)$$
$$\to H^1(U) \oplus H^1(V) \to H^1(S^{n-1}) \xrightarrow{\delta} H^2(S^n) \to \cdots$$

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \xrightarrow{\delta} H^1(S^n) \to 0 \to H^1(S^{n-1}) \xrightarrow{\delta} H^2(S^n) \to 0$$

$$0 \to H^2(S^{n-1}) \xrightarrow{\delta} H^3(S^n) \to 0 \qquad \cdots$$

$$0 \to H^3(S^{n-1}) \xrightarrow{\delta} H^4(S^n) \to 0$$

$$\vdots$$

$$0 \to H^{n-1}(S^{n-1}) \xrightarrow{\delta} H^n(S^n) \to 0 \to 0 \to 0 \to 0$$

So $H^k(S^{n-1}) \cong H^{k+1}(S^n)$ if $n \ge 2$, $k \ge 1$

It's not hard to see S^n, $n \geq 2$, is simply-connected. So $H^1(S^n) = 0$.

Inductively, we have the proposition:

$\dim H^k$	S^1	S^2	S^3	S^4
0	1	1	1	1
1	1	0	0	0
2	0	1	0	0
3		0	1	0
4			0	1
5				0
6				

Eg. Compute $H^k(S^1 \times S^2)$

$S^1 \times S^2 = U \cup V$, where $U = S^1 \setminus \{N\} \times S^2 \simeq S^2$

$V = S^1 \setminus \{S\} \times S^2 \simeq S^2$

$U \cap V \simeq \{E, W\} \times S^2 = S^2 \sqcup S^2$

k	U	V	$U \cap V$
0	1	1	2
1	0	0	0
2	1	1	2
3	0	0	0
⋮	⋮	⋮	⋮

So M.V. is:

$$0 \to H^0(S^1 \times S^2) \to \mathbb{R}^2 \xrightarrow{\Phi} \mathbb{R}^2 \xrightarrow{\delta} H^1(S^1 \times S^2) \to 0$$

$$0 \xrightarrow{\delta} H^2(S^1 \times S^2) \to \mathbb{R}^2 \xrightarrow{\Phi} \mathbb{R}^2 \xrightarrow{\delta} H^3(S^1 \times S^2) \to 0$$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ comes from the map $i^* - j^*$, but its target is $\mathbb{R}^2$ since $U \cap V$ is disconnected. So we can write it as:

$$\Phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x-y \end{pmatrix}$$

or, in matrix form, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. This matrix has rank 1, so $\dim \ker \Phi = 2 - 1 = 1$.

By exactness $H^0(S^1 \times S^2) = \ker \Phi$ has dim 1.

Similarly, $\operatorname{im} \Phi$ is one-dimensional, and by exactness, $\operatorname{im} \Phi = \ker \delta$. So $\dim \ker \delta = 1$, so $\dim \operatorname{im} \delta = 2 - 1 = 1$.

But $\operatorname{im} \delta = H^1(S^1 \times S^2)$.

The same analysis applies to $H^2(S^1 \times S^2)$ and $H^3(S^1 \times S^2)$.

k	0	1	2	3	4	
$\dim H^k(S^1 \times S^2)$	1	1	1	1	0	…

Def'n. Suppose $F: M^n \to N^n$ is a smooth map of connected, oriented, compact manifolds. Then there is a map $F^*: H^n(N) \to H^n(M)$ can be described as a linear map from $\mathbb{R}$ to $\mathbb{R}$ i.e. multiplication by a constant. We call that constant the degree of F.

$$\int_M F^*\omega = (\deg F) \int_N \omega$$

Theorem $\deg F$ is an integer.

So it counts something. What?

Def'n Given $F: M^m \to N^n$ a map of oriented manifolds, $p \in M$ is a critical point if $\operatorname{rank} F_*|_p < n$. The set of images of critical points is the set of critical values. Any other point of N is a regular value for F. (That is, a regular value is a point in N all of whose preimages have F_* onto.)

Lemma If $q \in N$ is a regular value, $F^{-1}(q) \subset M$ is a manifold of dimension $m-n$.

Sard's Theorem If $m \geq n$, the set of regular values is almost all of N.

If $m=n$, then for almost every $q \in N$, $F^{-1}(q)$ is a 0-manifold. If F is a proper map, then $F^{-1}(q)$ is a compact 0-manifold, i.e., a finite collection of points.

Def'n The Brouwer degree of a proper map $F: M \to N$ is $\sum_{p \in F^{-1}(q)} \operatorname{sign}(F^*_p)$.

Note that this a priori depends on q! It's possible to show directly that

Proposition The Brouwer degree is well-defined.

But we'll show this by:

Theorem $\deg F$ = Brouwer degree of F.

Proof. Let $q \in N$ be a regular value, and $F^{-1}(q) = \{p_1, \ldots, p_k\}$. We can find a coordinate nbhd W of q and nbhds $U_1, \ldots, U_k$ of $p_1, \ldots, p_k$ so that $F: U_i \xrightarrow{\cong} W$. Moreover we can take U_i small so $\operatorname{sign} F_*$ is constant.

Choose $\omega = \phi\, dy^1 \wedge \dots \wedge dy^n$ on W, and put coordinates $\{x_i^1, \dots, x_i^n\}$ on U_i so that $y \circ F = x_i$

Then $F|_{U_i}^* \omega = F \circ \phi\, dx_i^1 \wedge \dots \wedge dx_i^n$, and

$\int_{U_i} F^* \omega = \pm \int_W \omega$, with sign determined by whether

F is orientation-preserving on U_i.

Thus $\int_M F^* \omega = \sum_{i=1}^{k} \operatorname{sign} F^*_{p_i} \int_W \omega = (\text{Brouwer degree}) \int_N \omega$

$$\int_M F^*\omega = (\deg F) \int_N \omega$$

So since $\int_N \omega \neq 0$, $\deg F = \text{Brouwer degree}$.