MATH 600 Calculus Cheatsheet

This cheatsheet is not, of course, complete.

Given a natural number k, a function $u : \mathbb{R}^m \to \mathbb{R}$ is called C^k if u has all possible k^{th} derivatives and the k^{th} derivatives are continuous.

A function $u : \mathbb{R}^m \to \mathbb{R}^n$ can be written as $u = (u^1, \ldots, u^n)$ where each $u^i : \mathbb{R}^m \to \mathbb{R}$. We say u is C^k if each of its components u^i is C^k.

A function is C^∞ if it is C^k for every natural number k.

The **support** of a continuous function $u : \mathbb{R}^m \to \mathbb{R}$ is the closure of the set $\{x \in \mathbb{R}^m | u(x) \neq 0\}$. We write $C^k_c = \{u \in C^k | u \text{ has compact support}\}$.

The **total derivative** of $u : \mathbb{R}^m \to \mathbb{R}^n$ at $x \in \mathbb{R}^m$, denoted Du_x, is the linear map $Du_x : \mathbb{R}^m \to \mathbb{R}^n$ which best approximates u near x. In a coordinate system, Du_x is represented as a $n \times m$ matrix, which consists of the partial derivatives of the component functions.

The **directional derivative** of $u : \mathbb{R}^m \to \mathbb{R}$ at $x \in \mathbb{R}^m$ in the direction $V \in \mathbb{R}^m$ is $D_V u_x$, the derivative of the function $u_V(t) = u(x + tV)$ at $t = 0$. In a coordinate system, $D_V u_x = Du_x \cdot V$.

The **chain rule**: If $u : \mathbb{R}^m \to \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable, then: $D(v \circ u)_x = Dv_{u(x)} \cdot Du_x$, where $\cdot$ denotes composition of linear maps (in coordinates, matrix multiplication).

Inverse Function Theorem. *Let $x_0 \in \mathbb{R}^m$. Suppose $u : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and has Du_{x_0} invertible. Then there are a neighbourhoods W of x_0 and V of $z_0 = u(x_0)$, such that:*

- *$u : W \to V$ is invertible*
- *The inverse $v : V \to W$ is as smooth as u is.*

Implicit Function Theorem. *Let $(x_0, y_0) \in \mathbb{R}^{m+k}$. Suppose $u : \mathbb{R}^{m+k} \to \mathbb{R}^k$ is C^1 and that the $k \times k$-submatrix of $Du_{(x_0,y_0)}$, $\left(\frac{\partial u}{\partial y}\right)$ is invertible. Then there are neighbourhoods W of (x_0, y_0) and V of x_0, and a map $g : V \to \mathbb{R}^k$, such that:*

- *$g(x_0) = y_0$*
- *For any $x \in W$, $u(x, g(x)) = u(x_0, y_0) = z_0$*
- *For any $(x, y) \in V$, if $u(x, y) = z_0$, then $y = g(x)$.*
- *g is as smooth as u is.*