MATH 600 Exam 2

Name:

Instructions

- For the following questions, you may use all and only the following resources:
  1. your class notes
  2. John M. Lee, "Introduction to Smooth Manifolds"
  3. John W. Milnor, "Topology from the Differentiable Viewpoint"
  4. your returned graded homework assignments and exam 1

  In particular, you are **NOT** to consult other students, tutors, internet sources, or books other than the ones listed above.
- You should also be suspicious of using any result from the book which makes a problem seem trivial. If you have a question about whether a particular theorem can be used for a particular problem, please send me an email–the earlier the better!
- Please start each problem on a new page and write legibly.
- The exam is due at 6pm on Tuesday, 18 December 2012. You are to turn it in to my mailbox in the Mathematics Department office, 4W1 DRL. Email submission is also acceptable, provided it arrives before the deadline.
- Feel free to ask me questions, but I reserve the right of coyness.
- When you turn in your exam, include this page. Sign the statement that appears below these instructions.

I have read and understood the above instructions. In completing this exam, I have consulted only the allowed sources. I understand that the penalty for consulting other sources, including other students, is that I will receive a zero grade for this exam.

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1. A *symplectic structure* on a smooth manifold M^{2n} is a 2-form $\omega \in \mathcal{A}^2(M)$ which is closed and *nondegenerate*, that is, for all $p \in M$, if $X_p \in T_pM$ has $X_p \lrcorner \omega_p = 0$, then $X_p = 0$.
   (a) Show that $\omega^n = \omega \wedge \cdots \wedge \omega$ (n times) is a nowhere-vanishing $2n$-form. Conclude M is orientable.
   (b) Assume M is compact. Show that ω^n is not exact.
   (c) Assume M is compact. Show that $H^{2k}(M) \neq 0$ for $k = 0, \ldots, n$.
   (d) Show that S^n, $n \geq 3$, does not admit a symplectic structure.
   (e) Show that S^2 admits a symplectic structure.
   (f) *Bonus.* Give an example of a form which is closed and has the property that if $X \in \mathfrak{X}(M)$ has $X \lrcorner \omega = 0 \in \mathcal{A}^1(M)$, then $X = 0 \in \mathfrak{X}(M)$, but which is not nondegerate.
2. Compute the de Rham cohomology groups of $\mathbb{RP}^n$ as follows.
   (a) If $\alpha : S^n \to S^n$ is the antipodal map $\alpha(p) = -p$, and $\pi : S^n \to \mathbb{RP}^n$ is the quotient map, then we have a short exact sequence
   $$0 \to \mathcal{A}^k(\mathbb{RP}^n) \xrightarrow{\pi^*} \mathcal{A}^k(S^n) \xrightarrow{\mathrm{id} - \alpha^*} E^k \to 0$$
   where $E^k = \mathrm{im}(\mathrm{id} - \alpha^*) \subset \mathcal{A}^k(S^n)$.
   (b) Show that the E^k form a cochain complex, with the differential d as the map $E^k \to E^{k+1}$.
   (c) We call a form ω on S^n "invariant" if $\alpha^*\omega = \omega$ and "antiinvariant" if $\alpha^*\omega = -\omega$. Show that any form can be decomposed into the sum of invariant and antiinvariant forms, $\omega = \omega_{\mathrm{inv}} + \omega_{\mathrm{anti}}$. (*Hint.* $\omega_{\mathrm{inv}} = \frac{1}{2}(\omega + \alpha^*\omega)$.)
   (d) If $k < n$, then
   $$H^k(E) = \frac{\ker d}{\mathrm{im}\, d} = \frac{\{\omega - \alpha^*\omega | \omega \in \mathcal{A}^k(S^n), d(\omega - \alpha^*\omega) = 0\}}{\{d(\sigma - \alpha^*\sigma) | \sigma \in \mathcal{A}^{k-1}(S^n)\}}$$
   is trivial. (*Hint.* Since $H^k(S^n) = 0$, there is $\eta \in \mathcal{A}^{k-1}(S^n)$ with $d\eta = \omega - \alpha^*\omega$. Consider η_{anti}.)
   (e) There is a long exact sequence which relates $H^k(S^n)$ and $H^k(\mathbb{RP}^n)$ if $k < n$. Write it out and use it to compute $H^k(\mathbb{RP}^n)$ for $k < n$.
   (f) The antipodal map $\alpha : S^n \to S^n$ is orientation-preserving if n is odd and orientation-reversing if n is even. Prove this fact, and use it to compute $H^n(\mathbb{RP}^n)$.
3. A *Riemannian metric* g is a symmetric, positive definite covariant 2-tensor, that is, for any vectors $X, Y \in T_pM$, $g(X,Y) = g(Y,X)$ and $g(X,X) \geq 0$, with equality only if $X = 0$.
   (a) If we write a Riemannian metric g in local coordnates as $g_{ij} dx^i \otimes dx^j$, explain the properties of symmetry and positive definiteness in terms of the coefficients g_{ij}.
   (b) If we express a Riemannian metric in two different coordinate systems as $g = g_{ij} dx^i \otimes dx^j = \tilde{g}_{pq} dy^p \otimes dy^q$, give a formula for $\tilde{g}_{pq}$ in terms of g_{ij}.
   (c) Can you describe your formula from part (b) in terms of linear algebra?
   (d) For any vector field $X \in \mathfrak{X}(M)$, $(X \lrcorner g) \in \mathcal{A}^1(M)$. Show that this assignment gives isomorphisms $T_pM \cong T_p^*M$ and $\mathfrak{X}(M) \cong \mathcal{A}^1(M)$.
4. If $\gamma : [a,b] \to M^n$ is a smooth curve and X is a smooth vector field on M, define
   $$\int_\gamma X = \sum_{i=1}^n \int_a^b \frac{d\gamma^i}{dt}(t) X^i(\gamma(t)) dt$$
   where in coordinates $\{x^1, \ldots, x^n\}$, we express $\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt} \frac{\partial}{\partial x^i}$ and $X = X^j \frac{\partial}{\partial x^j}$.
   (a) Is $\int_\gamma X$ well defined? (*Hint.* No.)
   (b) Explain what additional structure on the manifold M you might use to define something like $\int_\gamma X$. (*Hint.* Start by thinking linear-algebraically.)