MATH 600 Homework 1

Due 14 September 2012

Lee 1-1 Let X be the set of all points $(x, y) \in \mathbb{R}^2$, such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff.

Lee 1-3 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct smooth structures. [*Hint.* Begin by constructing homeomorphisms from $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \setminus \{0\}$.]

Lee 1-5 Define $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}$$

Define $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a) Show that these σ and $\tilde{\sigma}$ are the stereographic projection maps described in class.

(b) Show that σ is bijective. (*Hint.* The inverse is given by $\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u, |u|^2 - 1)}{|u|^2 + 1}$.)

(c) Show that the transition map $\tilde{\sigma} \circ \sigma^{-1}$ is smooth.

(d) Show that the smooth structure on S^n given this way is the same as the one given by viewing S^n as a collection of graphs.

Lee 1-7 Complex projective space, $\mathbb{CP}^n$, is the set of 1-dimenstional complex-linear subspaces ("complex lines") of $\mathbb{C}^{n+1}$, with the quotient topology induced by the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-manifold, and give it a smooth structure analogous to the smooth structure for $\mathbb{RP}^n$.

1. Show that $\mathbb{RP}^1$ is homeomorphic to S^1. (*Hint.* Both are homeomorphic to the interval $[0, \pi]$ with its endpoints identified.)
2. We will prove that the dimension of a manifold is well-defined: Let M be a topological manifold. We start with the following deep theorem:
   Brouwer's Invariance of Domain. *Suppose $U \subset \mathbb{R}^k$ is open, and $f : U \to \mathbb{R}^k$ is one-to-one and continuous. Then $f(U) \subset \mathbb{R}^k$ is open.*
   (a) Use Invariance of Domain to show that there can be no homeomorphism between Euclidean spaces of different dimensions. (*Hint.* If $\rho : \mathbb{R}^n \to \mathbb{R}^m$ were such a homeomorphism, with $m < n$, then consider the map $\rho \circ \iota : \mathbb{R}^m \to \mathbb{R}^m$, where $\iota : \mathbb{R}^m \to \mathbb{R}^n$ is the natural inclusion.)
   (b) Let (U, φ) and (V, ψ) be two charts which both contain some $p \in M$. If $\varphi : U \tilde{\to} \mathbb{R}^n$ and $\psi : V \tilde{\to} \mathbb{R}^m$, show that $m = n$. That is, we can define *the dimension of M at the point p* to be this common value.
   (c) Suppose M is connected, and $p, q \in M$. Show that the dimension of M at p is equal to the dimension of M at q.
   (d) Give an example of a topological manifold which has different dimensions at different points.
3. Prove Lee's Lemma 1.10(b).