MATH 600 Homework 2

Due 28 September 2012

Lee 2-6 If M is a topological space define $C(M)$ as the space of continuous maps $M \to \mathbb{R}$.

(a) Note that pointwise operations make $C(M)$ an algebra over $\mathbb{R}$.

(b) Given $F : M \to N$ a continuous map, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$. Show that F^* is an algebra homomorphism.

(c) If M is a smooth manifold, $C^\infty(M)$ is a subalgebra of $C(M)$.

(d) If M, N are smooth manifolds, show that F is smooth iff $F^*(C^\infty(N)) \subset C^\infty(M)$.

(e) Suppose F is a homeomorphism. Show that F is a diffeomorphism iff $F^* : C^\infty(N) \to C^\infty(M)$ is an algebra isomorphism.

Lee 2-15 Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable iff M is paracompact and has countably many components.

1. Show that $\mathbb{CP}^1$ is homeomorphic to S^2. (*Hint.* Both are homeomorphic to $([0, 2\pi] \times [0, \infty)) \cup \{*\}/_\sim$, where $*$ is a point and $\sim$ is an appropriate equivalence relation.)
2. If M^n is a smooth manifold-with-boundary, then ∂M is a $(n-1)$-dimensional manifold (without boundary). ∂M is as smooth as M is.
3. Suppose M^m and N^n are topological manifolds with smooth structures $\mathcal{A}$, $\mathcal{B}$ respectively.

   (a) Show that the collection of all $\big(U \times V, (x^1, \ldots, x^m, y^1, \ldots, y^n)\big)$ where $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$ gives a smooth atlas (call it $\mathcal{A} \times \mathcal{B}$) for $M \times N$, of dimension $m + n$.

   (b) Show that the projection maps $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth with respect to $\mathcal{A}$, $\mathcal{B}$, and the smooth structure induced by $\mathcal{A} \times \mathcal{B}$

Lee 3-1 If $F : M^m \to N^n$ is a smooth map between smooth manifolds, with M connected, such that for each $p \in M$, $F_* : T_pM \to T_{F(p)}N$ is the zero map, show that F is a constant map.

Lee 3-2 Let M_1, M_2 be smooth manifolds, $p \in M_1, q \in M_2$. Show that $T_{(p,q)}(M_1 \times M_2)$ is naturally isomorphic to $T_pM_1 \oplus T_qM_2$. (*Hint.* The isomorphism is given by $((\pi_M)_*, (\pi_N)_*)$.)

4. Show that the $\{D_i|_{\mathbf{a}}\}_{i=1,\ldots,n}$ are linearly independent as elements of $T_{\mathbf{a}}\mathbb{R}^n$.
5. Fill in the details of the proof that if $p \in (U, x) \subset M$, then $\left\{\frac{\partial}{\partial x^i}|_p\right\}_{i=1,\ldots,n}$ are a basis for T_pM.