MATH 600 Homework 3.5

Not due, but do!

This assignment is intended to get you comfortable with operations involve tensors. Almost all of it comes down using the transformation law for tensors in coordinates.

1. (a) Suppose that T is a tensor field of type $\binom{k}{l}$. Show that its trace $\operatorname{tr} T$, defined in coordinates by

$$(\operatorname{tr} T)^{i_1 \cdots i_{l-1}}_{j_1 \cdots j_{k-1}} = T^{t i_1 \cdots i_{l-1}}_{t j_1 \cdots j_{k-1}}$$

   is a $\binom{k-1}{l-1}$ tensor field.

   (b) If δ is the Kronecker $\binom{1}{1}$ tensor, what is $\operatorname{tr} \delta$?

2. Suppose A and B are tensor fields of types $\binom{k}{l}$ and $\binom{r}{s}$ respectively.

   (a) Show that there is a $\binom{k+r}{l+s}$ tensor field whose coordinate representation is

$$(AB)^{i_1 \cdots i_{l+s}}_{j_1 \cdots j_{k+r}} = A^{i_1 \cdots i_l}_{j_1 \cdots j_k} B^{i_{l+1} \cdots i_{l+s}}_{j_{k+1} \cdots j_{k+r}}$$

   (b) What is the invariant name for this tensor field?

   (c) Show that there is a $\binom{k+r-1}{l+s-1}$ tensor field whose coordinate representation is

$$(A.B)^{i_1 \cdots i_{l+s-1}}_{j_1 \cdots j_{k+r-1}} = A^{i_1 \cdots i_{l-1} t}_{j_1 \cdots j_k} B^{i_l \cdots i_{l+s}}_{t j_{k+1} \cdots j_{k+r-1}}$$

3. Give invariant descriptions for the tensor field $\operatorname{tr} T$ and the tensor field mentioned in 2(c). (*Hint.* Your answer should involve pairs of cobases–but must be independent of any such choice.)

4. Show that $\wedge : \bigwedge^k(V) \times \bigwedge^l(V) \to \bigwedge^{k+l}(V)$ is associative as follows:

   (a) If $S \in T^k(V)$, $T \in T^l(V)$, and $\operatorname{Alt}(S) = 0$, then $\operatorname{Alt}(S \otimes T) = \operatorname{Alt}(T \otimes S) = 0$.

   (b) Let $\omega \in T^k(V), \eta \in T^l(V), \theta \in T^m(V)$. Show that

$$\operatorname{Alt}(\operatorname{Alt}(\omega \otimes \eta) \otimes \theta) = \operatorname{Alt}(\omega \otimes \operatorname{Alt}(\eta \otimes \theta)) = \operatorname{Alt}(\omega \otimes \eta \otimes \theta)$$

   (c) Conclude that, for any $\omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V), \theta \in \bigwedge^m(V)$, $(\omega \wedge \eta) \wedge \theta$ and $\omega \wedge (\eta \wedge \theta)$ are both equal to

$$\frac{(k+l+m)!}{k!l!m!} \operatorname{Alt}(\omega \otimes \eta \otimes \theta)$$