MATH 600 Homework 3

Due 12 October 2012

1. If V is a vector space, we define the *sphere* of V by $S(V) = (V \setminus \{0\}) / \sim$, where $v \sim w$ if $v = \alpha w$ for some $\alpha > 0$.
   (a) Show that $S(\mathbb{R}^{k+1})$ has a smooth structure which is diffeomorphic to S^k. (*Hint.* Mimic the construction of $\mathbb{RP}^k$.)
   (b) If $E \xrightarrow{\pi} M$ is a vector bundle, define
   $$S(E) = \bigsqcup_{p \in M} S(E_p)$$
   If the rank of E is k and the dimension of M is n, show that $S(E)$ is a smooth manifold of dimension $n+k-1$, which surjects smoothly onto M. (*Hint.* Use the smooth manifold construction lemma.)
2. If $E \xrightarrow{\pi} M$ is a vector bundle, show that the zero section gives a one-to-one immersion ("embedding") of M into E.
3. We call a vector bundle $E \xrightarrow{\pi} M$ *trivial* if we can find a local trivialisation (U, Φ) with $U = M$. Show that any two trivial vector bundles of the same rank over the same base are bundle-isomorphic.
4. If $E \xrightarrow{\pi} M$ is a vector bundle of rank k with sections $\{\sigma_1, \ldots, \sigma_k\}$ such that, for each $p \in M$, $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ are linearly independent in E_p, then $E \xrightarrow{\pi} M$ is trivial.
5. Show that the Möbius bundle is not trivial. (*Hint.* Use the converse of the above.)
6. Let $E \xrightarrow{\pi} M$ be a vector bundle. Define
   $$E^* = \bigsqcup_{p \in M} (E_p)^*$$
   (a) Show that E^* is a vector bundle over M. Express the transition maps for E^* in terms of the transition maps for E.
   (b) Show that $E^* \xrightarrow{\pi^*} M$ is trivial iff $E \xrightarrow{\pi} M$ is trivial.

Lee 5-4 Show that in the vector bundle construction lemma, it suffices to check the cocycle condition.

Lee 6-1 Show that the isomorphism between a finite-dimensional vector space V and the dual of its dual $(V^*)^*$ is natural, but that there is no natural isomorphism between V and V^*.

Lee 6-2 This problem, as stated, is nonsense. Can you give a condition on F which makes the problem sensible? Then do the problem.