MATH 600 Homework 4

Due 2 November 2012

1. Let V be a finite-dimensional vector space, $\omega^1, \dots, \omega^k, \eta^1, \dots, \eta^k \in V^*$

Lee 12-3 Let V be a finite dimensional vector space. Show that the $\{\omega^i\}|_{i=1,\cdots,k}$ are linearly dependent iff $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Lee 12-4 If the $\{\omega^i\}|_{i=1,\cdots,k}$ and the $\{\eta^i\}|_{i=1,\cdots,k}$ are linearly independent sets, show that $\operatorname{Span}\left\{\omega^1, \dots, \omega^k\right\} = \operatorname{Span}\left\{\eta^1, \dots, \eta^k\right\}$ iff $\omega^1 \wedge \cdots \wedge \omega^k = c\eta^1 \wedge \cdots \wedge \eta^k$.

2. $\omega \in T^k(V)$ is *decomposable* if $\eta = \omega^1 \wedge \cdots \wedge \omega^k$. Show that every element of $T^2(\mathbb{R}^3)$ is decomposable, but that there are indecomposable elements of $T^2(\mathbb{R}^n)$ for $n \geq 4$.

3. Let V be a finite-dimensional vector space. A $\omega \in T^2(V)$ is *nondegenerate* if $\omega(X, Y) = 0$ for all $Y \in V$ implies $X = 0$. Show that the following are equivalent
   (a) ω is nondegenerate.
   (b) If $\{\theta^1, \dots, \theta^n\}$ are a basis for V^*, and $\omega = \omega_{ij}\theta^i \otimes \theta^j$, then the matrix (ω_{ij}) is nonsingular.
   (c) The linear map $\overline{\omega} : V \to V^*$ given by $\overline{\omega}(X) : Y \mapsto \omega(X, Y)$ is invertible.

4. Let $\omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V)$, $X, Y \in V$.
   (a) Show that $X \lrcorner (Y \lrcorner \omega) = -Y \lrcorner (X \lrcorner \omega)$
   (b) If $\{e_1 \dots, e_n\}$ are a basis of V and $\{\omega^1, \dots, \omega^n\}$ are the dual cobasis, then

$$e_j \lrcorner \left(\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}\right) = \begin{cases} 0 & \text{if } j \text{ is not one of the } i_\alpha \\ (-1)^{\alpha-1}\omega^{i_1} \wedge \cdots \wedge \widehat{\omega^{i_\alpha}} \wedge \cdots \wedge \omega^{i_k} & \text{if } j = i_\alpha \end{cases}$$

   (c) Prove the Leibniz rule for $\lrcorner$ and $\wedge$:

$$X \lrcorner (\omega \wedge \eta) = (X \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (X \lrcorner \eta)$$

5. Given a smooth manifold M, consider the total space of its tangent bundle, i.e. TM as a manifold in its own right. Show that TM is always orientable. *Hint.* Show that if $\{x^1, \dots, x^n\}$ and $\{y^1, \dots, y^n\}$ are local coordinates on M, then $y_* \circ (x_*)^{-1}$ looks like

$$\begin{pmatrix} D_j(y^i \circ x^{-1}) & 0 \\ \circledast & D_j(y^i \circ x^{-1}) \end{pmatrix}$$

6. Given a vector bundle $E \xrightarrow{\pi} M$, define an *orientation* of E to be a choice of orientation for each E_p, which is continuous in p. If $E \xrightarrow{\pi} M$ admits an orientation, call it *orientable*.
   (a) Show that if $U \subset M$ has the property that the subbundle $\pi^{-1}(U) \xrightarrow{\pi} U$ is not orientable, then $E \xrightarrow{\pi} M$ is not orientable. (*Hint.* Consider the contrapositive of this statement.)
   (b) Construct a rank n vector bundle over S^1 in the following way: given a vector space isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$, let

$$E = ([0, 1] \times \mathbb{R}^n / \sim$$

   where $\sim$ is the equivalence relation $(0, v) \sim (1, Tv)$.
   (c) Show that $E \xrightarrow{\pi} S^1$ as constructed above is orientable iff T is orientation preserving.