MATH 600 Homework 5

Due 19 November 2012

Lee 17-2 Compute the flows of the following vector fields on $\mathbb{R}^2$. Recall that a flow is a pair $\mathcal{D}, \theta$, where $\mathcal{D} \subset R \times M$ and $\theta : \mathcal{D} \to M$ is a local group action of $\mathbb{R}$ on M.

(a) $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

(b) $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

(c) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

(d) $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$

Lee 17-5 We call a curve $\gamma : \mathbb{R} \to M$ *periodic* if there is a $T > 0$ so that $\gamma(t+kT) = \gamma(t)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Suppose $X \in \mathfrak{X}(M)$ and γ is a maximal integral curve for X.

(a) Show γ is exactly one of constant, injective, or nonconstant periodic.

(b) If γ is periodic and nonconstant, show that there is a unique positive $T > 0$ so that $\gamma(t) = \gamma(t')$ iff $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold, diffeomorphic to $\mathbb{R}^0$, $\mathbb{R}$, or S^1.

Lee 17-8 Suppose M is oriented and θ is a local flow on M. Show that θ_t is orientation-preserving where it is defined.

Lee 17-13 If M is a manifold-with-boundary, then the boundary ∂M has a *collar.* (*Hint.* Read Lemma 13.15 and Lemma 13.16.)

Proposition 18.9 Suppose X, Y are smooth vector fields, ω and τ are smooth differential forms. Then

(a) $\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X\omega) \wedge \tau + \omega \wedge (\mathcal{L}_X\tau)$

(b) $\mathcal{L}_X(Y \lrcorner \omega) = (\mathcal{L}_X Y) \lrcorner \omega + Y \lrcorner (\mathcal{L}_X\omega)$

Lee 18-6 Let $X \in \mathfrak{X}(M)$. So that the operator $\mathcal{L}_X : \mathcal{T}^k(M) \to \mathcal{T}^k(M)$ is uniquely defined by the properties:

(a) $\mathcal{L}_X f = Xf$ for any $f \in C^\infty(M)$.

(b) $\mathcal{L}_X$ satisfies the Leibniz rule with respect to $\otimes$.

(c) $\mathcal{L}_X$ satisfies the Leibniz rule with respect to contraction of vector fields into one-forms.

(d) $\mathcal{L}_X$ commutes with d on $C^\infty(M)$, i.e. $\mathcal{L}_X(df) = d(\mathcal{L}_X(f))$ for any smooth function f.

(*Part of this exercise is to write down what the second and third items mean without looking up the statement of the problem in Lee.*)

1. Prove that d commutes with the Lie derivative as follows. Let $\omega \in \bigwedge^k(M)$, $X \in \mathfrak{X}(M)$, θ_t the flow generated by X. Let $\text{supp}(X) = \overline{\{p \in M | X_p \neq 0\}}$.

   (a) Suppose $X_p \neq 0$. Let $\{x^1, \ldots, x^n\}$ be the coordinates provided by the Canonical Form Theorem, so that $\theta_t(x^1, \ldots, x^n) = (x^1+t, x^2, \ldots, x^n)$. If $\omega = \omega_I dx^I$ in these coordinates, give the coordinate expressions for $\theta_t^*\omega$ and $d(\theta_t^*\omega)$.

   (b) Give the coordinate expressions for $\frac{d}{dt}|_{t=0}\theta_t^*\omega$ and $d\left(\frac{d}{dt}|_{t=0}\theta_t^*\omega\right)$.

   (c) Conclude that $(\mathcal{L}_X d\omega)_p = (d(\mathcal{L}_X\omega))_p$.

   (d) Show that the same statement holds for any $p \in \text{supp}(X)$.

   (e) Show that $(\mathcal{L}_X d\omega)_p = (d(\mathcal{L}_X\omega))_p$ if $X \equiv 0$ in a neighbourhood of p.