MATH 600 Homework 6

Not due, but do!

Lee 14-1 Consider $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{R}^4$, defined by $w^2 + x^2 = y^2 + z^2 = 1$, and with the product orientation. Compute $\int_{\mathbb{T}^2} xyzdw \wedge dy$.

Lee 15-3 If M is a smooth manifold and $\omega \in \mathcal{A}^k(M), \eta \in \mathcal{A}^l(M)$ are closed forms, show that the cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η. (Thus there is a bilinear map $\cup : H^k(M) \times H^l(M) \to H^{k+l}(M)$ given by $[\omega] \cup [\eta] = [\omega \wedge \eta]$, called the *cup product*.)

Spivak 8-31 (more on the cup product)

(a) If $\alpha \in H^k, \beta \in H^l$, show that $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

(b) The cup product is natural: if $F : M \to N$, then $F^*(\alpha \cup \beta) = (F^*\alpha) \cup (F^*\beta)$.

(c) Show that the *cross product* $\times : H^k(M) \times H^l(N) \to H^{k+l}(M \times N)$ given by $[\omega] \times [\eta] = [\pi_M^* \omega \wedge \pi_N^* \eta]$ is well-defined (π_M and π_N are the projections to M and N respectively.)

(d) If $\Delta : M \to M \times M$ is the diagonal map $p \mapsto (p, p)$, then $\alpha \cup \beta = \Delta^*(\alpha \times \beta)$.

Spivak 8-3 If n is an integer and $R > 0$, define the singular 1-cell $c_{R,n} : [0, 1] \to \mathbb{R}^2 \setminus \{0\}$ by

$$c_{R,n}(t) = (R \cos(2n\pi t), R \sin(2n\pi t))$$

(a) For any R_1, R_2, n, show that there is a singular 2-cell $\sigma : [0, 1]^2 \to \mathbb{R}^2 \setminus \{0\}$ with $\partial \sigma = c_{R_1,n} - c_{R_2,n}$.

(b) If $\gamma : [0, 1] \to \mathbb{R}^2 \setminus \{0\}$ is any curve with $\gamma(0) = \gamma(1)$, show that there is some $n \in \mathbb{Z}$ with $\gamma - c_{1,n} = \partial \sigma$ for some singular 2-cell σ.

(c) Show that n is unique. (This n is the *winding number* of γ around 0.)

Spivak 8-5 We could also have defined integration using *singular simplices*. Let $\Delta_k = \left\{(x^1, \ldots, x^k) \middle| 0 \leq x^i \leq 1, \sum_i x^i \leq 1\right\}$; a singular k-simplex on a manifold M is a smooth map $\sigma : \Delta_k \to M$. A singular k-chain is a formal sum of singular k-simplices.

(a) For each $0 \leq i \leq k$, define $\partial_i : \Delta_{k-1} \to \Delta_k$ by:

$$\begin{cases} \partial_0(x^1, \ldots, x^{k-1}) & = \left(\left[1 - \sum_i x^i\right], x^1, \ldots, x^{k-1}\right) \\ \partial_i(x^1, \ldots, x^{k-1}) & = \left(x^1, \ldots, x^{i-1}, 0, x^i \ldots, x^{k-1}\right) \end{cases}$$

If c is a singular k-simplex, define the singular $k-1$-simplex $\partial\sigma$ by

$$\partial \sigma = \sum_{i=0}^{k} (-1)^i \sigma \circ \partial_i$$

Show that $\partial^2 = 0$.

(b) Define integration of an n-form on the standard singular n-simplex. Show that, if $\omega = f dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n$, then

$$\int_{\Delta_n} d\omega = \int_{\partial \Delta_n} \omega$$

(c) Define integration of k-forms on singular k-simplices.

(d) Prove Stokes' theorem for singular k-simplices. (Imitate the proof of, but do not use, Stokes' Theorem for k-cells.)

1. Given a manifold-with-boundary M^n, with an orientation μ, define the *outward orientation* $\partial\mu$ on ∂M by declaring, for any $p \in \partial M$, that an ordered basis $\{v_1, \ldots, v_{n-1}\}$ for $T_p(\partial M)$ is in $(\partial\mu)_p$ exactly if $\{\nu, v_1, \ldots, v_{n-1}\} \in \mu_p$, where ν is an outward-pointing vector at p.

   (a) Show that $\partial\mu$ is well-defined, that is, the choice of outward-pointing vector does not make a difference.

(b) Consider the n-dimensional Euclidean halfspace $\mathbf{H}^n = \{(x^1, \dots, x^n) | x^n \geq 0\}$, with its standard orientation μ_0. Show that $\partial\mu_0$, on $\partial\mathbf{H}^n = \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$ is $(-1)^n$ times the usual orientation on $\mathbb{R}^{n-1}$.

2. Suppose $\pi : \tilde{M}^n \to M^n$ is a k-sheeted smooth covering map, $\tilde{M}$ and M are oriented, and π is an orientation-preserving map. If $\omega \in \mathcal{A}_0^n(M)$, show $\int_{\tilde{M}} \pi^*\omega = k \int_M \omega$.