MATH 600 Topology Cheatsheet

This cheatsheet is not, of course, complete.

A **topological space** is a pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of **open subsets** of X so that:

1. $X \in \mathcal{O}, \varnothing \in \mathcal{O}$
2. For any collection $\{O_\alpha\}_{\alpha\in\Lambda}$ of open sets, $\bigcup_{\alpha\in\Lambda} O_\alpha$ is an open set.
3. For any finite collection $O_1, \ldots, O_N$ of open sets, $\bigcap_{i=1}^{N} O_i$ is an open set.

A **base** (or **basis**) for the toplogical space $(X, \mathcal{O})$ is a collection $\mathcal{B} \subseteq \mathcal{O}$ such that any $O \in \mathcal{O}$ has $O = \bigcup_{\alpha\in\Lambda} B_\alpha$ for some $\{B_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{B}$.

An **open cover** for X is a collection of open sets $\{O_\alpha\}_{\alpha\in\Lambda}$ with $X = \bigcup_{\alpha\in\Lambda} O_\alpha$.

A topological space $(X, \mathcal{O})$ is:

- **Hausdorff** if for every pair of points $p, q \in X$ there are open sets $U, V \in \mathcal{O}$ with $p \in U, q \in V$ and $U \cap V = \varnothing$.
- **second-countable** if it has a countable base.
- **connected** if there are not two disjoint, nonempty, open subsets which cover X
- **path-connected** if every pair of points can be joined by a path.
- **locally path-connected** if there is a base of path-connected sets.
- **compact** if every open cover $\{O_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover, i.e. there are $\alpha_1, \ldots, \alpha_N \in \Lambda$ for which $X = \bigcup_{i=1}^{N} O_{\alpha_i}$.
- **locally compact** if every point $p \in X$ lies in an open set O, which is contained in a compact set K.

Given a topological space $(X, \mathcal{O})$ and a subset $S \subseteq X$, we can form a topological space $(S, \mathcal{O} \cap S)$ by setting $\mathcal{O} \cap S = \{O \cap S | O \in \mathcal{O}\}$. We say that S **has the subspace topology** from $(X, \mathcal{O})$, and call $(S, \mathcal{O} \cap S)$ a **topological subspace** of $(X, \mathcal{O})$.

(As an exercise, prove that a topological subspace is a topological space in its own right.)

Given a topological space $(\tilde{X}, \mathcal{O})$ and an equivalence relation $\sim$ on $\tilde{X}$, let $X = \tilde{X}/_\sim = \{[x] | x \in X\}$. We can form a topological space $(X, \mathcal{O}/_\sim)$ by setting $\mathcal{O}/_\sim = \{U \subseteq X | \{x | [x] \in U\} \in \mathcal{O}\}$. We call $(X, \mathcal{O}/_\sim)$ **the quotient of** X **by** $\sim$.

Given topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we can form a topological space $(X \times Y, \mathcal{O}_X \times \mathcal{O}_Y)$ by setting $\mathcal{O}_X \times \mathcal{O}_Y$ to be the set of all unions of products $O_X \times O_Y$, where $O_X \in \mathcal{O}_X$ and $O_Y \in \mathcal{O}_Y$. We call $(X \times Y, \mathcal{O}_X \times \mathcal{O}_Y)$ the **product** of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

(Note: when the number of factors in the product is not finite, there are some subtleties as to how to generalise this definition! This won't concern us, however.)

Lemma. *In a locally path-connected space, path-connectedness and connectedness are equivalent.*

The following properties are quotient-hereditary, i.e. if a topological space $\tilde{X}$ has the property, then so does any quotient $X = \tilde{X}/_\sim$:

1. connectedness
2. path-connectedness
3. local path-connectedness
4. compactness

The following properties are subspace-hereditary, i.e. if a topological space X has the property, then so does any subspace $S \subset X$:

1. Hausdorffness
2. second-countability

The following properties are finite-product-herediary, i.e. if the topological spaces $X_1, \ldots, X_N$ have the property, then so does $X_1 \times \cdots \times X_N$:

1. connectedness
2. path-connectedness
3. local path-connectedness
4. compactness
5. Hausdorffness
6. second-countability

Topological manifolds are automatically locally path-connected and locally compact.

Given two topological spaces $(X_0, \mathcal{O}_0)$ and $(X_1, \mathcal{O}_1)$, a function $\phi : X_0 \to X_1$ is called a **continuous map** (or just **map**) if, for every $O \in \mathcal{O}_1$, $\phi^{-1}(O) \in \mathcal{O}_0$. A continuous map with an inverse that is also a continuous map is called a **homeomorphism**.