Some topics from 12.5 (see pages 7 and 8 of Lecture 2)

- Line of intersection of two planes
- Distance of a point to a plane
- Angle between planes

13.1 Vector-valued functions and motion in space

Suppose there's a particle moving through space during some time interval \( I \) (some period of time). Then the \( x, y \) and \( z \) coordinates of the particle are functions of time \( t \) (meaning, they depend on the instant of time under consideration).

\[
\begin{align*}
x &= f(t), \\
y &= g(t), \\
z &= h(t)
\end{align*}
\]

The collection of points \((x, y, z) = (f(t), g(t), h(t))\) form a curve in space, which is called the path of the particle.

The vector that points from the origin to the position \((f(t), g(t), h(t))\) of the particle, is called the position vector \( \vec{r}(t) \) of the particle.

\[
\vec{r}(t) = \overrightarrow{OP} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}
\]

This is something new! A vector which depends on time. As time varies, the vector \( \vec{r}(t) \) points at different points on the curve.

This is called a vector-valued function on the time interval \( I \).
• We can also have a vector-valued function defined on all of 3-space, which gives a vector at each point of 3-space. This is called a vector field. e.g. the gravitational force field.

• Real-valued functions, for example \( f(t), g(t) \) and \( h(t) \) from the previous page, are called scalar functions.

• Describing shapes by equations — the cylinder

\[ x^2 + y^2 = 1 \]

The equation \( x^2 + y^2 = 1 \) in the \( x-y \) plane defines a circle of radius 1, centered at the origin.

When we consider this equation in 3-space, it means that the \( x \) and \( y \) coordinates of the point must satisfy the equation, but the \( z \)-coordinate can be anything.

The collection of all such points is the cylinder in the picture.

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Example (from the textbook)

Draw the graph of the vector function \( \vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k} \)

Solution.

For any \( t \), \( \cos^2 t + \sin^2 t = 1 \)

so the \( x \) and \( y \) coordinates of points on the curve satisfy the equation \( x^2 + y^2 = 1 \).

Meanwhile, as \( t \) increases, the \( z \)-coordinate (which equals \( t \)) is increasing. Therefore the curve goes round the circle and keeps increasing in height. It is a \( \text{helix} \).
Derivatives of vector-valued functions

If \( \mathbf{r}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k} \) is the position vector of a particle moving in space, define:

\[
\frac{d\mathbf{r}}{dt} = \frac{df}{dt} \mathbf{i} + \frac{dg}{dt} \mathbf{j} + \frac{dh}{dt} \mathbf{k}
\]

Example. Suppose \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \) (helix), then \( \mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \).

Tangent vector: If \( \mathbf{r}'(t) \neq \mathbf{0} \), then \( \mathbf{r}'(t) \) is called the tangent vector to the curve at \( P \).

The line passing through \( P \), with direction parallel to \( \mathbf{r}'(t) \), is called the tangent line to the curve at \( P \).

Example. Find the parametric equations for the tangent line to the helix \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \) at the point \( P(0, 1, \frac{\pi}{2}) \).

Solution: What value of \( t \) does this point correspond to?

→ we must have \( \cos t_o = 0 \), \( \sin t_o = 1 \) and \( t_o = \frac{\pi}{2} \)

so \( t_o = \frac{\pi}{2} \)

What is \( \mathbf{r}'(\frac{\pi}{2}) \)?

\( \mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \)

so \( \mathbf{r}'(\frac{\pi}{2}) = -\mathbf{i} + 0 \mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{k} \)

The tangent line passes through \( (0, 1, \frac{\pi}{2}) \) and has direction \( -\mathbf{i} + \mathbf{k} \).

Therefore the parametric equations are:

\[
\begin{align*}
x &= 0 - t \\
y &= 1 \\
z &= \frac{\pi}{2} + t
\end{align*}
\]
If \( \vec{r}(t) \) = position vector, then \( \vec{v}(t) = \frac{d\vec{r}}{dt} \) is called the velocity vector of the particle. Speed is the magnitude of velocity, \( |\vec{v}(t)| \).

and \( \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \) is called the acceleration vector.

The unit vector \( \frac{\vec{v}(t)}{|\vec{v}(t)|} \) is called the direction of motion of the particle at time \( t \).

Example \( \vec{r}(t) = (3t+1)\hat{i} + \sqrt{3}t\hat{j} + t^2\hat{k} \) is the position vector of a particle at time \( t \). Find the velocity and acceleration of the particle. Also find the angle between velocity and acceleration vectors at time \( t=0 \).

Solution. Velocity \( \vec{v}(t) = \vec{v}'(t) = 3\hat{i} + \sqrt{3}\hat{j} + 2t\hat{k} \)

Acceleration \( \vec{a}(t) = \frac{d\vec{v}}{dt} = 0\hat{i} + 0\hat{j} + 2\hat{k} = 2\hat{k} \)

Velocity at time \( t=0 \) is \( \vec{v}(0) = 3\hat{i} + \sqrt{3}\hat{j} + 0\hat{k} = 3\hat{i} + \sqrt{3}\hat{j} \)

Acceleration at time \( t=0 \) is \( \vec{a}(0) = 2\hat{k} \)

Angle between \( \vec{v}(0) \) and \( \vec{a}(0) \):

Note that \( \vec{v}(0), \vec{a}(0) = (3\hat{i} + \sqrt{3}\hat{j}) \cdot 2\hat{k} \)

\[ = 6(\hat{i} \cdot \hat{k}) + 2\sqrt{3}(\hat{j} \cdot \hat{k}) \]

\[ = 6 \times 0 + 2\sqrt{3} \times 0 \quad (\hat{i}, \hat{j}, \hat{k} \text{ are mutually orthogonal}) \]

\[ = 0 \]

Therefore \( \vec{v}(0) \) and \( \vec{a}(0) \) are orthogonal (their dot product is zero) but neither one of them is the zero vector.

Therefore, angle \( \theta \) w.r.t \( \vec{v}(0) \) and \( \vec{a}(0) \) is \( 90^\circ \left( \frac{\pi}{2} \right) \).