some more topics from section 13.1

Recall: in a previous lecture we defined vector functions and saw how to differentiate them. Differentiation of vector functions follows some important rules:

1. \[ \frac{d}{dt} \overrightarrow{C} = 0 \]

2. \[ \frac{d}{dt} (c \overrightarrow{u}(t)) = c \overrightarrow{u}'(t) \]

3. \[ \frac{d}{dt} (\overrightarrow{u}(t) \pm \overrightarrow{v}(t)) = \overrightarrow{u}'(t) \pm \overrightarrow{v}'(t) \]

4. dot product rule: \[ \frac{d}{dt} (\overrightarrow{u}(t) \cdot \overrightarrow{v}(t)) = \overrightarrow{u}'(t) \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \overrightarrow{v}'(t) \]

5. cross product rule: \[ \frac{d}{dt} (\overrightarrow{u}(t) \times \overrightarrow{v}(t)) = \overrightarrow{u}'(t) \times \overrightarrow{v}(t) + \overrightarrow{u}(t) \times \overrightarrow{v}'(t) \]

6. chain rule: \[ \frac{d}{dt} (\overrightarrow{u}(f(t))) = f'(t) \overrightarrow{u}'(f(t)) \]

Example: \[ \overrightarrow{u}(t) = \cos t \hat{i} + \sin t \hat{j} + t^2 \hat{k} \]. Let \[ \overrightarrow{v}(t) = \overrightarrow{u}(t^2 + 2) \]. What is \[ \overrightarrow{v}'(t) \]?

Solution: (using chain rule). \[ \overrightarrow{u}'(t) = -\sin t \hat{i} + \cos t \hat{j} + 2t \hat{k} \]

\[ \overrightarrow{v}'(t) = \frac{d}{dt} \overrightarrow{v}(t) = \frac{d}{dt} [\overrightarrow{u}(t^2 + 2)] = ? \]

Let \[ f(t) = t^2 + 2 \], then \[ \frac{d}{dt} [\overrightarrow{u}(t^2 + 2)] = f'(t) \overrightarrow{u}'(f(t)) \]

\[ = 2t \cdot \overrightarrow{u}'(t^2 + 2) \]

\[ = 2t \cdot (-\sin (t^2 + 2) \hat{i} + \cos (t^2 + 2) \hat{j} + 2(t^2 + 2) \hat{k}) \]
An application of the dot product rule: vector functions of constant length.

Suppose we're given that \( \vec{r}(t) \) has constant magnitude \( a \).

i.e. \( |\vec{r}(t)| = a \)

so \( \vec{r}(t) \cdot \vec{r}(t) = a^2 \)

Differentiate both sides: \( \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0 \)

(since \( a^2 \) is a constant).

So, \( 2 \vec{r}'(t) \cdot \vec{r}(t) = 0 \)

so \( \vec{r}'(t) \cdot \vec{r}(t) = 0 \)

That is, if \( \vec{r}(t) \) has constant length, then \( \vec{r}(t) \) and its derivative \( \vec{r}'(t) \) are orthogonal.

An application of the chain rule:

Recall that at the end of the last lecture we talked about the arc length parameter, and the arc length parametrization of the helix, which was: \( \vec{r}(t(s)) = \cos \left( \frac{s}{\sqrt{2}} \right) \hat{i} + \sin \left( \frac{s}{\sqrt{2}} \right) \hat{j} + \frac{s}{\sqrt{2}} \hat{k} \)

we also noted that \( \frac{ds}{dt} = |\vec{r}(t)| \) (Speed)

Then, so long as we consider curves with speed not equal to zero, we have \( \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\vec{r}|} \)

Then, thinking of \( \vec{r} \) as a function of \( s \),

\( \frac{d}{ds} \left( \frac{d}{dt} \right)(\vec{r}(t(s))) = \frac{d}{dt} \frac{dt}{ds} = \frac{\vec{r}'(t)}{|\vec{r}|} \cdot \frac{1}{|\vec{r}|} = \frac{\vec{r}}{|\vec{r}|} = \hat{\vec{r}} \)

That is, \( \frac{d}{ds} \) is the unit tangent vector of the curve.
13.4 Curvature and Normal vectors of a curve

As a particle moves along a curve, its direction of motion, given by the unit tangent vector \( \vec{T} \), is changing. The rate at which \( \vec{T} \) changes per unit length of the curve, is called the curvature, \( \kappa \).

\[
\kappa = \left| \frac{d \vec{T}}{ds} \right|
\]

The quantity \( \kappa \) tells you how sharply the curve is turning at a given point on the curve.

Note: by the chain rule, \( \kappa = \left| \frac{d \vec{T}}{ds} \right| = \left| \frac{d \vec{T}}{dt} \cdot \frac{dt}{ds} \right| = \left| \frac{d \vec{T}}{dt} \right| \left| \frac{dt}{ds} \right| = \left| \frac{d \vec{T}}{dt} \right| \frac{1}{|\vec{T}|} \)

So

\[
\kappa = \frac{1}{|\vec{T}|} \left| \frac{d \vec{T}}{dt} \right|
\]

Example: \( \vec{r}(t) = \vec{r}_0 + t \vec{v} \) is the vector equation for a line.

\[
\vec{v}(t) = \frac{d \vec{r}}{dt} = \frac{d}{dt} (\vec{r}_0 + t \vec{v}) = \frac{d \vec{r}_0}{dt} + \frac{d}{dt} (t \vec{v}) = 0 + \vec{v} = \vec{v}
\]

a constant vector

\[
\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}}{|\vec{v}|} \quad \text{a constant unit vector in the direction of } \vec{v}
\]

Therefore \( \frac{d \vec{T}}{dt} = \vec{0} \)

so, curvature \( \kappa = \frac{1}{|\vec{v}|} \left| \frac{d \vec{T}}{dt} \right| = \frac{1}{|\vec{v}|} \cdot |\vec{0}| = 0 \)

That is, the curvature of a straight line is zero.
Example: \( \vec{r}(t) = a \cos t \hat{i} + a \sin t \hat{j} \) parametrizes a circle of radius \( a \) in the \( x-y \) plane.

\[
\vec{v}(t) = \frac{d\vec{r}}{dt} = -a \sin t \hat{i} + a \cos t \hat{j}
\]

\[
|\vec{v}(t)| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a
\]

so \( \vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = -\frac{a \sin t}{a} \hat{i} + \frac{a \cos t}{a} \hat{j} = -\sin t \hat{i} + \cos t \hat{j} \)

and \( \frac{d\vec{T}}{dt} = -\cos t \hat{i} - \sin t \hat{j} \)

\[
|\frac{d\vec{T}}{dt}| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1
\]

Therefore, curvature \( K = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right| = \frac{1}{a}, \quad 1 = \left| \frac{1}{a} \right| \)

A circle has constant curvature equaling \( \frac{1}{\text{radius}} \).

Unit normal vector for a curve

\[
\vec{N} = \frac{1}{K} \frac{d\vec{T}}{ds}
\]

Since \( K = \left| \frac{d\vec{T}}{ds} \right| \), \( \vec{N} \) is indeed a unit vector.

Also, since \( \vec{T} \) is a vector function of constant length \( = 1 \), \( \vec{T} \) is orthogonal to its derivative \( \frac{d\vec{T}}{ds} \).

so \( \vec{N} \) is a unit vector defined along the curve, which at each point on the curve is orthogonal to the unit tangent vector at that point.
Formula for calculating $\vec{N}$:

\[
\vec{N} = \frac{1}{k} \frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}}{\left| \frac{d\vec{T}}{dt} \right| \left| \frac{dt}{ds} \right|} = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}
\]

so \[\vec{N} = \frac{\vec{d\vec{T}}/dt}{\left| \vec{d\vec{T}}/dt \right|}\]

**Example:** helix \[\vec{\gamma}(t) = 3\cos t \hat{i} + 3\sin t \hat{j} + 4t \hat{k}\]

Find the curvature $k$ and the unit normal vector $\vec{N}$.

**Solution:** \[\vec{\gamma}'(t) = \frac{d\vec{\gamma}}{dt} = -3\sin t \hat{i} + 3\cos t \hat{j} + 4 \hat{k}\]

\[|\vec{\gamma}'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9 + 16} = 5\]

\[\vec{T}(t) = \frac{\vec{\gamma}'(t)}{|\vec{\gamma}'(t)|} = -\frac{3}{5} \sin t \hat{i} + \frac{3}{5} \cos t \hat{j} + \frac{4}{5} \hat{k}\]

\[\frac{d\vec{T}}{dt} = -\frac{3}{5} \cos t \hat{i} - \frac{3}{5} \sin t \hat{j}\]

Curvature $k = \frac{1}{|\vec{\gamma}'|} \left| \frac{d\vec{T}}{dt} \right|$

\[= \frac{1}{5} \left| -\frac{3}{5} \cos t \hat{i} - \frac{3}{5} \sin t \hat{j} \right| = \frac{1}{5} \sqrt{\left(\frac{3}{5}\cos t\right)^2 + \left(\frac{3}{5}\sin t\right)^2}
\]

\[= \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}\]

Unit normal vector $\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{-\frac{3}{5} \cos t \hat{i} - \frac{3}{5} \sin t \hat{j}}{\frac{3}{5}}$

\[= -\cos \hat{i} - \sin t \hat{j}\]
13.5 Tangential and Normal components of acceleration

- Binormal vector \( \vec{B} = \vec{T} \times \vec{N} \) (Definition)
  is a unit vector defined at each point of the curve, that is orthogonal to both \( \vec{T} \) (unit tangent vector) and \( \vec{N} \) (unit normal vector)

- together \( \vec{T}, \vec{N} \) and \( \vec{B} \) give rise to a vector frame, that is moving along the curve. This is called the TNB frame.

- Any vector that is defined at each point of the curve can be rewritten in terms of its \( \vec{T}, \vec{N} \) and \( \vec{B} \) components, just like you would usually write a vector in terms of its \( \hat{i}, \hat{j}, \hat{k} \) components.

In particular, acceleration

Note that \( \vec{a} = \left| \vec{a} \right| \vec{T} = \frac{d\vec{s}}{dt} \vec{T} \)

Differentiating, \( \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \vec{T} \right) = \frac{d^2 s}{dt^2} \vec{T} + \frac{ds}{dt} \frac{d\vec{T}}{dt} \)

\( = \frac{d^2 s}{dt^2} \vec{T} + \frac{ds}{dt} \frac{d\vec{T}}{ds} \frac{ds}{dt} \)

\( = \frac{d^2 s}{dt^2} \vec{T} + \left( \frac{ds}{dt} \right)^2 \vec{a} \vec{N} \)

So, the acceleration vector has a \( \vec{T} \) component and an \( \vec{N} \) component but no \( \vec{B} \) component.

The \( \vec{T} \) component is \( a_T = \frac{d^2 s}{dt^2} \) (tangential component of acceleration)

The \( \vec{N} \) component is \( a_N = \left( \frac{ds}{dt} \right)^2 k \) (normal component of acceleration)