14.1 Functions of several variables

- A real-valued function $f$ is a rule, which for each element in a given set $D$, assigns a unique real number.

![Diagram](image)

- The set $D$ is called the **domain** of $f$, and the collection of real numbers that appear as values of the function, is called the **range** of $f$.

- If the domain $D$ consists of pairs of numbers $(x, y)$ then $f$ is called a function of two variables.

- If the domain $D$ consists of 3-tuples $(x, y, z)$ then $f$ is called a function of three variables.

**Finding the domain and range of a function**

If the function $f$ is given by a formula, then the domain consists of all elements $(x, y)$ or $(x, y, z)$ for which the formula "makes sense", i.e. gives a real number as its value.

E.g. $f(x, y) = \sqrt{1-x^2-y^2}$

This makes sense only when $1-x^2-y^2 \geq 0$

i.e. when $x^2 + y^2 \leq 1$
E.g. \( f(x, y) = \frac{1}{xy} \). This makes sense when \( xy \neq 0 \).

i.e. when neither \( x \) nor \( y \) is zero.

E.g. \( g(x, y) = \sin^{-1} \left( \frac{x+y}{5} \right) \)

\( \sin^{-1} \) is defined only for values between \(-1\) and \(1\).

Therefore, the domain is given by \(-1 \leq \frac{x+y}{5} \leq 1\).

i.e. \(-5 \leq x+y \leq 5\)

E.g. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \)

domain = all of 3-space

range = \([0, \infty)\)

E.g. \( f(x, y, z) = \ln z + \sqrt{y-x^2} \)

domain = (the half-space \( z > 0 \)) intersected with the region \((y > x^2)\)

Some definitions:

For a region \( R \) in the \( xy \) plane, a point \( p = (x_0, y_0) \) is called an interior point

if it's possible to make a small disk centered at \( p \), which lies completely within \( R \).

\( q = (x, y, z) \) is called a boundary point if any disk centered at \( q \) contains points of \( R \) as well as points outside \( R \).
• If $R$ consists of only interior points (no boundary points) we say $R$ is an open set.

  e.g. the set $x^2 + y^2 < 1$ (the interior of the disk)

• If $R$ contains all its boundary points we say $R$ is a closed set.

  e.g. the set $x^2 + y^2 \leq 1$ (the disk along with its boundary)

• If $R$ is contained inside a disk of finite radius we say $R$ is bounded. If not, we say $R$ is unbounded.

  e.g. bounded set:

  ![bounded set diagram]

  unbounded set

  ![unbounded set diagram]

• A function of two variables can also be represented as a graph. This is just the collection of points $(x, y, f(x, y))$.

  It is also called the surface $z = f(x, y)$.

  e.g. $f(x, y) = 10 - x^2 - y^2$

  Its graph is what's called a paraboloid.
• On the other hand, let's think about functions of three variables \( f(x, y, z) \)

\[ f(x, y, z) = c \]

The set of points in 3-space where \( f \) has a constant value \( f(x, y, z) = c \), is called a \underline{level surface} of \( f \).

E.g. What are the level surfaces of the function

\[ f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \]

\[ \rightarrow \text{They are spheres centered at the origin!} \]

• Note: We can define the notions of \underline{interior}, \underline{boundary}, \underline{open} and \underline{closed} for regions in 3-space, analogous to the definitions for regions in the \( x, y \) plane.

14.2 Limits and Continuity in Higher dimensions

• The meaning of a limit of a function \( f(x, y) \)

If there is a number \( L \) so that as the point \((x, y)\) gets very close to \((x_0, y_0)\), the value \( f(x, y) \) approaches \( L \), then we say

\[ \lim_{{(x, y) \rightarrow (x_0, y_0)}} f(x, y) = L \]

"the limit of \( f(x, y) \) as \((x, y)\) approaches \((x_0, y_0)\), equals \( L \)"

• It is important that \((x, y)\) is allowed to approach \((x_0, y_0)\) in all directions!
Properties

If \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \) and \( \lim_{(x, y) \to (x_0, y_0)} g(x, y) = M \), then

1. \( \lim_{(x, y) \to (x_0, y_0)} (f(x, y) \pm g(x, y)) = L \pm M \)

2. \( \lim_{(x, y) \to (x_0, y_0)} kf(x, y) = kL \)

3. \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) \cdot g(x, y) = LM \)

4. \( \lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \) provided \( M \neq 0 \)

5. \( \lim_{(x, y) \to (x_0, y_0)} (f(x, y))^n = L^n \) if \( n \) a positive integer

6. \( \lim_{(x, y) \to (x_0, y_0)} (f(x, y))^\frac{1}{n} = L^\frac{1}{n} \) if \( n \) a positive integer, and if \( n \) even, we require \( L > 0 \).

Example.

1. \( \lim_{(x, y) \to (3, -4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \)

2. \( \lim_{(x, y) \to (1, 2)} \frac{x^3 + 3y - 6}{2x - y^3} = \frac{1^3 + 3 \cdot 2 - 6}{2 \cdot 1 - 2^3} = \frac{1 + 6 - 6}{2 - 8} = \frac{1}{-6} = -\frac{1}{6} \)

3. \( \lim_{(x, y) \to (0, 0)} \frac{x^2 - xy}{\sqrt{x} + \sqrt{y}} \) Plugging in doesn't work because we get \( \frac{0}{0} \).
so, try to factorize: \[ \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \frac{x((\sqrt{x})^2-(\sqrt{y})^2)}{\sqrt{x}-\sqrt{y}} \]

\[ = \frac{x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}} = x(\sqrt{x}+\sqrt{y}) \]

so the limit becomes \[ \lim_{(x,y) \to (0,0)} x(\sqrt{x}+\sqrt{y}) = 0. \]

4. Does \[ \lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^3+y^3} \] exist?

Again, plugging-in doesn't help since it gives \[ \frac{0}{0} \].

Let's try approaching \((0,0)\) along different paths.

**Path 1:** approach along the \(x\)-axis, i.e. through points \((t,0)\).

Then, we get \[ \frac{2 \cdot t^2 \cdot 0}{t^3 + 0^3} = 0 \]

**Path 2:** approach along the path \(y=x\), i.e. through points \((t,t)\).

Then, we get \[ \frac{2 \cdot t^2 \cdot t}{t^3 + t^3} = \frac{2t^3}{2t^3} = 1 \]

Since the function approaches different values along different paths, the limit does not exist!

A function \(f(x,y)\) is continuous at \((x_0, y_0)\) if

1. \(f\) is defined at \((x_0, y_0)\)
2. \(\lim_{(x,y) \to (x_0, y_0)} f(x,y)\) exists
3. \(\lim_{(x,y) \to (x_0, y_0)} f(x,y) = f(x_0, y_0)\)

A function is continuous if it is continuous at every point in its domain.
Example: Show that \( f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)

is not continuous at \((0, 0)\).

**Solution.** We'll show that the limit
\[
\lim_{(x, y) \to (0, 0)} f(x, y)
\]
does not exist.

Path 1. Approach through points \((t, 2t)\) (line of slope 2)

then \[
\frac{2xy}{x^2 + y^2} = \frac{2 \cdot t \cdot 2t}{t^2 + (2t)^2} = \frac{4t^2}{t^2 + 4t^2} = \frac{4t^2}{5t^2} = \frac{4}{5}
\]

Path 2. Approach through points \((x, 5x)\) (line of slope 5)

then \[
\frac{2xy}{x^2 + y^2} = \frac{2 \cdot x \cdot 5x}{x^2 + (5x)^2} = \frac{10t^2}{x^2 + 25t^2} = \frac{10t^2}{26t^2} = \frac{5}{13}
\]

Since the function approaches different values along different paths, the limit does not exist!

So \( f \) is not continuous at \((0, 0)\).