1. **Problem 6.** A space $X$ is said to be contractible if the identity map $1_X : X \to X$ is homotopic to a constant map.
   (a) Show that any convex open set in $\mathbb{R}^n$ is contractible.
   (b) Show that a contractible space is path connected.
   (c) Show that if $Y$ is contractible, then all maps of $X \to Y$ are homotopic to one another.
   (d) Show that if $X$ is contractible and $Y$ is path connected, then all maps of $X \to Y$ are homotopic.

**Solution:**
(a) Let $X \subset \mathbb{R}^n$ be convex, and let $c \in X$. Let $f : X \to X$ be the constant map $f(x) = c$. Then $H : X \times I \to X$ defined by:

$$H(x,t) = tc + (1-t)x$$

is a homotopy between $1_X$ (at $t=0$) and $f$ (at $t=1$). Note that the map makes sense as $H(x,t) \in X$ for every $x \in X$ since $X$ is convex.

(b) Let $X$ be contractible, i.e. there’s a homotopy $H$ s.t. $H(x,0) = x$ and $H(x,1) = c$ for every $x \in X$, and for some fixed $c \in X$. Then, for each $x \in X$, $f_x(t) = H(x,t)$ is a path between $x$ and $c$ in $X$. So every point of $X$ is connected to the fixed point $c$, by a path. Therefore any two points $x_1$ and $x_2$ of $X$ can be joined via a path through $c$.

(c) Let $Y$ be contractible, i.e. there is a homotopy $H$ between $1_Y$ and a constant map $f(y) = c$. Let $g : X \to Y$. Then $H \circ g : X \times I$ is a homotopy between $g$ and the constant map $\tilde{f} : X \to Y$, $\tilde{f}(x) = c$ for each $x$. So every map $g : X \to Y$ is homotopic to the constant map $\tilde{f}$. Since homotopy is an equivalence relation, this implies all maps $X \to Y$ are homotopic.

(d) $X$ is contractible, i.e. there’s a homotopy $H$ as in (b). Let $f : X \to Y$. Then $f \circ H : X \times I \to Y$ is a homotopy between $f$ and the constant map $\tilde{f} \equiv f(c)$. Now, let $f_1 \equiv y_1$ and $f_2 \equiv y_2$ be any two constant maps $X \to Y$. Let $\alpha : I \to Y$ be a path between $y_1$ and $y_2$. Then $F(x,t) = \alpha(t)$ is a homotopy between $f_1$ and $f_2$. So we have:
1) Every map $X \to Y$ is homotopic to some constant map.
2) Any two constant maps $X \to Y$ are homotopic.

Since homotopy is an equivalence relation, these two facts together imply that any two maps $X \to Y$ are homotopic.
2. **Problem 8.** Show that the operation \( * \) of composition of paths induces a corresponding well-defined operation \( * \) of composition of fixed end point homotopy classes of paths, that is associative, has left and right identities, and has inverses.

**Solution:** The operation on fixed endpoint homotopy classes of paths in a space \( X \) is defined as:

\[
[\alpha] * [\beta] = [\alpha * \beta]
\]

**Well-defined:** Suppose \( \alpha_1 \) homotopic to \( \alpha_2 \) and \( \beta_1 \) homotopic to \( \beta_2 \) via \( F,G : I \times I \to X \) respectively. Then \( H : I \times I \to X \) given by

\[
H(x,t) = \begin{cases} 
F(2t,s) & \text{if } 0 \leq t \leq \frac{1}{2} \\
G(2t-1,s) & \text{if } \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

is a homotopy between \( \alpha_1 * \beta_1 \) and \( \alpha_2 * \beta_2 \). Therefore \([\alpha_1 * \beta_1] = [\alpha_2 * \beta_2]\) and the operation \( * \) is well-defined on homotopy classes of paths in \( X \).

**Associative:** We need to show that for paths \( f,g,h \) with contiguous images, the path \((f * g) * h\) is homotopic to the path \( f * (g * h)\). Note that

\[
((f * g) * h)(s) = \begin{cases} 
f(4s) & \text{if } 0 \leq s \leq \frac{1}{4} \\
g(4s-1) & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\
h(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 
\end{cases}
\]

and also

\[
(f * (g * h))(s) = \begin{cases} 
f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\
g(4s-2) & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\
h(4s-3) & \text{if } \frac{3}{4} \leq s \leq 1 
\end{cases}
\]

Then \( H : I \times I \to X \) defined by

\[
H(t,s) = \begin{cases} 
f\left(\frac{4s}{1+t}\right) & \text{if } 0 \leq s \leq \frac{1+t}{4} \\
g(4s-1-t) & \text{if } \frac{1+t}{4} \leq s \leq \frac{2+t}{4} \\
h\left(\frac{4s}{2-t} - \frac{2+t}{2-t}\right) & \text{if } \frac{2+t}{4} \leq s \leq 1 
\end{cases}
\]

is a homotopy between \((f * g) * h\) and \( f * (g * h)\). That \( H \) is continuous can be seen by the pasting lemma.

**Identities:** For homotopy classes of paths between points \( x \) and \( y \) in \( X \), we claim that the constant maps \( c_1(s) = p \) and \( c_2(s) = q \) are left and right identities of the \( * \) operation respectively. To see this, construct an explicit homotopy between \( f \) and \( c_1 * f \) or between \( f \) and \( f * c_2 \) as done above for showing associativity, or use the idea of reparametrization of paths described in Hatcher, page 27.

**Inverses:** For a path \( f \), the path \( f^{-1} \) defined by \( f^{-1}(s) = f(1-s) \) is an inverse for \( f \) with respect to the operation \( * \). Again, to check that \( f * f^{-1} \) is homotopic to \( c_1 \) or that \( f^{-1} * f \) is homotopic to \( c_2 \), explicitly construct homotopies or use the idea of reparametrization of paths described in Hatcher.

3. **Problem 12.** Show that a continuous map \( f : (X,x_0) \to (Y,y_0) \) induces a group homomorphism \( f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0) \).
**Solution:** Define the map $f_*$ as:

$$f_*[\alpha] = [f \circ \alpha]$$

for $\alpha$ a representative of a homotopy class of loops.

**Well-defined:** If $\alpha$ is homotopic to $\beta$ via $H : I \times I \to X$ then $f \circ \alpha$ is homotopic to $f \circ \beta$ via $f \circ H$.

**$f_*$ is a group homomorphism:** We need to show that

$$f_*([\alpha] * [\gamma]) = f_*[\alpha] * f_*[\gamma] \quad (1)$$

Recall that

$$([\alpha] * [\gamma])(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\gamma(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 
\end{cases}$$

So

$$f_*([\alpha] * [\gamma])(t) = \begin{cases} 
 f \circ \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
 f \circ \gamma(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 
\end{cases}$$

But this is the same as $[f \circ \alpha] * [f \circ \gamma]$, which by the definition of $f_*$, is the right hand side of eqn (1).