1. **Problem 1.** Show that the map $p: \mathbb{R}^1 \to S^1$ defined by

\[ p(x) = (\cos 2\pi x, \sin 2\pi x) \]

is a covering map.

**Solution:**

For each point $q_0 \in S^1$, $q_0 = (\cos 2\pi x_0, \sin 2\pi x_0)$ for some $x_0 \in [0, 1)$. Let $U$ be an open neighbourhood of $q_0$, $U = \{(\cos 2\pi x, \sin 2\pi x) : x \in (x_0 - \delta, x_0 + \delta)\}$, for a fixed $0 < \delta < \frac{1}{2}$. Then $p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (n + x_0 - \delta, n + x_0 + \delta)$. Restricted to each $U_n = (n + x_0 - \delta, n + x_0 + \delta)$, $p$ is a homeomorphism onto $U$. Therefore $U$ is an evenly covered neighbourhood of $q_0$. We can get such a set $U$ for every point $q_0 \in S^1$, so $p$ is a covering map. $\square$

2. **Problem 6.** Let $p: E \to B$ be a covering map, and suppose that $B$ is connected. Show that if $p^{-1}(b)$ has $k$ elements for some $b \in B$, then it has $k$ elements for every $b \in B$. In this case, we say that $E$ is a $k$-fold covering of $B$.

**Solution:**

Let $B_n = \{b \in B| p^{-1}(b) \text{ has } n \text{ elements}\}$. Then $B = \bigcup_n B_n$. We’re given that $B_k \neq \emptyset$. It’s enough to show that each $B_n$ is an open set, because the fact that $B$ is connected would then imply that all but one of the $B_n$’s is empty.

Let $b \in B_n$, and let $U \subseteq B$ be an evenly covered neighbourhood of $b$, $p^{-1}(U) = \bigcup_j U_j$ such that $p$ restricted to each $U_j$ is a homeomorphism onto $U$. Then, since $p^{-1}(b)$ has $n$ elements, it implies that the number of sets $U_j$ is $n$. Thus for every $x \in U$, its preimages lie in the $U_j$’s, and hence $p^{-1}(x)$ has $n$ elements for each $x \in U$. Therefore $U \subseteq B_n$, so $B_n$ is open. $\square$

3. **Problem 12.** Let $f: S^1 \to S^1$ be defined by $f(z) = z^n$. Note that $f(1) = 1$. Compute the induced homomorphism $f*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$.

**Solution:**

First observe that $\pi_1(S^1, 1)$ is isomorphic to the additive group $\mathbb{Z}$. To see this one can define a function $\phi: \pi_1(S^1, 1) \to \mathbb{Z}$ by defining it on loops as follows. Let $p: \mathbb{R} \to S^1$ be the covering map from Problem 1. For a loop $\alpha: I \to S^1$, let $\tilde{\alpha}: I \to \mathbb{R}$ be its lift with initial point $0 \in \mathbb{R}$. Then define $\phi([\alpha]) = \tilde{\alpha}(1)$. (Checking that this map is well-defined and is actually a group isomorphism, was the content of Problem 10.)

So, $f*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is actually a group homomorphism $F$ from $\mathbb{Z}$ to $\mathbb{Z}$, and this is described by specifying the image of the generator $1 \in \mathbb{Z}$ under $F$. Therefore consider the loop $\gamma(t) = e^{2\pi it}$, which is a generator of $\pi_1(S^1, 1)$. (It
corresponds to \(1 \in \mathbb{Z}\) under the isomorphism \(\phi\)). Under the map \(f_\ast\), it gets sent to the loop \(f \circ \gamma(t) = (e^{2\pi it})^n = e^{2\pi int}\). The lift of this path starting at \(0 \in \mathbb{R}\) is the path \(\tilde{\gamma} : I \to \mathbb{R}, \tilde{\gamma}(t) = nt\). Observe that \(\tilde{\gamma}(1) = n\), so by the correspondence \(\phi\), this implies that \(f \circ \gamma\), i.e., \(f_\ast[\gamma]\), corresponds to \(n\) times the generator in \(\pi_1(S^1, 1)\).

We conclude that the homomorphism \(f_\ast : \pi_1(S^1, 1) \to \pi_1(S^1, 1)\) is the homomorphism \(\mathbb{Z} \to \mathbb{Z}\) that sends \(1\) to \(n\). \(\square\)

**Note**: If you want to do the proof without going to the universal cover \(\mathbb{R}\), you will need to show (by indicating an explicit homotopy) that the loop \(f \circ \gamma(t) = e^{2\pi int}\) is homotopic to the loop \(\gamma \ast \cdots \ast \gamma\), where \(\ast\) means concatenation of loops. This claim needs a proof!

4. (Hatcher, page 38, Problem 9) Let \(A_1, A_2, A_3\) be compact sets in \(\mathbb{R}^3\). Use the Borsuk-Ulam theorem to show that there is one plane \(P\) in \(\mathbb{R}^3\) that simultaneously divides each \(A_i\) into two pieces of equal measure.

**Solution**: We’re going to define a map \(S^3 \to \mathbb{R}^3\) on which we’ll use the conclusion of the Borsuk-Ulam theorem.

For this, we claim that every point on \(S^3\) is in one-to-one correspondence with an oriented two-plane in \(\mathbb{R}^3\). Any point \((a, b, c, d) \in S^3 - \{(0, 0, 0, \pm 1)\}\) corresponds to a two-plane \(P : ax + by + cz + d = 0\). And given a two-plane \(P : ax + by + cz + d = 0\), we have the corresponding point \(\frac{1}{\sqrt{a^2 + b^2 + c^2 + d^2}}(a, b, c, d) \in S^3 - \{(0, 0, 0, \pm 1)\}\).

Let’s also associate to the two-plane \(ax + by + cz + d = 0\), the direction vector \(v = (a, b, c) \in \mathbb{R}^3\). Observe that \(P\) divides \(\mathbb{R}^3\) into two half-spaces. Translating \(v\) so its base point lies on the plane \(P\), we denote by \(A\) the half-space that the endpoint of \(v\) lies in.

Now, define a function \(f : S^3 \to \mathbb{R}^3\), as :

\[
f(a, b, c, d) = (m(A_1 \cap A), m(A_1 \cap A), m(A_1 \cap A)),
\]

where \(m\) is the Lebesgue measure on \(\mathbb{R}^3\). The function \(f\) is certainly continuous on \(S^3 - \{(0, 0, 0, \pm 1)\}\), and can be extended to a continuous function on all of \(S^3\), as follows.

As \((a, b, c, d)\) tends to \((0, 0, 0, 1)\), the plane \(P\) moves away from the origin, but its direction vector \((a, b, c)\) points towards the origin. Since all the \(A_i\)’s are compact, eventually, for the plane \(P\) being far enough out (i.e. \((a, b, c, d)\) near enough to \((0, 0, 0, 1)\)), each \(A_i\) is completely contained in the half-space \(A\). Therefore eventually \(m(A_i \cap A) = m(A_i)\) for each \(i\), for all \((a, b, c, d)\) near enough to \((0, 0, 0, 1)\). So we can define \(f((0, 0, 0, 1)) = (m(A_1), m(A_2), m(A_3))\). Similar analysis shows we can define \(f((0, 0, 0, -1)) = 0\), since all \(A_i\)’s lie in the complement of \(A\), for \((a, b, c, d)\) near enough to \((0, 0, 0, -1)\).
Now that we’ve got the continuous function $S^3 \to \mathbb{R}^3$, the Borsuk-Ulam theorem tells us that there’s a pair of antipodal points $\{x, -x\}$ in $S^3$ with $f(x) = f(-x)$. But $f(x), f(-x)$ respectively give the measures of the portions of each $A_i$ that lie in the two half-spaces of the two-plane $P$ corresponding to $x$. So $f(x) = f(-x)$ means that the plane $P$ divides each $A_i$ into two sets of equal measure. □