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Asymptotic Pseudodifferential Operators And Index Theory

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Abstract. We introduce the notion of an asymptotic pseudodifferntial operator on spin bundles over a compact manifold and develop a calculus for these operators. We then use the the formula of Jaffe, Lesniewski, and Osterwalder for theta-summable Fredholm modules, along with the asymptotic calculus, to compute the cyclic cocycle that Connes attaches to the Dirac operator on a compact spin manifold.

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INTRODUCTION

This paper had its origins in a seminar held at MSRI in the Fall of 1987. Our goal was to understand Connes's computation of the cyclic cocycle corresponding to the Dirac operator on a compact spin manifold ([**C-1**], [**C-2**]). In particular, we needed to understand the ideas in the paper by Getzler [**G**]. It soon became clear that a change of view would simplify the situtation. Of course, this change is implicit in the letter from Connes to Quillen.

Getzler introduced an algebra of pseudodifferential operators ($\psi D0$'s) on a compact spin manifold that incorporated the quantization map of Bokobza-Haggiag [**B**] with some supersymmetric ideas from quantum field theory. These ideas suggest that Clifford multiplication contributes to the degree of a differential operator. If p denotes a symbol, $\theta(p)$ will be the associated pseudodifferential operator. Given a ψDO Q, there is a symbol map, $\sigma(Q)$, which was first introduced by Widom ([**W1**], [**W2**]). Next, a scaling was introduced, which is essentially the traditional scaling of asymptotic analysis, but extended to take into account the Clifford variables [**MF**]. A family of compositions was defined by:

$$p \circ_t q = \sigma(\theta(p_t) \circ \theta(q_t))_{t^{-1}}.$$

This scaling had the important property that $\lim_{t\to 0} p \circ_t q$ was computable. While the $\theta(p)$'s generate an algebra of ψDO's, it is clear that $\theta(p_t) \circ \theta(q_t)$ is not the scaled version of some other operator. It was this observation that led us to enlarge the class of symbols, moving closer to Widom's work and the classical asymptotic methods.

Let $p(x, \xi, t)$ be a symbol of degree n such that $p(x, \xi, t)$ has an asymptotic expansion in t with respect to the natural topology on the symbol space:

$$p(x, \xi, t) \sim \sum_{l=o}^{\infty} t^l p_l(x, \xi) \quad \text{with } p_l \text{ a symbol of degree } n - l.$$

We define an asymptotic operator to be (essentially) $\theta(p(x, \xi, t)_t) = P_t$. In this setting Getzler's rescaled operator is just an asymptotic operator whose symbol has the trivial asymptotic expansion $p(x, \xi, t) \sim p(x, \xi)$. Once the notion of an asymptotic pseudodifferential operator ($A\psi DO$) has been defined, the development of a calculus of $A\psi DO$'s paralleling the calculus of ψDO's is straightforward.

As we were working to understand Connes's work, we realized that a formula for the character of a Fredholm module in which all of the terms were asymptotic operators would be extremely useful. One of the main difficulties Connes needed to address was that $D_t = tD$ is not an $A\psi DO$ so D_t^{-1} is not an $A\psi DO$ either. We then received a preprint of Getzler and Szenes [**GS**], in which they presented such a formula, which is due to Jaffe, Lesniewski, and Osterwalder [**JLO**]. This formula has three beautiful attributes:

(1) one does not have to cup with the small complex to make D invertible;
(2) all terms are manifestly asymptotic operators; and
(3) it applies to the larger context of θ-summable Fredholm modules.

Using this formula for the Chern character, we write out a computation of the cyclic cocycle attached to the Dirac operator on a spin manifold.

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1. Review of Clifford Algebras and Spinors.

In this section a brief review of the construction of the Clifford algebra and spinors is presented. (For more details see **[Gi]**, **[ABP]**.) Let V be an even-dimensional vector space with an inner product, $\langle \cdot, \cdot \rangle$. Denote by V^n the n-fold tensor product of V with itself, so

$$V^n = V \otimes V \otimes \cdots \otimes V \text{ n-factors.} \tag{1.1.}$$

(We set V^0 to be $\mathbf{R}$.) The tensor algebra of V is the graded algebra

$$T(V) = \oplus_{n=0}^{\infty} V^n. \tag{1.2}$$

Let I be the ideal generated by the relations $v \otimes v - \langle v, v \rangle$. It should be noted that the ideal is not homogeneous under the Z grading but is homogeneous under the Z_2 grading. We have the following.

Definition 1.1. *The Clifford algebra of the pair $(V, \langle \cdot, \cdot \rangle)$ is $Cliff(V) = T(V)/I$.*

Since the ideal I is homogeneous under the Z_2 grading, we have that $Cliff(V)$ is a Z_2 graded algebra. Let $e_1, e_2, \cdots, e_n$ be an orthonormal basis for V; then $Cliff(V)$ is the algebra generated by the $e_i^{\prime}$s subject to the relations:

$$\begin{aligned} e_i e_j + e_j e_i &= 0 \quad \text{if } i \neq j \\ e_i^2 &= 1. \end{aligned} \tag{1.3}$$

Using the above relations we can write down a basis for $Cliff(V)$. Let $I = (i_1 < i_2 < \cdots < i_p)$ be a multi-index and set $e_I = e_{i_1} \cdot e_{i_2} \cdots e_{i_p}$; then $Cliff(V)$ has the 2^n monomials e_I as a basis. Denote by $\hat{\otimes}$ the graded tensor product. By considering the basis of $Cliff(V)$ as above we see that:

$$Cliff(V \otimes W) = Cliff(V) \hat{\otimes} Cliff(W). \tag{1.4}$$

Define the transpose map $a \to a^t$ for $a \in Cliff(V)$ by defining it on a basis and then extending it by linearity, for e_I we have:

$$(e_{i_1} \cdots e_{i_p})^t = e_{i_p} \cdots e_{i_1}. \tag{1.5}$$

A simple computation shows that $(e_{i_1} \cdots e_{i_p})^t = (-1)^{p(p-1)/2}(e_{i_1} \cdots e_{i_p})$. Next define $\tau = (i)^{n/2} e_1 \cdots e_n$ where $e_1, \ldots, e_n$ is an oriented orthonormal basis of V. Then we see that $\tau^2 = 1$.

Let $Cliff(V)_C$ denote the complexified Clifford algebra, $Cliff(V)_C = Cliff(V) \otimes_R C$. If V is a two-dimensional vector space, we can define the well-known isomorphism of $Cliff(V)_C$ with the two-by-two complex matrix algebra. This isomorphism is implemented by the Pauli matrices:

$$
\begin{aligned}
e_1 &\longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
e_2 &\longrightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
\tau = ie_1e_2 &\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{1.6}
$$

If the dimension of the vector space V is $n = 2m$, then it follows from (1.4) and (1.6) that $Cliff(V)_C$ is isomorphic to the matrix algebra $M_{2^m}(C)$ of $2^m \times 2^m$ matrices. In particular, it is a simple algebra. Left multiplication of $Cliff(V)_C$ on itself decomposes into a direct sum of irreducible submodules. After choosing an isomorphism of $Cliff(V)_C$ with $M_{2^m}(C)$, each column of $M_{2^m}(C)$ yields an irreducible submodule. All of these are isomorphic, and in fact there is only one irreducible $Cliff(V)_C$ module up to isomorphism.

DEFINITION 1.2. *Let Δ be an irreducible submodule for $Cliff(V)_C$ acting on itself by left multiplication; then Δ is called the space of spinors.*

DEFINITION 1.3. *Define $Spin(V)$ by:*

$$
Spin(V) = \{w \in Cliff(V) | w = v_1 \cdots v_{2j} \quad and |v_k| = 1\}. \tag{1.7}
$$

Note that $Spin(V)$ is a group with $w^{-1} = w^t$.

Define a map $\rho : Cliff(V) \to End(Cliff(V))$ as follows. If $w \in Cliff(V)$, then:

$$
\rho(w)(v) = wvw^t. \tag{1.8}
$$

Suppose that $e_1, \ldots, e_n$ is an orthonormal basis for V. Then:

$$
\begin{cases} \rho(e_1)(e_1) = e_1 \\ \rho(e_1)(e_i) = -e_i \end{cases} \Rightarrow \rho(e_1) = \begin{cases} \text{reflection through the} \\ \text{hyperplane orthogonal to } e_1 \end{cases}. \tag{1.9}
$$

We see that if $v \in V \subset Cliff(V)$, then $\rho(v)$ takes V to V. Furthermore, if $|v| = 1$, then $\rho(v) \in O(V)$, the orthogonal group of V. Consequently, ρ maps $Spin(V)$ into $SO(V)$, and since every element in $SO(V)$ can be written as a product of an even number of reflections, we see that ρ maps onto $SO(V)$. If $\rho(w) = 1$, then $wvw^t = v$ for all $v \in V$. It is easy to check that this implies that w is in the center of $Cliff(V)$. Since the center of $Cliff(V)$ consists of the scalars and $|w| = 1$, we have that $w = \pm 1$. Therefore, we have the short exact sequence of groups:

$$
1 \to Z_2 \to Spin(V) \to SO(V) \to 1. \tag{1.10}
$$

To show that $Spin(V)$ is connected, we only have to produce a path from -1 to 1 in $Spin(V)$. We simply write one down:

$$\gamma(\theta) = (cos(\theta)e_1 - sin(\theta)e_2)e_1. \tag{1.11}$$

Thus $\gamma(0) = 1$, $\gamma(\pi) = -1$, and we conclude that $Spin(V)$ is a two fold-cover of $SO(V)$.

Once we fix an orthonormal basis of V, $e_1, \cdots, e_n$, we can identify $SO(V)$ with $SO(n, R)$. The Lie algebra of $SO(n, R)$ and of $Spin(V)$ is then identified with the skew symmetric matrices of $M_n(R)$, which we denote by $so(n)$. If $A = (a_{ij}) \in so(n)$, then the action of A on Δ is given by Clifford multiplication by:

$$\frac{1}{4}\sum_{ij} a_{ij}e_ie_j. \tag{1.12}$$

For the details of this computation, see [**G**, p.173].

Definition 1.4. *Let V be an oriented vector space and $e_1, \cdots, e_n$ be an oriented basis for V. If $\tau = (i)^{n/2}e_1 \cdots e_n$, then we define the half-spin representations of $Spin(V)$ by $\Delta^{\pm} = \{x \in \Delta | \tau \cdot x = \pm x\}$.*

Note that τ commutes with every element of $Spin(V)$, so $\Delta^{\pm}$ are invariant under the action of $Spin(V)$. If $v \in V$, let $c(v)$ be the operator on Δ which is Clifford multiplication by v. Then it is clear that:

$$c(v) : \Delta^{\pm} \to \Delta^{\mp}. \tag{1.13}$$

Let V be an oriented inner-product space with oriented orthonormal basis $e_1, \cdots, e_n$. Let $\Lambda(V)$ be the associated exterior algebra of V.

Definition 1.5. *Define a linear isomorphism σ from $Cliff(V)$ to $\Lambda(V)$ as follows : for each multi-index $I = (i_1 < \cdots < i_p)$ we have*

$$\sigma(e_{i_1} \cdots e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}. \tag{1.14}$$

It is easy to check that this map is independent of the basis and that

$$\sigma(a \cdot b) = \sigma(a) \wedge \sigma(b) \quad \text{mod } \Lambda^{deg(a)+deg(b)-1}. \tag{1.15}$$

Recall that a Z_2 graded vector space, $V = V^+ \oplus V^-$, is called a supervector space. Let ϵ be the operator that is $+1$ on V^+ and -1 on V^-; then ϵ is called the grading operator for V. If V is Z_2 graded, then $End(V)$ is also Z_2 graded with the grading operator $Ad(\epsilon)$. If V is finite dimensional, we define the supertrace of $a \in End(V)$ by;

$$tr_s(a) = tr(\epsilon a). \tag{1.16}$$

The fundamental property of a trace is that $tr([a, b]) = 0$, where $[a, b] = ab - ba$. There is an analogous property for the supertrace, but one has to replace the commutator $[a,b]$ by the graded commutator. If $a \in End(V)$, we say that $deg(a) = 0$ or 1 when $Ad(\epsilon)(a) = \pm a$, and we write $(-1)^a$ for $(-1)^{deg(a)}$. For $a, b \in End(V)$, we define the graded commutator as:

$$[a,b]_s = ab - (-1)^{deg(a)deg(b)}ba. \tag{1.17}$$

For the supertrace we have:

$$tr_s([a,b]_s) = 0. \tag{1.18}$$

Let V and W be two Z_2 graded vector spaces, and recall that $\hat{\otimes}$ represents the graded tensor product. We have:

$$End(V\hat{\otimes}W) = End(V)\hat{\otimes}End(W). \tag{1.19}$$

Under this isomorphism we have:

$$a\hat{\otimes}b(v\hat{\otimes}w) = (-1)^{deg(a)deg(v)}a(v)\hat{\otimes}b(w). \tag{1.20}$$

It follows from (1.20) that:

$$tr_s(a\hat{\otimes}b) = tr_s(a)tr_s(b). \tag{1.21}$$

The space of spinors Δ is naturally a super vector space with the grading operator τ, and $Cliff(V)_C = End(\Delta)$ is a super algebra. Equation (1.21) makes it easy to compute the value of the supertrace on elements of the Clifford algebra acting on the spinors. We start off with a two-dimensional vector space spanned by the orthonormal basis e_1 and e_2. Recall the isomorphism of equation (1.6) implemented by the Pauli matrices; then we see that $tr_s(e_1) = tr_s(e_2) = 0$ and $tr_s(e_1e_2) = \frac{2}{i}$. More generally, for a real vector space of dimension $n = 2m$, $Cliff(V)_C$ is identified with $M_{2^m}(C) = \otimes^m(M_2(C))$ and the spinors are identified with 2^m-dimension column vectors. By equation (1.21) we have:

$$tr_s(e_I) = \begin{cases} 0 & I \neq \{1,\ldots,n\} \\ (\frac{2}{i})^m & I = \{1,\ldots,n\} \end{cases}. \tag{1.22}$$

2. Pseudodifferential Operators.

This section introduces the main ingredients for the asymptotic pseudodifferential operators. We first discuss the spin bundle and then talk about the standard symbol spaces and the ordinary (that is, non-asymptotic) pseudodifferential operators.

The Spin Bundle.

Let M be a compact, oriented, Riemannian manifold with tangent bundle $TM \xrightarrow{\pi} M$. Since M is oriented, we can find an open cover, $\{U_\alpha\}$, of M that trivializes the tangent bundle such that the transition functions $g_{\alpha\beta}$ are in $C^\infty(U_\alpha \cap U_\beta; SO(n))$. Assume that M is equipped with a Riemannian connection ∇. If $e_1, \ldots, e_n$ is a local orthonormal frame, then

$$\nabla_{e_i}(e_j) = \sum_k \Gamma_{ij}^k e_k. \tag{2.1}$$

Set $f^i =$ to the one-form dual to e_i and define $\omega_i^k = \sum_i \Gamma_{ij}^k f^i$, then

$$\nabla(e_i) = \sum_k \omega_i^k e_k. \tag{2.2}$$

The ω_i^k are called *the local connection one forms* defining the connection. From

$$0 = d\langle e_i, e_j\rangle = \langle \nabla(e_i), e_j\rangle + \langle e_i, \nabla(e_j)\rangle \tag{2.3}$$

we see that $\omega = (\omega_j^k)$ is an $so(n)$-valued one-form. The curvature of ∇ is defined by

$$\Omega = d\omega - \omega \wedge \omega. \tag{2.4}$$

If we evaluate $\Omega(e_i, e_j) \in Hom(TM)$, then

$$\Omega(e_i, e_j) = \sum R_{lij}^k e_k \otimes f^l. \tag{2.5}$$

The tensor R_{lij}^k is skew in the ij indices (that is, Ω is a two form) and also skew in the kl indices (that is, $\Omega(e_i, e_j) \in so(n)$) **[H]**.

Recall that $\rho : Spin(n) \to SO(n)$ is a two-fold covering map. If the U_α are suitably chosen (contractible will certainly do), then we can lift the transition functions $g_{\alpha\beta}$ to functions $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)$ such that $\rho(\bar{g}_{\alpha\beta}) = g_{\alpha\beta}$. However, we may not be able to lift them so that the cocycle condition $\bar{g}_{\alpha\beta}\bar{g}_{\beta\gamma} = \bar{g}_{\alpha\gamma}$ holds. If we can, then we say that the manifold is spinable, and a choice of liftings is called a spin structure (see **[Gi]**). We will assume that a spin structure has been chosen for M; then we can form the bundle of spinors over M, which we denote by $\mathbf{S}$. Let us recall how $\mathbf{S}$ is constructed. First, form the disjoint union $S = \cup_\alpha U_\alpha \times \Delta$. Then define an equivalence relation on S as follows: if $(m, x) \in U_\alpha \times \Delta$ and $(n, y) \in U_\beta \times \Delta$, then $(m, x) \sim (n, y)$ iff $m = n$ and $\bar{g}_{\alpha\beta}(x) = y$. The bundle of spinors is then equal to $\mathbf{S} = S/\sim$. It is clear that $\mathbf{S}$ trivializes over U_α and that $\bar{g}_{\alpha\beta}$ are the transition functions.

Let $e_1, \ldots, e_n$ be an orthonormal frame for TM over U_α, and let ω be the associated connection one-form. We can define a local connection one-form on $\mathbf{S}$ restricted to U_α by using the homomorphism from $so(n)$ into $Cliff(R^n)$:

$$\bar{\omega} = \frac{1}{4}\sum_{ij} \omega_j^i e_i e_j \in Cliff(TM|_{U_\alpha}). \tag{2.6}$$

Denote by $\bar\nabla$ the corresponding connection on $\mathbf{S}$. The curvature of this connection is given by:

$$\bar\Omega(e_i, e_j) = \frac{1}{4}\sum_{kl} R^k_{lij} e_k e_l \in Cliff(TM|_{U_\alpha}). \tag{2.7}$$

Since M is assumed to be oriented, we have a decomposition of $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ corresponding to the decomposition of Δ into $\Delta^+ \oplus \Delta^-$. In fact, $\tau = (i)^{n/2} e_1 \cdots e_n$ defines a global section of $Cliff(TM)$ and $\Gamma(\mathbf{S}^\pm) = \{s \in \Gamma(\mathbf{S}) | \tau s = \pm s\}$. Since $\tau\omega = \omega\tau$ the connection $\bar\nabla$ restricts to a connection on $\mathbf{S}^\pm$, which we still denote by $\bar\nabla$.

If $\mathbf{E}$ is any vector bundle over M with connection $\nabla_{\mathbf{E}}$, then we will always use the tensor product connection on $\mathbf{E} \otimes \mathbf{S}$:

$$\nabla_{\mathbf{E}\otimes\mathbf{S}} = \nabla_{\mathbf{E}} \otimes Id + Id \otimes \nabla_{\mathbf{S}}. \tag{2.8}$$

Symbols and Their Calculus.

Recall the standard notion of a symbol for a vector bundle $\mathbf{E}$ over a compact manifold M. Let $\pi : T^*M \to M$ be the natural map and let $S = \pi^*(Hom(\mathbf{E}, \mathbf{E}))$ be the pull-back of the bundle $Hom(\mathbf{E}, \mathbf{E})$ to a bundle over T^*M.

DEFINITION 2.1. *A section $p \in S$ is called a symbol of order m if for every multi-index α and β we have the estimates:*

$$\| \partial_x^\alpha \partial_\xi^\beta p(x,\xi) \| \le C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|}. \tag{2.9}$$

We denote by $\Sigma^m(\mathbf{E})$ the symbols of order m.

Note that $\Sigma^m(\mathbf{E})$ is a Fréchet space with respect to the semi-norms:

$$\rho_{\alpha\beta}(p) = inf\{\ C_{\alpha\beta} \mid \ \ \| \partial_x^\alpha \partial_\xi^\beta p(x,\xi) \| \le C_{\alpha\beta}(1 + |\xi|)^{m-\beta}\ \}. \tag{2.10}$$

Following Getzler [G], we will introduce a new filtration on the space of symbols of $\mathbf{E} \otimes \mathbf{S}$. Recall that we have a linear isomorphism of vector bundles $\sigma : Cliff(TM) \to \Lambda(TM) \cong \Lambda(T^*M)$, where we use the inner product to implement the isomorphism between TM and T^*M. Consequently, we have:

$$\begin{aligned} Hom(\mathbf{E}\otimes\mathbf{S}) &\cong Hom(\mathbf{E}) \otimes Hom(\mathbf{S}) \\ &\cong Hom(\mathbf{E}) \otimes Cliff(TM) \cong Hom(\mathbf{E}) \otimes \Lambda(T^*M). \end{aligned} \tag{2.11}$$

We will denote by $\bar\sigma$ the linear isomorphism from $Hom(\mathbf{E}) \otimes Cliff(TM)$ to $Hom(\mathbf{E}) \otimes \Lambda(T^*M)$ and by $\bar\theta$ its inverse. We denote by $\Omega^j(M)$ the smooth forms of degree j on M. Let $\mathbf{L}$ be the pull-back to T^*M of the bundle $Hom(\mathbf{E}) \otimes \Lambda(T^*M)$ via the map $\pi : T^*M \to M$.

DEFINITION 2.2. *A section $p \in \Gamma(\mathbf{L})$ will be called an s-symbol of order l if*

$$p = \sum_{j=0}^{m} p_j(x,\xi) \otimes \omega_j \quad \text{with } p_j \in \Sigma^{l-j} \text{ and } \omega_j \in \Omega^j(M). \tag{2.12}$$

We will denote the collection of s-symbols of order l by $S\Sigma^l(\mathbf{E})$.

It should be noted that $S\Sigma^l$ is also a Fréchet space when we give it the projective tensor product topology, since $\Omega(M)$ is a Fréchet space.

Remark. This definition of an s-symbol seem a bit strange and its ultimate justification will be the resulting asymptotic calculus that we develop. However, if one looks at the papers by the physicist Alvarez-Gaumé [**AG**], and Witten [**W**], one sees that they were led to this definition in a relatively straightforward manner from the study of supersymmetric quantum field theory.

Given a s-symbol $p \in S\Sigma^l$, we want to define an operator $P = \theta(p)$. To do this we follow Bokobza-Haggiag's definition. First, we need to recall the definition of normal coordinates.

Let m be a point in the manifold M and T_mM be the tangent space to M at m. Since we have a connection on TM, there is an exponential map $Exp_m : T_mM \to M$. If $X \in T_mM$, let $\gamma_X(t)$ be the unique geodesic through m such that $\gamma'_X(0) = X$; then $Exp_m(X) = \gamma_X(1)$. Choose an orthonormal basis, $X_1, \ldots, X_n$, for T_mM, then $Exp(x_1X_1 + \cdots + x_nX_n) = (x_1, \cdots, x_n)$ defines a system of normal coordinates in a neighborhood of m_o. Exp can also be viewed as a map $Exp : TM \to M \times M$ which is a diffeomorphism of a neighborhood of the zero section onto a neighborhood of the diagonal in $M \times M$:

$$Exp(m, X_m) = (m, Exp_m(X_m)). \tag{2.13}$$

Let α be a function that is identically one in a neighborhood of the diagonal of $M \times M$ such that the exponential map is a diffeomorphism on the support of α.

Let $(m,x) \in Supp(\alpha)$. There is a unique geodesic from m to x; if $x = Exp_m(X)$ then that geodesic is $Exp_m(tX)$. Let $\tau(m,x) : (\mathbf{E} \otimes \mathbf{S})_m \to (\mathbf{E} \otimes \mathbf{S})_x$ be parallel translation along the unique geodesic from m to x. If $s \in \Gamma(\mathbf{E} \otimes \mathbf{S})$, then we define:

$$\hat{s}_m(x) = \alpha(m,x)\tau(x,m)s(x). \tag{2.14}$$

Notice that $\hat{s}_m(x)$ is really a function on $T_m(M)$, since if x is not in the image of the exponential map, $\alpha(x,m) = 0$ and $\hat{s}_m(x) = 0$. We will write $\hat{s}_m(X)$ for $\hat{s}_m(Exp_m(X))$ when there is no chance of confusion.

DEFINITION 2.3. *Let $p \in S\Sigma(\mathbf{E})$ and $s \in \Gamma(\mathbf{E} \otimes \mathbf{S})$; then we define:*

$$\theta(p)(s)(m) = \int_{T_m(M)\times T_m^*(M)} e^{-i\langle X,\xi\rangle}\bar{\theta}(p)(m,\xi)\hat{s}_m(X)\,dX\,d\xi. \tag{2.15}$$

Remark. The existence of a Riemannian structure on M yields a smooth choice of Lebesque measure on $T_m(M)$. We choose the measure $d\xi$ to be dual to dX in the sense that Fourier inversion holds:

$$\phi(o) = \int_{T_m^*(M)} \int_{T_m(M)} e^{-i\langle X,\xi\rangle} \phi(X)\, dX\, d\xi. \tag{2.16}$$

Remark. The operator $\theta(p)$ depends on the choice of the cut-off function α; however, when we introduce the notion of an asymptotic operator, we will see that the asymptotic operator is independent of the choice of α.

The set of all such operators, along with all infinitely smoothing operators, will be denoted by $Op^a(\mathbf{E})$, and we set $Op(\mathbf{E}) = \cup_a Op^a(\mathbf{E})$.

Definition 2.4. *Given $s \in \Gamma(\mathbf{E} \otimes \mathbf{S})$, define $\bar{s}_m(x) = \alpha(m,x)\tau(m,x)s(m)$. Let $P \in Op(\mathbf{E})$ and $s \in \Gamma(\mathbf{E} \otimes S)$; define $\mu(P) \in End(\mathbf{E})_m \otimes End(\mathbf{S})_m$ by:*

$$\mu(P)(m,\xi)(s(m)) = P_y(e^{i\langle Exp_m^{-1}(y),\xi\rangle} \bar{s}_m(y))|_{y=m}. \tag{2.17}$$

Recall the map $\bar{\sigma} : Cliff(TM) \to \Omega(M)$ from equation 2.10. We define the s-symbol of P by:

$$\sigma(P)(m,\xi) = \bar{\sigma}(\mu(P)(m,\xi)). \tag{2.18}$$

Thus, $\sigma(P)(m,\xi)$ is a form-valued endomorphism of $\mathbf{E}_m$.

Remark. The two maps θ and σ are not inverses of each other; however, at the level of asymptotic operators, they will be inverses.

The following two examples will be key steps in obtaining the formula for the s-symbol of the composition of two operators.

Fix a point, $m \in M$, and choose an orthonormal basis, $X_1, \ldots, X_n$, of $T_m(M)$; then we have normal coordinates, $(x_1, \ldots, x_n) \to Exp_m(x_1 X_1 + \cdots + x_n X_n)$, in a neighboorhood of m. We also have that $(x_1, \ldots, x_n)$ provide coordinates on $T_m(M)$. Let $(\xi_1, \ldots, \xi_n)$ be dual coordinates to $(x_1, \ldots, x_n)$ on $T_m^*(M)$, so $\xi = \sum \xi_i X_i^*$. Let $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_p)$ be a basis for $(\mathbf{E} \otimes \mathbf{S})_m$ and extend this to a sychronous frame around m. Thus, $\mathbf{s}(x)$ is parallel translation of $\mathbf{s}$ along the unique geodesic from m to x. If $s(m) = \sum s_i(m) \cdot f_i(m)$, then $s_m(x) = \sum s_i(m) \cdot \mathbf{s}_i(x)$; that is, the coeffients of $\bar{s}_m(x)$ with respect to the frame $\mathbf{s}$ are constant.

Example 1.

Let $X = \sum c_i \frac{\partial}{\partial x_i}$ with $c_i \in \mathbf{R}$. (This is only valid in a neighborhood of m.) We can then compute:

$$\begin{aligned} \sigma(\nabla_X)(s)(m) &= \nabla_X(e^{i\langle Exp_m^{-1}(y),\xi\rangle} \bar{s}_m(y))|_{y=m} \\ &= i\langle X,\xi\rangle_m s(m) + \sum_{ij} s_i(m)\omega_i^j(X)_m(\mathbf{s}_j)_m \end{aligned} \tag{2.19}$$

In a synchronous frame around m, $\omega_i^j(X)_m = 0$; so we have:

$$\sigma(\nabla_X)(m,\xi) = i\langle X,\xi\rangle_m. \tag{2.20}$$

Example 2.
Next, we let $X = \sum c_i \frac{\partial}{\partial x_i}$ and $Y = \sum d_i \frac{\partial}{\partial x_i}$ with $d_i, c_i \in \mathbf{R}$. We have:

$$
\begin{aligned}
&\sigma(\nabla_X \nabla_Y)(m,\xi)s(m) = \nabla_X \nabla_Y (e^{i\langle Exp_m^{-1}(y),\xi\rangle}\bar{s}_m(y))|_{y=m} = \\
&\nabla_X (i\langle Y.\xi\rangle e^{i\langle Exp_m^{-1}(y),\xi\rangle}\bar{s}_m(y) + e^{i\langle Exp_m^{-1}(y),\xi\rangle}(\nabla_Y \bar{s}_m)(y))|_{y=m} = \\
&(i\langle Y,\xi\rangle i\langle X,\xi\rangle e^{i\langle Exp_m^{-1}(y),\xi\rangle}\bar{s}_m(y) + i\langle Y,\xi\rangle e^{i\langle Exp_m^{-1}(y),\xi\rangle}\nabla_X \bar{s}_m(y) \\
&\quad + i\langle X,\xi\rangle e^{i\langle Exp_m^{-1}(y),\xi\rangle}(\nabla_Y \bar{s}_m(y) + e^{i\langle Exp_m^{-1}(y),\xi\rangle}\nabla_X\nabla_Y \bar{s}_m(y))|_{y=m}.
\end{aligned}
\tag{2.21}
$$

As we saw above, $(\nabla_X \bar{s}_m(y))|_{y=m} = 0$, and it follows from appendix II of [**ABP**] that in this coordinate system, $\nabla_X\nabla_Y \bar{s}_m(y)|_{y=m} = \Omega(X,Y)s(m)$, where Ω is the curvature of the bundle $\mathbf{E}\otimes\mathbf{S}$. We can break Ω into two pieces, $\Omega_{\mathbf{E}} + \Omega_{\mathbf{S}}$. As we saw before, $\Omega_{\mathbf{S}}(X,Y) = \frac{1}{4}\sum R^k_{lij}c_i d_j e_k e_l$; thus we have:

$$
\sigma(\nabla_X\nabla_Y)(m,\xi) = i\langle X,\xi\rangle i\langle Y,\xi\rangle + \frac{1}{4}\sum R^k_{lij}c_i d_j\, f_k \wedge f_l + \Omega_{\mathbf{E}}(X,Y). \tag{2.22}
$$

Note that the first two terms are of order two while the last term, $\Omega_{\mathbf{E}}$, is of order zero.

Given two polynomial s-symbols, p, q, with associated differential operators, $\theta(p)$ and $\theta(q)$, we want to compute $\sigma(\theta(p)\circ\theta(q))$. The idea behind the computation is very simple. Recall that on the real line $\frac{d}{dx}$ generates a one-parameter group, $e^{t\frac{d}{dx}}$ and $(e^{t\frac{d}{dx}}\cdot f)(x) = f(t+x)$. Clearly, we have:

$$
e^{t\frac{d}{dx}}(f\cdot g) = (e^{t\frac{d}{dx}}f)(e^{t\frac{d}{dx}}g). \tag{2.23}
$$

There is an analogous construction when we have a vector bundle with a connection over a compact manifold. Suppose that $\mathsf{F}\to M$ is a vector bundle over M with connection ∇. If $X\in\Gamma(TM)$, denote by ψ^X_t the one-parameter group of diffeomorphisms generated by X. Given a point, $m\in M$, the map $t\to\psi^X_t(m)$ is a curve in M. Let $s\in\Gamma(\mathsf{F})$: then we have $e^{t\nabla_X}s$ is parallel translation of s along the integral curve $t\to\psi^X_t(m)$. If we have $\phi\in C^\infty(M)$ and $s\in\Gamma(\mathsf{F})$, then:

$$
e^{t\nabla_X}(\phi\cdot s)(m) = \phi(\psi^X_t(m))(e^{t\nabla_X}s)(m). \tag{2.24}
$$

Lemma 2.1. *In a normal coordinate system centered at m, let $X = \sum c_i\frac{\partial}{\partial x_i}$; then $\sigma(e^{t\nabla_X})(x,\xi) = e^{t\sigma(\nabla_X)(m,\xi)}(1+r(t))$, where $r(t)\in\Omega(M)$. If we write $r(t) = \sum t^i r_i, r_i\in\Omega(M)$, then $deg(r_i) < i$.*

Proof: This is just an application of Taylor's theorem with remainder as well as the Campbell-Baker-Hausdorff (CBH) formula. For $s\in\Gamma(\mathbf{E}\otimes\mathbf{S})$, we have:

$$
\begin{aligned}
&\sigma(e^{t\nabla_X})(m,\xi)s(m) = e^{t\nabla_X}(e^{i\langle Exp_m^{-1}(y),\xi\rangle}\cdot\bar{s}_m(y))|_{y=m} = \\
&e^{i\langle Exp_m^{-1}(\psi^X_t(y)),\xi\rangle}\cdot e^{t\nabla_X}(\bar{s}_m(y))|_{y=m} = \\
&e^{it\langle X,\xi\rangle_m}(\bar{s}_m + t\nabla_X\bar{s}_m + \frac{t^2}{2!}\nabla^2_X\bar{s}_m + r(t))|_{y=m}.
\end{aligned}
\tag{2.25}
$$

Now we saw before that $\nabla_X \bar{s}_m|_{y=m} = 0$ and $(\nabla_X^2 \bar{s}_m)|_{y=m} = \Omega(X,X)s(m)$; but Ω is skew so we have $\Omega(X,X) = 0$. One should note that $r(t)$ has no ξ dependence, $r(t) \in \Gamma(\bigwedge^{even}(M) \otimes End(\mathbf{E}))$.

Next, we need to recall the CBH formula. This formula says that if $c_i(A,B)$ are defined by:

$$exp(A)exp(B) = exp\left(\sum_{j=1}^{\infty} c_j(A,B)\right), \tag{2.26}$$

then if $c_1(A,B) = A + B$, the c_j's are uniquely determined by the following recursion formula:

$$\begin{aligned}(n+1)c_{n+1}(A:B) &= \frac{1}{2}[A-B, c_n(A,B)]+ \\ \sum_{p\geq 1, 2p\leq n} K_2p \sum_{\substack{k_1,\ldots,k_{2p}>0 \\ k_1+\cdots+k_2p=n}} &[c_{k_1}(A,B), [\cdots[c_{k_{2p}}(A,B), A+B]\cdots]\cdot\end{aligned} \tag{2.27}$$

Let s be the synchronous frame for $\mathbf{E}\otimes\mathbf{S}$ associated with our coordinate system. Thus we have $\bar{s}_m(y) = \sum s_i(m)\mathbf{s}_i(y)$, the point being that the coefficients in this frame for $\bar{s}_m$ are constants.

In this frame we have $\nabla_X = \partial_X + \omega(X)$, where $\omega(X)$ is the connection one-form. Thus we have:

$$e^{t\nabla_X} = e^{t\partial_X + t\omega(X)} = e^{t(\partial_X+\omega(X))}e^{-t\partial_X}e^{-t\omega(X)}e^{t\omega(X)}e^{t\partial_X}. \tag{2.28}$$

(1) Set $A = \partial_X$;
(2) Set $B = \omega(X)$.

If we apply the CBH formula to equation 2.28, we get:

$$e^{t(\nabla_X+\omega(X))} = e^{t\omega+\{tA,tB\}}e^{-t\omega(X)}e^{t\omega(X)}e^{t\partial_X}. \tag{2.29}$$

The term in the brackets, $\{tA, tB\}$, is of the form $[E_1[E_2, \ldots, [E_l, F], \ldots,]]$, where the E_i's are either tA or tB. Now tB is in $so(n)$ and $[tA, tB] = t^2\partial_X(B) \in so(n)$. Since $so(n)$ is closed under brackets, we see that $\{tA, tB\}$ is an $so(n)$-valued section of $Cliff$; hence, as a differential operator, it is of degree two. Now assign to t the weight -1 and assign to a Clifford operator the weight that is its degree. We see then that the weight of the term $\{tA, tB\}$ is less than or equal to zero. If we apply CBH once more, we have:

$$e^{t(\nabla_X+\omega(X))} = e^{G(tA,tB)}e^{t\omega(X)}e^{t\partial_X}. \tag{2.30}$$

Again, the explicit form of the CBH formula tells us that now the weight of $G(tA, tB)$ is strictly less than zero. Finally, if we evaluate this at the point m, the term $e^{t\omega(X)}|_m = 1$, since $\omega(X)|m = 0$. Thus we see that:

$$\left(e^{t\nabla_X}\bar{s}_m\right)|_{y=m} = e^{G(tA,tB)}\bar{s}(m). \tag{2.31}$$

Since the weight of $G(tA, tB)$ is strictly less than zero, we have that the weight of $e^{G(tA,tB)}$ is also strictly less than zero. If we write $e^{G(tA,tB)} = \sum t^i r_i$, then this says that $deg(r_i) < i$.

One should also note that $\psi_t^X(m) = exp_m(tX)$ for small t, since solving the flow equation is purely a local matter. ∎

Corollary 2.2. *Let $m \in M$ and X be the vector field defined in a neighborhood of m as in Lemma 2.1. Then in that neighborhood, we have:*

$$\sigma(\nabla_X^n) = (i\langle X, \xi\rangle)^n + p(m,\xi) \tag{2.32}$$

where $p(m,\xi) = \sum_{i\geq 1} \omega^i \otimes p_i(m,\xi)$ and $p_i \in \Sigma^{n-i}(\mathsf{E})$.

Proof: One has that $\sigma(\nabla_X^n)$ is equal to $n!$ times the coefficient of t^n in the expansion of $\sigma(e^{t\nabla_X})$. This now follows from lemma 2.1. ∎

Recall that an s-symbol is a section of the vector bundle over $T^*(M)$ defined by $\pi^*(\Omega(M) \otimes Hom(\mathsf{E}))$ with certain growth properties. In [G] a proof of the following theorem was outlined. The following proof is similar to the one in [G] but relies on Taylor's theorem instead of the algebra of formal symbols.

Let L be the bundle over $T^*(M)$ given by:

$$\mathsf{L} = \pi^*(End(\mathsf{E}) \otimes Cliff(T^*(M))). \tag{2.33}$$

The natural map $c : Cliff(T^*(M)) \otimes Cliff(T^*(M)) \to Cliff(T^*(M))$ given by the Clifford product induces a map from $\mathsf{L} \otimes \mathsf{L} \to \mathsf{L}$, which we will also call c.

Theorem 2.1. *Let $p \in S\Sigma^l(\mathsf{E})$ and $q \in S\Sigma^k(\mathsf{E})$ be two symbols whose dependence on the ξ variable is polynomial. There exist differential operators $\bar{a}_n : \Gamma(\mathsf{L}\otimes\mathsf{L}) \to \Gamma(\mathsf{L})$ such that if $a_n(p,q) = \sigma(\bar{a}_n(p,q))$, then:*

(1) $a_n(p,q) \in S\Sigma^{l+k-n}(\mathsf{E})$;
(2) $p \circ q = \sum_{n=0}^{\infty} a_n(p,q)$ *(finite sum); and*
(3) $a_0(p,q) = e^{-\frac{1}{4}R(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta})} p(x,\xi) \wedge q(x,\eta)|_{\xi=\eta}$.

We have used the notation:

$$-\frac{1}{4}R(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta})(p,q) = -\frac{1}{4}\sum_{ijkl} R^k_{lij}\, f_k \wedge f_l \wedge \frac{\partial p}{\partial\xi_i}(x,\xi) \wedge \frac{\partial q}{\partial\eta_j}(x,\eta). \tag{2.34}$$

Remark. There are two important parts to this theorem. First, the order of the symbol of the composition of two operators is less than or equal to the sum of the orders of the operators. While this is true for ordinary differential operators, the presence of the Clifford variables makes this far from trivial for supersymmetric differential operators. It will be this fact that is crucial in developing the calculus of asymptotic operators. The second part will be the formula for $a_0(p,q)$.

Proof: We will work locally; thus let $m \in M$ and let X and Y be the vector fields defined in a neighborhood of m as in example 2. Let $\wp_X^n = \sigma(\nabla_X^n)$; then by corollary 2.2 we have $\wp_X^n = (i\langle X,\xi\rangle)^n$ plus lower order terms in ξ. An easy induction shows that if $p \in S\Sigma(\mathsf{E})$ is an s-symbol with polynomial dependence in ξ, then we can write p as :

$$p(x,\xi) = \sum_i h_i(x)\wp_{X_i}^{n_i}, \quad h_i \in End(\mathsf{E}) \otimes \Omega(M). \tag{2.35}$$

Since the map $(p,q) \to \sigma(\theta(p) \circ \theta(q))$ is bilinear, it suffices to prove the theorem for symbols of the form $h(x)\wp_X^n$.

Let $h, g \in End(\mathbf{E}) \otimes \Omega(M)$; then we have:

$$\begin{aligned} &\sigma(\theta(h(\wp_X)^n\theta(g(\wp_Y)^m)) = \sigma(\bar{\theta}(h)(\nabla_X)^n\bar{\theta}(g)(\nabla_Y)^m) = \\ &\sigma(\bar{\theta}(h)\left(\sum_k \bar{\theta}(\nabla_X^{n-k})(g)(\nabla_X^k\nabla_Y^m)\right)) = \\ &\sum_k \sigma(\bar{\theta}(h)\bar{\theta}(\nabla_X^{n-k}g))\sigma\left((\nabla_X)^k(\nabla_Y)^m\right) \quad \text{mod lower order terms.} \end{aligned} \tag{2.36}$$

In the last line we have used equation 1.15. Thus, we are reduced to proving the theorem for the operators $(\nabla_X)^n$ and $(\nabla_Y)^m$.

Next, we need to recall the Campbell-Baker-Hausdorff formula again.

Note that it follows from 2.27 that $c_n(tA, tB) = t^n c_n(A, B)$, so $c_n(A, B)$ is a homogeneous polynomial in A and B of degree n. If we apply 2.26 with $A = t\nabla_X$ and $B = s\nabla_Y$, then we have:

(1) $c_1(t\nabla_X, s\nabla_Y) = t\nabla_X + s\nabla_Y$; and
(2) $c_2(t\nabla_X, s\nabla_Y) = \frac{1}{2}ts[\nabla_X, \nabla_Y] = \frac{1}{2}ts\Omega(X,Y)$.

Recall that $\Omega = \Omega_{\mathbf{S}} + \Omega_{\mathbf{E}}$, where $\Omega_{\mathbf{S}}$ is the curvature of the spin bundle and $\Omega_{\mathbf{E}}$ is the curvature of $\nabla_{\mathbf{E}}$. As a ψDO, $\Omega_{\mathbf{S}}(X,Y)$ is of order two, while $\Omega_{\mathbf{E}}(X,Y)$ is of order zero.

Now suppose that ϕ is a section of $Cliff(T^*(M))$ such that $\phi(x) \in so(n) \subset Cliff(T^*(M)_x)$ for all $x \in M$; then $[\nabla_X, \phi] = \nabla_X(\phi) \in so(n)$. Since $so(n)$ is closed under brackets, we conclude from the Campbell-Baker-Hausdorff formula, (2.32), that $c_j(\nabla_X, \nabla_Y)$ is of order two as a ψDO for $j \geq 2$.

Remark. It will be important to notice that as a ψDO, the operators $c_j(X,Y)$ contain NO differentiation; they are purely Clifford variable operators. This follows from the explicit form of the CBH formula.

In what follows, associate with t and s weight -1 and associate with ξ and e weight 1. We have that $\sigma((t\nabla_X)^a \circ (s\nabla_Y)^b)$ is the coefficient of $t^a s^b$ in the expression $\sigma(e^{t\nabla_X} \circ e^{s\nabla_Y})$. We need to show that the weight of each term of the coefficient of $t^a s^b$ is less than or equal to zero and to determine the term of weight zero. Applying the CBH formula we have:

$$\begin{aligned} &\sigma(e^{t\nabla_X} \circ e^{s\nabla_Y}) \equiv \sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2}st\Omega(X,Y) + C(X,Y)} \equiv \\ &\sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2}\Omega_{\mathbf{s}}(X,Y) + C(X,Y)}). \end{aligned} \tag{2.37}$$

Define:

(1) $A = t\nabla_X + s\nabla_Y + \frac{1}{2}st\Omega_{\mathbf{S}}(X,Y)$ — this is weight 0;
(2) $B = C(X,Y)$— this is weight strictly less than 0.

We have:

$$e^{A+B} = e^{A+B}e^{-A}e^{A} = (e^{B+\{A,B\}})e^{A} \quad \text{by the CBH formula.} \tag{2.38}$$

We see from CBH that if the weight of B is less than zero, then the weight of the term$\{A, B\}$ is also less than zero; also the operator $e^{B+\{A,B\}}$ is purely a Clifford operator. Since this term is a Clifford operator, we can use equation 1.15 again to conclude that:

$$\sigma(e^{B+\{A,B\}}\, e^{A}) = \sigma(e^{B+\{A,B\}}) \wedge \sigma(e^{A}) \quad \text{mod lower order terms.} \tag{2.39}$$

Now $\sigma(e^{B+\{A,B\}})$ is of the form $1+f$, where f has weight strictly less than zero. Thus to finish the theorem, we need to show that $\sigma(e^A)$ has the desired form and compute the term of weight 0. To do this, we use the CBH formula again.

$$\begin{aligned}\sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2}st\Omega_s(X,Y)}) =& \sigma(e^{\frac{1}{2}ts\Omega_s(X,Y)}e^{t\nabla_X+s\nabla_Y}) \\ & \text{mod Clifford terms of negitive weight .}\end{aligned} \tag{2.40}$$

Finally, using lemma 2.1, we have equation 2.40 becomes:

$$\begin{aligned}\sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2}st\Omega_s(X,Y)}) =& e^{\frac{1}{2}ts\Omega_s(X,Y)}e^{it\langle X,\xi\rangle}e^{is\langle Y,\xi\rangle} \\ & \text{mod terms of negative weight .}\end{aligned} \tag{2.41}$$

Replacing X by iX, Y by iY and equating the powers of t and s, we obtain the conclusion of the theorem. ∎

3. Asymptotic Pseudodifferential Operators and their Calculus.

Widom developed a theory of asymptotics for pseudodifferential operators. If $p(x,\xi) \in \Sigma^m(\mathsf{E})$ is an ordinary symbol, Widom considered the family of pseudodifferential operators defined by $P_t = \theta(p(x,t\xi))$. When one considers this family instead of just the single operator P, one finds that the asymptotic expansions normally associated with heat kernels (as well as other expansions) follow naturally from the calculus that he developed. In our setting, we want to extend Widom's ideas to the class of s-symbols. The s stands for supersymmetric, and in supersymmetry Clifford multiplication is viewed as a differential operator.

Let $p \in S\Sigma^m(\mathsf{E})$, so that $p(x,\xi) = \sum_{i=0}^m p_i(x,\xi)\otimes\omega_i$, with $p_i \in \Sigma^{m-i}(\mathsf{E})$ and $\omega_i \in \Omega^i(M)$. We have the following definition, which was introduced in [G].

Definition 3.1. *If $p(x,\xi) \in S\Sigma^m(\mathsf{E})$ then:*

$$p_t(x,\xi) = \sum_{i=0}^{m} p_i(x,t\xi)\otimes t^i\omega_i. \tag{3.1}$$

Next we need to recall the classical notion of an asymptotic expansion.

Definition 3.2. *Let $p(x,\xi,t) \in S\Sigma^m(\mathsf{E})$ be a family of symbols; then $p(x,\xi,t)$ is called an asymptotic family if there exist symbols $p_n \in S\Sigma^{m-n}(\mathsf{E})$ such that the following asymptotic expansion holds:*

$$p(x,\xi,t) \sim \sum_{n=0}^{\infty} t^n p_n(x,\xi). \tag{3.2}$$

In other words, given $N > 0$, we have (recalling that the symbol spaces are Fréchet spaces):

$$\lim_{t\to 0} t^{-N}\left(p(x,\xi,t) - \sum_{i=0}^{N} t^i p_i(x,\xi)\right) = 0 \quad in S\Sigma^{m-n}(\mathbf{E}). \tag{3.3}$$

We will call the first term, $p_0(x,\xi)$, the leading symbol of p (not to be confused with the principal symbol).

As usual, the notion of an asymptotic expansion gives us an equivalence relation on the space of asymptotic families of symbols. This leads to the following definition:

DEFINITION 3.3. *Given an asymptotic family of symbols, $p(x,\xi,t)$, we call the equivalence class that p_t determines an asymptotic symbol. We will call a representative of the equivalence class an asymptotic symbol if no confusion is likely to arise.*

Let $H^s(\mathbf{E})$ be the s-th Sobolev space formed from the sections of $\mathbf{E}\otimes\mathbf{S}$, so $H^s(\mathbf{E})$ is the completion of $\Gamma(\mathbf{E}\otimes\mathbf{S})$ (see **[Gi]**). If P is an infinitely smoothing operator, then $P : H^s(\mathbf{E}) \to H^0(\mathbf{E})$, and we denote by $\|P\|_s$ the corresponding norm. This turns the space of smoothing operators into a Fréchet space.

DEFINITION 3.4. *Let P_t be a family of smoothing operators; then P_t is called asymptotically zero if $P_t \sim 0$ in the Fréchet space topology. Thus, given $N > 0$, we have for all s:*

$$\lim_{t\to 0} t^{-N}\|P_t\|_s = 0. \tag{3.4}$$

DEFINITION 3.5. *Given two families of operators, P_t and Q_t, we will say that P_t is equivalent to Q_t if the difference $P_t - Q_t$ is asymptotically zero. If P_t is equivalent to $\theta(p(x,\xi,t)_t)$ for p an asymptotic symbol, then we will call the equivalence class that P_t determines an asymptotic pseudodifferential operator, $A\psi DO$. As before, we will call a representative of this class an $A\psi DO$ if no confusion is likely to arise.*

Remark. Since the map θ depends on the choice of a cut-off function α, we should write $A\psi DO^\alpha$ for the corresponding collection of operators. However, the next lemma shows that this collection is independent of the choice of α .

LEMMA 3.6. *Let $\phi \in C^\infty(M\times M)$ which is identically zero in a neighborhood of the diagonal $\Delta \subset M\times M$ and whose support is contained in the neighborhood of Δ where the exponential map is a diffeomorphism. Given any asymptotic symbol, $p(x,\xi,t)$, define $\bar{P}_t$ by:*

$$\bar{P}_t s(m) = \iint e^{i\langle X_m,\xi_m\rangle}\bar{\theta}(p(m,\xi,t)_t)\phi(m, Exp_m(X))\hat{s}_m(X_m)\, dX_m\, d\xi_m. \tag{3.5}$$

Then $\bar{P}_t$ is an asymptotically zero operator.

PROOF: This is similar to lemma 1.2.6 of **[Gi]**. We can assume that $p(x,\xi,t) \in \Sigma^k(\mathbf{E})$, since p will be a finite sum of elements of this form times a form. Define a Laplican by $\Delta_\xi = \sum_j (i\frac{\partial}{\partial \xi_j})^2$; then $\Delta_\xi e^{i\langle X_m,\xi_m\rangle} = |X_m|^2 e^{i\langle X_m,\xi_m\rangle}$. Since $|x_m|^{-2l}\phi(m,X_m) \in C^\infty(T(M))$ for all l, we can integrate by parts to get:

(3.6)
$$\bar{P}_t s(m) = \iint e^{i\langle X_m,\xi\rangle}|X_m|^{-2l}\phi(m,X_m)\bar{\theta}(\Delta_\xi^l p(m,t\xi,t))\hat{s}_m(X_m)\,dX_m\,d\xi_m.$$

Now $\Delta_\xi^l p(x,t\xi,t) = t^{2l}(\Delta_\xi^l p)(x,t\xi,t) \in \Sigma^{k-2l}(\mathsf{E})$. Note that we get powers of t coming out from the repeated integration by parts. This is sufficient to prove the lemma. ∎

COROLLARY 3.7. *Let $p(x,\xi,t)$ be an asymptotic symbol; then the asymptotic operator $P_t = \theta(p(x,\xi,t)_t)$ is independent of the choice of the cut-off function α that was chosen to define θ.*

PROOF: This follows from lemma 3.6 . ∎

LEMMA 3.8. *Let $r(m,\xi_m,y,t) \in C^\infty(M, S\Sigma^d(\mathsf{E}))$ such that for all α,β,γ we have :*
$$||\partial_m^\alpha \partial_\xi^\beta \partial_y^\gamma r(m,\xi_m,y,t)|| \leq C_{\alpha,\beta,\gamma}(1+|\xi_m|)^{d-|\beta|}$$
where $C_{\alpha,\beta,\gamma}$ is independent of t. We denote the space of such functions by $\mathcal{S}$, which is a Fréchet space with the semi-norms:
$$\rho_{\alpha,\beta,\gamma}(r) = inf\{C_{\alpha,\beta,\gamma}|\quad ||\,\partial_m^\alpha \partial_\xi^\beta \partial_y^\gamma r(m,\xi_m,y,t)|| \leq C_{\alpha,\beta,\gamma}(1+|\xi_m|)^{d-|\beta|}\,\}.$$
Furthermore, assume that $r(m,\xi,y,t)$ has an asymptotic expansion:
$$r(m,\xi,y,t) \sim \sum_{n=0}^\infty t^n r_n(m,\xi,y).$$

If we define:
$$R_t(s)(m) = \int e^{-i\langle X_m,\xi_m\rangle} r(m,\xi_m,exp_m(X_m),t)_t \hat{s}(X_m)\,dX_m\,d\xi_m$$
then R_t is an $A\psi DO$, and if $R_t \sim \theta(p_t)$ we have:
$$p(m,\xi_m,t) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!}\partial_{X_m}^\alpha \partial_{\xi_m}^\alpha r(m,\xi_m,exp_m(X_m),t)|_{X_m=0}.$$

PROOF: We will write $r(m,\xi_m,X_m,t)$ for $r(m,\xi_m,exp_m(X_m),t)$. In what follows we will assume that r has no form component. Note that since $\hat{s}(X_m)$ vanishes off a neighborhood of $\Delta \subset M\times M$, R_t depends only on $r(m,\xi_m,X_m,t)$ for X_m in a neighborhood of zero. Thus we will suppose WLG that $r(m,\xi_m,X_m,t)$ has compact support in the X_m variable. We will write
$$\tilde{s}(\xi_m) = \int e^{-i\langle X_m,\xi_m\rangle}\hat{s}(X_m)\,dX_m \quad \text{and}$$
$$q(m,\xi_m,\eta_m,t) = \int e^{-i\langle X_m,\eta_m\rangle}\hat{s}(X_m)\,dX_m.$$

Using the fact that the Fourier transform of the product is the convolution of the Fourier transforms, we get:

$$(3.7)\qquad \begin{aligned}(R_t s)(m) = \int e^{-i\langle X_m,\xi_m\rangle} r(m, t\xi_m, X_m, t)\hat{s}(X_m)\, dX_m\, d\xi_m = \\ \int q(m, t\xi_m, \xi_m - \eta_m, t)\tilde{s}(\eta_m)\, d\eta_m\, d\xi_m.\end{aligned}$$

We estimate $|q(m, t\xi_m, \xi_m - \eta_m, t)|$ in the usual way. The estimates on r and the fact that $q(m, \xi, \eta, t)$ is rapidly decreasing in the eta variable yields:

$$(3.8)\qquad |q(m, t\xi_m, \xi_m - \eta_m, t)| \leq C_k(1 + |t\xi|)^d(1 + |\xi - \eta|)^{-k}.$$

Now Peetre's inequality says: $(1 + |x + y|)^s \leq (1 + |x|)^s(1 + |y|)^{|s|}$. Using this inequality and the fact that for all l $|\tilde{s}(\eta)| \leq C_l(1 + |\eta|)^{-l}$, we get that $|q(m, t\xi_m, \xi_m - \eta_m, t)\tilde{s}(\eta_m)|$ is integrable; thus we can interchange the order of integration . Define p by:

$$(3.9)\qquad \begin{aligned}p(m, t\eta_m, t) = \int q(m, t\xi_m, \xi_m - \eta_m, t)\, d\xi_m = \\ \int q(m, t\xi_m + t\eta_m, \xi_m, t)\, d\xi_m.\end{aligned}$$

Then we have :

$$(3.10)\qquad R_t(s)(m) = \int e^{-i\langle X_m,\xi_m\rangle} p(m, t\eta_m, t)\hat{s}(X_m)\, dX_m\, d\xi)_m.$$

Thus we only need to check that $p(m, \eta_m, t)$ is a symbol of order d and that it has the requisite asymptotic expansion.

Define a linear map $T : \mathcal{S} \to S\Sigma(\mathsf{E})$ by $T(r) = p$, where p is defined as above. We have that T is continuous from the Fréchet space $\mathcal{S}$ to the Fréchet space $S\Sigma(\mathsf{E})$. We have:

$$(3.11)\qquad \begin{aligned}&|\, \partial_m^\alpha \partial_{\eta_m}^\beta q(m, t\xi_m + \eta_m, \xi_m, t)\, | = \\ &|\, \int e^{-i\langle X_m,\xi_m\rangle} \partial_m^\alpha \partial_{\eta_m}^\beta r(m, t\xi_m + \eta_m, X_m, t)\, dX_m\, | = \\ &|\, \int e^{-i\langle X_m,\xi_m\rangle} \xi_m^{-\gamma} \partial_m^\alpha \partial_{\eta_m}^\beta \partial_{X_m}^\gamma r(m, t\xi_m + \eta_m, \xi_m, t)\, dX_m \leq \\ &KC_{\alpha,\beta,\gamma}(1 + |\xi_m|)^{-|\gamma|}(1 + |t\xi_m + \eta_m|)^{d-|\beta|} \leq \\ &KC_{\alpha,\beta,\gamma}(1 + |\xi_m|)^{-|\gamma|}(1 + |t\xi_m|)^{d-|\beta|}(1 + |\eta_m|)^{d-|\beta|}.\end{aligned}$$

The constant K is independent of r and bounds the volume of the compact set that $r(m,\xi,X_m,t)$ has for its X_m support. If $|\gamma|$ sufficiently large we can integrate the above inequality to get:

$$\rho_{\alpha,\beta}(T(r) \leq K\rho_{\alpha,\beta,\gamma}(r)). \tag{3.12}$$

This shows that $T(r)$ is a symbol and that if $r(m,\xi,X_m,t)$ has an asymptotic expansion, then, $T(r)$ also has an asymptotic expansion.

Using Taylor's formula we have:

$$q(m,t\xi_m+\eta_m,t) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!}\frac{1}{t^{|\alpha|}}\partial^{\alpha}_{\xi_m} q(m,\eta_m,t)(t\xi_m)^{\alpha}. \tag{3.13}$$

Thus we have:

$$\begin{aligned} &p(m,\eta_m,t) \sim \sum_{\alpha\geq 0}\frac{1}{\alpha!}\iint e^{-i\langle X_m,\xi_m\rangle}\xi_m^{\alpha}\partial^{\alpha}_{\xi_m} r(m,\eta_m,X_m,t)\,dX_m\,d\xi_m \sim \\ &\sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial^{\alpha}_{X_m}\partial^{\alpha}_{\xi_m} r(m,\eta_m,X_m,t)|_{X_m=0}. \end{aligned} \tag{3.14}$$

∎

LEMMA 3.9. *If $P_t \sim \theta(p_t)$ is an $A\psi DO$, then the adjoint, P_t^*, is also an $A\psi DO$.*

PROOF: Recall that $\hat{s}_m(y)$ is parallel translation of $s(y)$ from $(\mathbf{E}\otimes\mathbf{S})_y$ to $(\mathbf{E}\otimes\mathbf{S})_m$times $\phi(m,y)$ where $\phi(m,y)$ is zero off of a neighborhood of $\Delta \subset M\times M$, where the exponential map is a diffeomorphism. Let s,h be sections and set $y = exp_m(X_m)$; then we have :

(3.15)

$$\begin{aligned} &\langle P_t s,h\rangle = \int e^{-i\langle X_m,\xi_m\rangle}\langle p(m,t\xi_m,t)\hat{s}(X_m),h(m)\rangle\,dX_m\,d\xi_m\,dvol(m) = \\ &\int e^{-i\langle X_m,\xi_m\rangle}\langle \phi(m,y)\tau(m,y)s(y),p(m,t\xi_m,t)^*h(m)\rangle\,dX_m\,d\xi_m\,dvol(m). \end{aligned}$$

Recall that parallel translation is an isometry. Thus we have:

(3.16)

$$\begin{aligned} &\langle \tau(m,y)s_m(y),p(m,t\xi_m,t)^*h(m)\rangle|_m = \langle s(y),\tau(y,m)\,(p(m,t\xi_m,t)^*h(m))\rangle|_y = \\ &\langle s(y),(\tau(y,m)p(m,t\xi_m,t)^*)\,(\tau(y,m)h(m))\rangle|_y = \\ &\langle s(y),(\tau(y,m)p(m,t\xi_m,t)^*)\,\hat{s}(X_y)\rangle|_y. \end{aligned}$$

Let $dX_m = J(m,y)dvol(y)$, and $dvol(m) = J^{-1}(y,m)dX_y$; then we have $dX_m\,dvol(m) = a(m.y)dvol(y)\,dX_y$, where the function $a(m,y)$ has all of its derivatives bounded on the support of $\phi(m,y)$.

Now $exp_m(sX_m) = \gamma(s)$ is the unique geodesic with $\gamma(0) = m$ and $\gamma(1) = y$, thus $\gamma(1-s)$ is the unique geodesic from y to m. Since $\gamma\prime(0) = -\tau(y,m)(X_m)$, we see that $exp_y^{-1}(m) = -\tau(y,m)(X_m)$. Now $\langle X_m, \xi_m\rangle = \langle \tau(y,m)X_m, \tau(y,m)\xi_m\rangle$, so 3.15 equals:

$$\begin{aligned}(3.17)\qquad &\int e^{-i\langle X_y,\xi_y\rangle}\phi(m,y)a(m,y)\\ &\langle s(y), \tau(y,m)p(m,\tau(m,y)t\xi_y,t)^*\hat{h}_y(X_y)\rangle\, dX_y\, d\xi_y\, dvol(y).\end{aligned}$$

If we set $r(y,\xi_y,X_y,t) = \phi(m,y)a(m,y)\tau(y,m)p(m,\tau(m,y)t\xi_y,t)^* = \phi(m,y)a(m,y)\tau(y, exp_y(X_y))p(exp_y(X_y),\tau(m,y)t\xi_y,t)^*$, then we see that :

$$(3.18)\qquad (P_t^*h)(y) = \int e^{-i\langle X_y,\xi_y\rangle} r(y,t\xi_y,X_y,t)\hat{h}(X_y)dX_y\, d\xi_y.$$

Once we show that r satisfies the conditions of lemma 3.8 we are done. ∎

LEMMA 3.9. *Suppose that $r(m,\xi_m,t)$ is a family of symbols in $S\Sigma^n(\mathsf{E})$ such that the symbol estimates hold uniformly in t. If $R_t = \theta(r_t)$, then $\sigma(R_t)_{t^{-1}}(m,\xi_m) \sim r(m,\xi_m,t)$.*

(It should be noted that we are NOT assuming that r is an asymptotic symbol.)

PROOF: Without loss of generality, we can assume that r has no form component. Now we compute, If $y = Exp_m(X_m)$, then we have:

$$\begin{aligned}&\sigma(R_t)(\xi_m/t,m)s(m) = R_t\left(e^{-i\langle Exp_m^{-1}(y),\xi_m/t\rangle}\bar{s}(y)\right)\big|_{y=m} =\\ (3.19)\qquad &\left(\int e^{i\langle X_m,\eta_m\rangle}e^{-i\langle X_m,\xi_m/t\rangle}r(m,t\eta_m,t)\phi(m,X_m)\,dX_m\,d\eta_m\right)s(m) =\\ &\left(\int e^{i\langle X_m,\eta_m\rangle}r(m,t\eta_m+\xi_m,t)\phi(m,X_m)\,dX_m\,d\xi_m\right)s(m).\end{aligned}$$

Now we use Taylor's theorem on $r(m,t\eta_m+\xi_m,t)$ to get:

$$\begin{aligned}&r(m,t\eta_m+\xi_m,t) = \sum_{|\alpha|<m}\frac{t^{|\alpha|}}{\alpha!}\partial_\eta^\alpha r(m,\xi_m,t) + E\\ (3.20)\qquad &\text{where } E = \sum_{|\alpha|=m}\frac{m}{\alpha!}(t\alpha)^m\int_0^1 u^{m-1}\partial_\xi^\alpha r(m,\xi+(1-u)t\eta)\,du.\end{aligned}$$

Recall that the function $\phi(m,X_m)$ is identically one in a neighborhood of zero; so we have:

$$\begin{aligned}&\frac{t^{|\alpha|}}{\alpha!}\int e^{i\langle X_m,\eta_m\rangle}\eta_m^\alpha\partial_\eta^\alpha r(m,\xi_m,t)\,dX_m\,d\eta_m =\\ (3.21)\qquad &\frac{t^{|\alpha|}}{\alpha!}\partial_\eta^\alpha r(m,\xi_m,t)\partial_{X_m}\phi(m,X_m)|_{X_m=0} = \begin{cases} r(m,\xi_m,t) \text{ if } \alpha = 0\\ 0 \text{ if } \alpha \neq 0\end{cases}.\end{aligned}$$

To finish, we have to estimate the remainder term. First we note:

$$\begin{aligned}(3.22)\qquad &\| \partial_\xi^\alpha r(m, \xi + (1-u)t\eta, t) \| \leq C(1 + |\xi + (1-u)t\eta|)^{n-m} \leq \\ &C(1+|\xi|)^{n-m}(1 + |(1-u)t\eta|)^{|n-m|} \leq C(1+|\xi|)^{n-m}(1+|\eta|)^{|n-m|}.\end{aligned}$$

Therefore, integrating this estimate and remembering that $\int e^{i\langle X_m, \eta_m\rangle}\phi(m, X_m)\, dX_m$ is rapidly decreasing in η_m yields:

(3.23)

$$\begin{aligned}&\| \int e^{i\langle X_m,\eta_m\rangle} E(m, \xi_m, \eta_m, t)\phi(m, X_m)\, dX_m\, d\eta_m \| \leq \\ &Ct^m(1+|\xi_m|)^{n-m} \int \left| \int e^{i\langle X_m,\eta_m\rangle}\phi(m, X_m)\, dX_m \right| (1 + |\eta_m|)^{|n-m|}\, d\eta_m \leq \\ &Ct^m(1+|\xi_m|)^{n-m}\end{aligned}$$

One can similarly estimate the derivatives of the error term. We conclude that $\forall m$:

$$(3.24)\qquad \| r(m, \xi_m, t) - \sigma(R_t)_{t^{-1}}(m, \xi_m) \| \leq Ct^m(1+|\xi_m|)^{n-m}.$$

∎

LEMMA 3.10. *Let $p(m, \xi, t)$ and $q(m, \xi, t)$ be two asymptotic symbols and let $P_t = \theta(p_t)$ and $Q_t = \theta(q_t)$ be the respective pseudodifferential operators; then $P_t \circ Q_t = \theta(r_t)$, where r is also an asymptotic symbol. Furthermore, the leading symbol of $P_t \circ Q_t$ is given by $a_0(p_0, q_0)$, where p_0 and q_0 are the leading symbols of p and q respectively.*

PROOF: We recall how $\theta(p_t)$ was defined. Let $\alpha(m, y)$ be a smooth function defined in a neighborhood of the diagonal $\Delta \subset M \times M$, where the exponential map is a diffeomorphism. Let $\tau(m, y)$ be parallel translation along the unique geodesic from m to y for $(m, y) \in supp(\alpha)$. For a section s we defined $\hat{s}_m(y) = \alpha(m, y)\tau(m, y)s(y)$. If $y = exp_m(X_m)$, then we wrote $\hat{s}(X_m)$ for $\hat{s}_m(y)$. The point to remember is that $\hat{s}(X_m)$ vanishes outside a neighborhood of zero.

In what follows we will omit the cut-off function α from most of the calculations to make the notation clearer. Now we compute. Let $y = exp_m(X_m)$ and $z = exp_y(X_y)$; then :

(3.25)

$P_t \circ Q_t(s)(m) =$

$$\int_{T^*_m}\int_{T_m} e^{i\langle X_m,\xi_m\rangle} p(m, t\xi_m)\tau_m^y \left(\int_{T^*_y}\int_{T_y} e^{i\langle X_y,\eta_y\rangle} q(y, t\eta_y)\tau_y^z s(z)\, dX_y\, d\eta_y \right) dX_m\, d\xi_m.$$

Let Y_m be the parallel translate along the geodesic from y to m of X_y and ζ_m the parallel translate of η_y along the geodesic from y to m. Since parallel

translation is an isometry, we have that $\langle X_y, \eta_y\rangle_y = \langle Y_m, \zeta_m\rangle_m$, and the measures are also the same. Thus (3.21) becomes:
(3.26)
$$\int e^{i\langle Y_m, \zeta_m\rangle}\left(\int e^{i\langle X_m \cdot \xi_m\rangle} p(m, t\xi_m)\tau_m^y q(y, t\tau_y^m(\zeta_m))\tau_m^y \tau_y^z \tau_z^m \, dX_m \, d\xi_m\right) \tau_m^z s(exp_m(\phi(Y_m)))\, dY_m \, d\zeta_m$$

In the above equation we have written, $\phi(Y_m) = exp_m^{-1}(exp_y(\tau_m^y(Y_m)))$; note that $\phi(0) = X_m$.

Changing variables, with J the Jacobian of the transformation, yields:
(3.27)
$$\int e^{i\langle \phi^{-1}(Y_m), \zeta_m\rangle}\left(\int e^{i\langle X_m \cdot \xi_m\rangle} p(m, t\xi_m)\tau_m^y q(y, t\tau_y^m(\zeta_m))\tau_m^y \tau_y^z \tau_z^m \, dX_m \, d\xi_m\right) \tau_m^z s(exp_m(Y_m))J(Y_m)\, dY_m \, d\zeta_m.$$

Next note that $\phi^{-1}(0) = -X_m$ so by a result in **[Gi]** (bottom of the pg. 25); $\phi^{-1}(Y_m) = -X_m + A(Y_m)\cdot Y_m$, where $A(Y_m)$ is a linear map that is invertable in a neighborhood of zero. Thus equation (3.23) becomes:
(3.28)
$$\int e^{i\langle -X_m + A(Y_m)\cdot Y_m, \zeta_m\rangle} \int e^{i\langle X_m \cdot \xi_m\rangle} p(m, t\xi_m)\tau_m^y q(y, t\tau_y^m(\zeta_m))\tau_m^y \tau_y^z \tau_z^m \, dX_m \, d\xi_m \; \tau_m^z s(exp_m(\phi(Y_m)))J(Y_m)\, dY_m \, d\zeta_m.$$

Make another change of variables to get:

$$\int e^{\langle Y_m, \zeta_m\rangle} r(m, Y_m, t\zeta_m)\hat{s}_m(Y_m)\, dY_m \, d\zeta_m \qquad \text{where } r(m, t\zeta_m, Y_m) =$$
$$\int e^{i\langle X_m, \xi_m - A(Y_m)^t(\zeta_m)\rangle} p(m, t\xi_m)\tau_m^y q(y, t\tau_y^m(A(Y_m)^{-1t}(\zeta_m))) \tau_m^y \tau_y^z \tau_z^m K(Y_m)\, dX_m \, d\xi_m. \tag{3.29}$$

Finally, we have:

$$r(m, t^{-1}\zeta_m, Y_m)_t = \int e^{i\langle X_m, \xi_m\rangle} p(m, t\xi_m + B(Y_m)\zeta_m)\tau_m^y(q(y, \tau_y^m(C(Y_m))\zeta_m)S(m, y, z) K(Y_m) dX_m \, d\xi_m. \tag{3.30}$$

Now standard estimates show that r is a symbol (see **[T]**,vol. 1, pp. 23-29 theorem 3.2 and 3.3).

Next we neeed to show that the symbol of $P_t \circ Q_t$ has the requisite asymptotic expansion.

To do this, we appeal to Widom's calculation of the symbol of the composition of two pseudodifferential operators. Recall that in definition 2.4, given a pseudodifferential operator P, we defined $\mu(P) \in End(\mathbf{E}) \otimes End(\mathbf{S})$, which would be

the symbol that is defined by Widiom. Widiom then shows that:
(3.31)

$$\mu(P \circ Q) \sim \sum_{m_1 \dots m_k \geq 2} \frac{i^{k - \sum_0^m p_i - \sum_1^k m_i}}{k!p_0!p_1! \dots !p_k!m_1! \dots !m_k!} \nabla^{p_1+m_1} l \dots \nabla^{p_k+m_k} l \nabla^{p_0} \times$$
$$D^{\sum m_i} \mu(P) D^{\sum p_i} \mu(Q).$$

We can rephrase this in terms of theorem 2.1. Let $\mathbf{L}$ be the bundle over $T^*(M)$ defined in equation 2.7; then there exist differential operators $\bar{a}_n : \Gamma(\mathbf{L} \otimes \mathbf{L}) \to \Gamma(\mathbf{L})$, such that if $a_n = \sigma \circ \bar{a}_n$:

$$\sigma(\theta(p_t) \circ \theta(q_t)) \sim \sum t^n a_n(p,q)_t. \tag{3.32}$$

The degree of $a_n(p,q)$ is determined by two factors:

(1) how many times p and q are differentiated in the ξ variable; and
(2) the degree of the coefficents of a_n (since the coefficients of a_n will be in $End(\mathbf{E}) \otimes End(\mathbf{S}) \equiv End(\mathbf{E}) \otimes Cliff(T(M))$).

The a_n are determined by what they do on symbols that are polynomial in ξ (that is on differential operators). Suppose that $p = \omega \cdot \bar{p}$ with $\omega \in \Omega^s(M)$and $\bar{p}$ a homogeneous polynomial of degree $k - s$ in ξ. Then p is a symbol of degree k and $p_t = t^k p$. We make the same assumption about q, so that $q_t = t^l q$. Theorem 2.1 says:

$$\sigma(\theta(p_t) \circ \theta(q_t))_{t^{-1}} = t^{l+k} \sum_{n=0}^{\infty} a_n(p,q)_{t^{-1}}. \tag{3.33}$$

Now $a_n(p,q) \in S\Sigma^{l+k-n}(\mathbf{E})$, thus $a_n(p,q)_{t^{-1}} = t^{-(l+k-n)} a_n(p,q)$. If we put this into equation 3.33 we have:

$$\sigma(\theta(p_t) \circ \theta(q_t))_{t^{-1}} = \sum_{n=0}^{\infty} t^n a_n(p,q). \tag{3.34}$$

This shows that we have the requisite asymptotic expansion and identifies the first term of the expansion as $a_0(p,q)$. ∎

Lemma 3.11. *Let $p_n \in S\Sigma^{l-n}(\mathbf{E})$, n=n $= 1, 2, \dots$, then there exists an asymptotic symbol $p(m,\xi,t)$ such that $p(m,\xi,t) \sim \sum_{n=0}^{\infty} t^n p_n(m,\xi)$.*

Proof: This is a standard argument (see [**W1**], lemma 4.2). One chooses a C^∞ function ϕ such that $\phi(x) = 0$ if $x \leq 1$ and $\phi(x) = 1$ if $x \geq 2$. Then choose ϵ_n, positive numbers tending to zero as $n \to \infty$. Set $p(m,\xi,t) = \sum_{n=0}^{\infty} t^n \phi(\epsilon_n(\frac{1}{t^2} + |\xi|^2)) p_n(m,\xi)$. Then it is easy to show that $p(m,\xi,t) \sim \sum_{n=0}^{\infty} t^n p_n(m,\xi)$. ∎

Definition 3.1. *Let P_t be an $A\psi DO$ with symbol $p(m,\xi,t) \sim \sum t^n p_n(m,\xi)$; then P_t is called asymptotically elliptic if the map $q \to a_0(p_0,q)$ is invertible.*

Remark. The notion of ellipticity for $A\psi DO's$ is slightly more subtle than it is for ordinary pseudodifferential operators. For example, let e_i be a local frame for $T^*(M)$ in a neighborhood U, and consider the operator $C(e_i)$, Clifford multiplication by e_i. Then $C(e_i)$ is an invertible operator in the neighborhood U. The asymptotic operator $tC(e_i)$, which is of order one, is, however, not invertible as an asymptotic operator, since it's inverse would be $t^{-1}C(e_i)$, which is not an $A\psi DO$. It is also clear that the map $p \longrightarrow a_0(p, e_i)$ is not invertible, since $e_i \wedge e_i = 0$.

THEOREM 3.1. *If P_t is an asymptotically elliptic operator, then there exist Q_t, an asymptotic operator, such that $P_t \circ Q_t \sim 1$.*

PROOF: Let $\sigma(P_t)_{t^{-1}} \sim \sum_{n=0}^{\infty} t^n p_n$; then we find a symbol, $q \sim \sum_{n=0}^{\infty} t^n q_n$, by solving for the q_i's recursively. Using:

$$\sigma(\theta((p_1))_t \circ \theta((p_2)_t))_{t^{-1}} \sim \sum_{n=0}^{\infty} t^n a_n(p_1, p_2) \tag{3.35}$$

we have:

$$\sum_{n=0}^{\infty} t^n \left(\sum_{l+s+k=n} a_l(p_s, q_k) \right) - 1 = 0. \tag{3.36}$$

This says that $a_0(p_0, q_0) = 1$, or $q_0 = \{a_0(p_0, \cdot)\}^{-1}(1)$. For $n \geq 0$ we have :

$$\begin{aligned} &\sum_{l+s+k=n} a_l(p_s, q_k) = 0 \text{ so we have:} \\ &a_0(p_0, q_n) + \sum_{\substack{l+k+s=n \\ k<n}} a_l(p_s, q_k) \text{ or:} \\ &q_n = \{a_0(p_0, \cdot)\}^{-1} \left\{ - \sum_{\substack{l+k+s=n \\ k<l}} a_l(p_s, q_k) \right\}. \end{aligned} \tag{3.37}$$

Lemma 3.11 now says that there is an asymptotic symbol with the desired expansion. If we set $Q_t = \theta(q_t)$, then Q_t will be the desired inverse. ∎

THEOREM 3.2. *If P_t is an $A\psi DO$ of order less than zero, then $Tr_s(P_t)$ exist and:*

$$Tr_s(P_t) = (2\pi)^{-n} \left(\frac{2}{i}\right)^{\frac{n}{2}} \int_{T^*(M)} tr^{\mathbf{E}}(\sigma(P_t))_{t^{-1}} \, d\xi. \tag{3.38}$$

PROOF: For the most part, the proof is standard and is obtained from computing the trace by integrating the kernel of an integral operator along the diagonal. For some of the details one can consult [G], (theorem 3.7). However, some explanation of notation is called for at this point.

Recall that we had an intermediate map, $\bar{\theta} : S\Sigma(\mathbf{E}) \longrightarrow \Sigma(\mathbf{E} \otimes \mathbf{S})$, where $\Sigma(\mathbf{F})$ is the space of (ordinary) symbols for the bundle $\mathbf{F}$. The map $\bar{\theta}$ sent a form-valued symbol of $\mathbf{E}$ to a regular symbol of $\mathbf{E} \otimes \mathbf{S}$, where $\mathbf{S}$ is the bundle of spinors, by sending the form part to the corresponding operator in $End(\mathbf{S}) \cong Cliff(T^*(M))$.

Suppose that $P_t = \theta(p_t)$. If we denote by tr_s the supertrace on the bundle $\mathbf{E} \otimes \mathbf{S}$, then it is standard ([**W1**],pp. 46) that:

$$(3.39) \qquad Tr_s(P_t) = (2\pi)^{-n} \int_{T^*(M)} tr_s(\bar{\theta}(p_t))\, dx\, d\xi.$$

Let $tr^{\mathbf{E}}(T)$ denote the trace of the endomorphism T on the bundle $\mathbf{E}$; using equation 1.22 we have that $tr_s(\bar{\theta}(p_t)) = (\frac{2}{i})^{\frac{n}{2}} top[tr^{\mathbf{E}}(p_t)]$. Here, $top[\omega]$ means the top dimensional piece of the form ω. Since only the top dimensional piece of a form contributes to the integral over the manifold M, equation 3.39 becomes:

$$(3.40) \qquad Tr_s(P_t) = (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_{T^*(M)} tr^{\mathbf{E}}\sigma(P_t)\, d\xi.$$

The symbol has not yet been inverse scaled. If we inverse scale the top piece, we introduce a factor of t^{-n} in front of the integral. However, the cotangent fiber also gets scaled by a factor of t^{-1} which introduces a factor of t^n in front of the integral, and the two factors cancel. Thus, we get exactly the formula 3.38 for the supertrace of P_t. ∎

Corollary 3.3. *If P_t is an $A\psi DO$ with leading symbol p_o of degree less than zero, then $Tr_s(P_t)$ admits an asymptotic expansion:*

$$(3.41) \qquad Tr(P_t) \sim \sum_{m=0}^{\infty} b_m t^m, \quad \textit{with} \quad b_0 = (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int tr^{\mathbf{E}}(p_o)\, d\xi.$$

Proof: By assumption we have $\sigma(P_t)_{t^{-1}} \sim \sum_{m=0}^{\infty} t^m p_m$. Recalling that this expansion holds in the topology of the symbol space and using theorem 3.2 yields the desired expansion. ∎

Now we want to appy the above results to the Dirac operator on a compact spin manifold. Let $\mathbf{E}$ be a vector bundle over the compact spin manifold M and let $\nabla_{\mathbf{E}}$ be a connection on $\mathbf{E}$. We will denote by ∇ the tensor product connection on $\mathbf{E} \otimes \mathbf{S}$ (or $\mathbf{E} \otimes \mathbf{S}^{\pm}$). Let $\alpha : \mathbf{E} \otimes \mathbf{S} \otimes T^*(M) \to \mathbf{E} \otimes \mathbf{S} \otimes T(M)$ be defined by the inner product on $T(M)$. Let $C(X)$ denote Clifford multiplication by X, and let $c : \mathbf{E} \otimes \mathbf{S} \otimes T(M) \to \mathbf{E} \otimes \mathbf{S}$ be defined as follows:

$$(3.42.) \qquad c(e \otimes s \otimes X) = e \otimes c(X) \cdot s.$$

The Dirac operator is defined by:

$$(3.43) \qquad D(s) = (c \circ \alpha \circ \nabla)(s).$$

If $e_1, \cdots, e_n$ is a local orthonormal frame for the tangent bundle, then locally we have:

$$(3.44) \qquad D(s) = \sum_{i=0}^{n} c(e_i) \cdot (\nabla_{e_i} s).$$

Remark. Note that the order of D is two, so tD is not an asymptotic operator; however, t^2D is an $A\psi DO$.

Next we need to know D^2. This is a standard computation, so we will only outline it. Fix a point $m \in M$ and choose a synchronous frame in a neighborhood of x ($\nabla_{e_i}(e_j)(x) = 0$, or $\Gamma^l_{ij}(x) = 0$). At the point x one computes:

$$D^2 = \sum_{i=1}^{n}(\nabla_{e_i})^2 + \frac{1}{4}\sum_{ijkl} R^i_{jkl}e_ie_je_ke_l + \frac{1}{2}\sum_{ij} F_{ij}e_ie_j \tag{3.45}$$

where F_{ij} is the curvature of the bundle E. By using the Bianchi identities and with K the scalar curvature of M, this reduces to :

$$D^2 = \sum_{i=1}^{n}(\nabla_{e_i})^2 + \frac{1}{2}\sum_{ij} F_{ij}e_ie_j + \frac{1}{4}K. \tag{3.46}$$

LEMMA 3.12. *Let $D_t = tD$, and let $\phi \in C^\infty(M)$; then:*

(1) *D_t^2 is an $A\psi DO$ with leading symbol $(-|\xi|^2 + \frac{1}{2}\sum F_{ij}f_i \wedge f_j)$; and*
(2) *$[D_t, \phi]$ is an $A\psi DO$ with leading symbol $d\phi$.*

PROOF: We have :

$$\sigma(t^2D^2) = -t^2|\xi|^2 + \frac{1}{2}t^2\sum F_{ij}f_i \wedge f_j + \frac{1}{4}t^2K. \tag{3.47}$$

Inverse scale to get:

$$\sigma(t^2D^2)_{t^{-1}} = (-|\xi|^2 + \frac{1}{2}\sum F_{ij}f_i \wedge f_j) + \frac{1}{4}t^2K. \tag{3.48}$$

For the other part we have:

$$[D_t, \phi] = t\sum_{i=0}^{n} d\phi(e_i)\cdot e_i. \tag{3.49}$$

Consequently , $\sigma([D_t, \phi]) = td\phi$; so

$$\sigma([D_t, \phi])_{t^{-1}} = d\phi. \tag{3.50}$$

∎

Remark. We noted that D_t is not an $A\psi DO$; consequently, D_t^{-1} is not an $A\psi DO$ either. It is precisely this fact that makes Connes's original computation of the Chern character so difficult. One should note that in the letter to Quillen, Connes isolates the expression tD_t^{-1}, the point being that tD_t^{-1} is an $A\psi DO$. Specifically,

$$\sigma(tD_t^{-1}) = \sum tf_i\frac{1}{t\xi_i}. \tag{3.51}$$

Thus if we inverse scale we get:

$$\sigma(tD_t^{-1})_{t^{-1}} = \sum \frac{1}{\xi_i} f_i. \tag{3.52}$$

The crucial point here is that $c(e_i)^{-1} = c(e_i)$.

Given a connection and a "linear "function $l \in C^\infty(T^*(M) \times M)$, Widom defines a complete symbol for a pseudodifferential operator on M. It was shown in [**R**] that the function $\langle exp_m^{-1}(x), \xi_m\rangle = l(\xi_m, x)$ is such a linear function. If P_t is an $A\psi DO$ then we see that P_t is a family of pseudodifferential operators in the sense of Widom (the treatment of the Clifford variables as differential operators only adds a dependence on t that is uniform for $t \in [0,1]$, so Widoms theory works for our asymptotic operators.) We see that if we choose for the linear function $\langle exp_m^{-1}(x), \xi_m\rangle$, then Widom's symbol is exactly:

$$\sigma(P_t)(m, t^{-1}\xi). \tag{3.53}$$

Lemma 4.10 of [**W1**] can be stated as follows:

LEMMA 3.13. *The operator $e^{t^2D^2}$ is a $A\psi DO$.*

PROOF: Applying the function calculus to $e^{t^2D^2}$ gives:

$$e^{t^2D^2} = \frac{1}{2\pi}\int_\Gamma (t^2D^2 - \lambda)^{-1} e^\lambda \, d\lambda. \tag{3.54}$$

Next , Widom shows that :

$$\sigma(e^{t^2D^2}) = \frac{1}{2\pi}\int_\Gamma \sigma((t^2D^2 - \lambda)^{-1}) e^\lambda \, d\lambda. \tag{3.55}$$

The delicate point is that the following expansion is uniform in λ:

$$\sigma((t^2D^2 - \lambda)^{-1}) \sim \sum_{n=0}^{\infty} t^n (p_n(\lambda))_t. \tag{3.56}$$

Thus we can integrate the expansion term by term to obtain the desired asymptotic expansion for $\sigma(e^{t^2D^2})$. ∎

LEMMA 3.14. *Let p be the leading symbol of t^2D^2; then the leading symbol of $e^{t^2D^2}$ is $e^{a_0(p,\cdot)}(1)$.*

PROOF: Note that $(t^2D^2 - \lambda)^{-1}$ has the leading symbol $a_0(p - \lambda, \cdot)^{-1}(1) = (a_0(p.\cdot) - \lambda)^{-1}(1)$. Thus $e^{t^2D^2}$ has the leading symbol:

$$\left(\frac{1}{2\pi}\int_\Gamma (a_0(p,\cdot) - \lambda)^{-1} e^\lambda \, d\lambda\right)(1) = e^{a_0(p,\cdot)}(1). \tag{3.57}$$

∎

COROLLARY 3.15. *With notation as above we have:*

$$Tr_s(e^{t^2D^2}) = (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_{T^*(M)} e^{a_0(p,\cdot)}(1)\, d\xi + O(t), \qquad \textit{hence}$$

(3.58)

$$ind(D) = \lim_{t\to 0} Tr_s(e^{t^2D^2}) = (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_{T^*(M)} e^{a_0(p,\cdot)}(1)(x,\xi)\, d\xi.$$

Next we want to compute the integral in equation 3.58. Recall that R was the curvature of $T(M)$. Thus if X, Y are vector fields, $R(X,Y) \in End(T(M))$ and $R(X,Y)$ is an $so(n)$-valued endomorphism. Choose a local orthonormal frame, $e_1, \cdots, e_n$, for the tangent bundle and let $f_1, \cdots, f_n$ be the corresponding dual frame for the cotangent bundle. If the matrix of $R(X,Y)$ with respect to this frame is $(R(X,Y)_{ij})$, then the curvature of the spin bundle is given by:

$$\bar{R}(X,Y) = \frac{1}{2}\sum_{ij} R(X,Y)_{ij} e_i e_j \in Cliff(T(M)). \tag{3.59}$$

Thus we have:

$$\sigma(\bar{R}(X,Y)) = \frac{1}{2}\sum_{ij} R(X,Y)_{ij} f_i \wedge f_j. \tag{3.60}$$

The frame $f_1, \cdots, f_n$ determines local coordinates $(\xi_1, \cdots, \xi_n)$ on $T^*(M)$, and with respect to these coordinates we have the operator:

$$R(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}) = 2\sum_{ij} \sigma(\bar{R}(e_i, e_j)) \frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\eta_j}. \tag{3.61}$$

In what follows we will simply write: $\sum_{ij} R(e_i, e_j)\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\eta_j}$, where $R(e_i, e_j)$ is the two-form $\sum_{kl} R^k_{lij} f_k \wedge f_l$. The formula for $a_0(p,q)$ is :

$$a_0(p,q) = e^{-\frac{1}{4}R(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta})} p(x,\xi) \wedge q(x,\eta)|_{\xi=\eta}. \tag{3.62}$$

Recall that F is the curvature of the bundle E, so $\sigma(F) = \sum F_{ij} f_i \wedge f_j$. In what follows we will write $-|\xi|^2 + \frac{1}{2}F$ for $-|\xi|^2 + \frac{1}{2}\sigma(F)$. Given an arbitrary symbol q we have:

$$\begin{aligned} & a_0(p,q) = (-|\xi|^2 + \frac{1}{2}F) \wedge q(x,\xi) - \\ & \frac{1}{4}R(\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta})(-|\xi|^2 + \frac{1}{2}F) \wedge q(x,\eta)|_{\eta=\xi} + \\ & \frac{1}{2}\frac{1}{16}R(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta})\left(R(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta})\right)(-|\xi|^2 + \frac{1}{2}F) \wedge q(x,\eta)|_{\eta=\xi}. \end{aligned} \tag{3.63}$$

All other terms are zero. First we consider the last two terms. We have:

$$R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta})(-|\xi|^2 + \frac{1}{2}F) \wedge q(x,\eta)|_{\eta=\xi} = -2\sum_{ij} R(e_i, e_j)\xi_i \frac{\partial}{\partial \xi_i} q(x,\xi). \tag{3.64}$$

We will denote the operator $\sum_{ij} R(e_i, e_j)\xi_i \frac{\partial}{\partial \xi_i}$ by $R(\xi, \frac{\partial}{\partial \xi})$.

For the next term we have:

$$R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta})R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta})(-|\xi|^2 \frac{1}{2}F) \wedge q(x,\eta) = \sum_{i,j,k,l} R(e_i, e_j)R(e_k, e_l)\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \eta_j}\frac{\partial}{\partial \xi_k}\frac{\partial}{\partial \eta_l}(-|\xi|^2 + \frac{1}{2}F) \wedge q(x,\eta). \tag{3.65}$$

Now $-|\xi|^2 = -(\xi_1^2 + \cdots + \xi_n^2)$ so $\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_k}(-|\xi|^2) = 0$ unless $i = k$ in which case it equals -2. Also $R(e_k, e_l) = -R(e_l, e_k)$; hence equation 3.65 becomes:

$$2\sum_{ijk} R(e_i, e_k)R(e_k, e_j)\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} q(x,\xi). \tag{3.66}$$

We will write this as:

$$2\,(R \wedge R)\,(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi})(q(x,\xi)). \tag{3.67}$$

Thus equation 3.67 becomes:

$$a_0(p,q) = \left(-|\xi|^2 \frac{1}{2}F + \frac{1}{2}R(\xi, \frac{\partial}{\partial \xi}) + \frac{1}{16}R \wedge R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi})\right)(q(x,\xi)). \tag{3.68}$$

Recall that we are writing 1 for the constant symbol $1(x,\xi) = 1$, and we need to compute $e^{a_0(p,\cdot)}(1)$. Note that $\frac{1}{2}R(\xi, \frac{\partial}{\partial \xi})(-|\xi|^2) = -|\xi|^2$, so we have:

$$e^{a_0(p,\cdot)}(1) = e^{\left(-|\xi|^2 \frac{1}{2}F + \frac{1}{2}R(\xi, \frac{\partial}{\partial \xi}) + \frac{1}{16}R \wedge R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi})\right)}(1) = e^{\frac{1}{2}F} e^{(-|\xi|^2 + \frac{1}{16}R \wedge R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}))}(1). \tag{3.69}$$

Remark. Of course, 3.69 is only true for the constant symbol 1.

Next, we appeal to Mehler's formula,**[GJ]**. The operator $e^{-\frac{1}{2}(\xi^2 - a^2(\frac{d}{d\xi})^2)}$ is an integral operator with kernel given by:

$$k_a(\xi,\eta) = (2\pi a \sinh(a))^{-\frac{1}{2}} exp\left(-\frac{1}{2a\sinh(a)}(\cosh(a)(\xi^2 + \eta^2) - 2\xi\eta)\right). \tag{3.70}$$

If A is a skew-symmetric matrix with eigenvalues $\pm i a_j$, then we can form the corresponding operator, $H = (|\xi|^2 - A^2(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}))$. The kernel of e^{-H} satisfies:

$$\int k_H(\xi,\eta)\, d\xi\, d\eta = \prod_{j=1}^{n} \frac{2\pi a_j}{\sinh(a_j)} = \left(\det \frac{-2\pi i A}{\sinh(-iA)}\right)^{\frac{1}{2}}. \tag{3.71}$$

If we apply this to equation 3.58 and 3.69 we have:

$$\begin{aligned} Ind(D) = \lim_{t\to 0} Tr_s(e^{\frac{1}{2}t^2 D^2}) = \\ (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_M e^{F/4} \left(\det \frac{-2\pi i R/4}{\sinh(-iR/4)}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.72}$$

Given a matrix, B, recall that, ([**G**] pp. 99):

$$\hat{A}(B) = \left(\det \frac{B/4\pi}{\sinh(B/2\pi)}\right)^{\frac{1}{2}}. \tag{3.73}$$

Therefore, we have:

$$\begin{aligned} \left(\det \frac{-2\pi i(R/4)}{\sinh(-i(R/4))}\right)^{\frac{1}{2}} = (2\pi)^{\frac{n}{2}} 2^{\frac{n}{2}} \left(\det \frac{-i(R/4)/2}{\sinh(-i(R/4))}\right)^{\frac{1}{2}} = \\ (2\pi)^{\frac{n}{2}} 2^{\frac{n}{2}} \hat{A}(-2\pi i(R/4)). \end{aligned} \tag{3.74}$$

Next, recall that the Chern character, $Ch(B)$, is equal to $tr(e^{-B/2\pi i})$, so we have $tr(e^{F/4}) = Ch(-2\pi i(F/4))$. Thus:

$$Ind(D) = (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}}(2\pi)^{\frac{n}{2}} 2^{\frac{n}{2}} \int_M Ch(-2\pi i(F/4)) \wedge \hat{A}(-2\pi i(R/4)). \tag{3.75}$$

Recalling that the top degree piece of the intergrand is homogeneous of degree $\frac{n}{2}$ we have:

$$\begin{aligned} Ind(D) =& (2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}}(2\pi)^{\frac{n}{2}} 2^{\frac{n}{2}} \left(\frac{-\pi i}{2}\right)^{\frac{n}{2}} \int_M Ch(F) \wedge \hat{A}(R) = \\ & (-1)^{\frac{n}{2}} \int_M Ch(F) \wedge \hat{A}(R). \end{aligned} \tag{3.76}$$

4. Calculation of the Cocycle for the Dirac Operator.

In this section we will apply the asymptotic calculus to the calculation of the cyclic cocycle corresponding to the Dirac operator on a compact spin manifold M. See [**C3**] and [**JLO**] for details on theta-summable fredholm modules. Let $H = H^+ \oplus H^-$ denote the Hilbert space of square-integrable spinors over M, and let ϵ be the corresponding grading operator, so $\epsilon|_{H^+} = 1$ and $\epsilon|_{H^-} = -1$. Let D be the Dirac operator on the space of spinors; thus:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \tag{4.1}$$

Here $D^+ : H^+ \to H^-$ and $D^- : H^- \to H^+$. Let $\mathfrak{A} = C^1(M)$; then $\mathfrak{A}$ is a Banach algebra (it is the completion of $C^\infty(M)$ with respect to the norm $||f|| = ||f||_\infty + ||df||_\infty$). Let $\tilde{\mathfrak{A}} = \mathfrak{A}/\mathbf{C}$ and define $C_n(\mathfrak{A}) = \mathfrak{A} \otimes \tilde{\mathfrak{A}}^{\otimes_\pi n}$ where $\otimes_\pi$ denotes the projective tensor product. The bar complex is $\sum_{n=0}^\infty C_N(\mathfrak{A})$ with a suitable topology. The cobar complex is the topological dual and the Chern character of a theta-summable Fredholm module is a cochain in the cobar complex, closed with respect to the coboundary operator $\partial = b + B$. Thus , for appropriate $B_i \in C_i(\mathfrak{A})^*$:

$$Ch((D,H)) = \sum_{i=0}^{\infty} B_i. \tag{4.2}$$

The Chern character of the module (H, D) as an entire cyclic cocycle is given by:

$$\begin{aligned}&(Ch((D,H)),(f_0,\dots,f_{2k})) = B_{2k}(f_0,\dots,f_{2k}) = \\ &\int_{\Delta_n} Tr_s\left(f_0 e^{-t_1 D^2}[D,f_1]e^{-(t_2-t_1)D^2}\cdots[D,f_{2k}]e^{-(1-t_{2k})D^2}\right) dt_1\cdots dt_{2k}.\end{aligned} \tag{4.3}$$

Let $D_\alpha = \alpha D$. As noted before, D_α is not an asymptotic operator; however, $D_\alpha^2 = \alpha^2 D^2$ and $[D_\alpha, f] = \alpha C(df)$ are asymptotic operators. (Recall that $C(df)$ is Clifford multiplication by df.)

THEOREM 4.1.

(1) *$(Ch((D_\alpha,H)),(f_0,\dots,f_{2k}))$ is convergent for $\alpha \to 0$ and for $f_i \in \mathfrak{A}$.*
(2) *The limit $\tau_k(f_0,\dots,f_{2k}) = \lim_{\alpha\to 0}(Ch((D_\alpha,H)),(f_0,\dots,f_{2k})$ gives a cocyle (considering all of the τ_k's together)in the cobar complex, and:*

$$\tau_k(f_0,\dots,f_{2k}) = vol(\Delta_{2k})(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_M \hat{A}(-2\pi R/4) f_0\, df_1 \wedge\cdots\wedge df_{2k}. \tag{4.4}$$

PROOF: That the collection of τ_k's form a cyclic cocycle follows from the existence of the limit and the cooresponding fact for $Ch((D,H))$, which can be found in **[GS]**. Consider the operator:

$$P_\alpha = f_0 e^{-t_1 D_\alpha^2}[D_\alpha, f_1]e^{-(t_2-t_1)D_\alpha^2}\cdots[D_\alpha,f_{2k}]e^{-(1-t_{2k})D_\alpha^2}. \tag{4.5}$$

By the above remarks, each of the individual terms is an asymptotic operator; hence, so is their product. To compute $\lim_{\alpha\to 0} Tr_s(P_\alpha)$, it suffices to compute the leading symbol. Recall that the leading symbols of $e^{\frac{-t}{2}D_\alpha^2}$ and $[D_\alpha, f]$ are:

$$\begin{aligned}&(a)\quad e^{-t(|\xi|^2 - \frac{1}{16}R\wedge R(\partial_\xi,\partial_\xi))}(1)\\ &(b)\quad df.\end{aligned} \tag{4.6}$$

By the formula for the composition of symbols, 4.6a and 4.6b commute, since df contains no differentiation and 4.6a is a form of even order. Thus we have:
(4.7)

$$\lim_{\alpha\to 0} Tr_s(P_\alpha) = \lim_{\alpha\to 0}(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int (\sigma(P_\alpha))_{\alpha^{-1}} d\xi =$$

$$(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int \sigma(f_0 e^{-t_1 D_\alpha^2}[D_\alpha, f_1]e^{-(t_2-t_1)D_\alpha^2} \cdots [D_\alpha, f_{2k}]e^{-(1-t_{2k})D_\alpha^2})_{\alpha^{-1}} =$$

$$(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int f^0 df_1 \wedge \cdots \wedge df_{2k} \wedge e^{-t_1-(t_2-t_1)-\cdots-(1-t_{2k})(|\xi|^2-\frac{1}{16}R\wedge R(\partial_\xi,\partial_\xi))} =$$

$$(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_M f_0 df_1 \wedge \cdots \wedge df_{2k} e^{a(p,\cdot)}(1) =$$

$$(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_M \hat{A}(-2\pi i R/4) f_0 df_1 \wedge \cdots \wedge df_{2k}.$$

∎

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