Asymptotic Pseudodifferential Operators
And Index Theory

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Abstract. We introduce the notion of an asymptotic pseudodifferential operator on spin bundles over a compact manifold and develop a calculus for these operators. We then use the the formula of Jaffe, Lesniewski, and Osterwalder for theta-summable Fredholm modules, along with the asymptotic calculus, to compute the cyclic cocycle that Connes attaches to the Dirac operator on a compact spin manifold.

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INTRODUCTION

This paper had its origins in a seminar held at MSRI in the Fall of 1987. Our goal was to understand Connes's computation of the cyclic cocycle corresponding to the Dirac operator on a compact spin manifold ([C-1], [C-2]). In particular, we needed to understand the ideas in the paper by Getzler [G]. It soon became clear that a change of view would simplify the situation. Of course, this change is implicit in the letter from Connes to Quillen.

Getzler introduced an algebra of pseudodifferential operators ($\psi DO$'s) on a compact spin manifold that incorporated the quantization map of Bokobza-Haggiag [B] with some supersymmetric ideas from quantum field theory. These ideas suggest that Clifford multiplication contributes to the degree of a differential operator. If $p$ denotes a symbol, $\theta(p)$ will be the associated pseudodifferential operator. Given a $\psi DO$ $Q$, there is a symbol map, $\sigma(Q)$, which was first introduced by Widom ([W1], [W2]). Next, a scaling was introduced, which is essentially the traditional scaling of asymptotic analysis, but extended to take into account the Clifford variables [MF]. A family of compositions was defined by:

$$p \circ_t q = \sigma(\theta(p_t) \circ \theta(q_t))_{t_1}.$$  

This scaling had the important property that $\lim_{t \to 0} p \circ_t q$ was computable. While the $\theta(p)$'s generate an algebra of $\psi DO$'s, it is clear that $\theta(p_t) \circ \theta(q_t)$ is not the scaled version of some other operator. It was this observation that led us to enlarge the class of symbols, moving closer to Widom's work and the classical asymptotic methods.

Let $p(x, \xi, t)$ be a symbol of degree $n$ such that $p(x, \xi, t)$ has an asymptotic expansion in $t$ with respect to the natural topology on the symbol space:

$$p(x, \xi, t) \sim \sum_{l=0}^{\infty} t^l p_l(x, \xi) \quad \text{with } p_l \text{ a symbol of degree } n - l.$$  

We define an asymptotic operator to be (essentially) $\theta(p(x, \xi, t), t) = P_t$. In this setting Getzler's rescaled operator is just an asymptotic operator whose symbol has the trivial asymptotic expansion $p(x, \xi, t) \sim p(x, \xi)$. Once the notion of an asymptotic pseudodifferential operator ($A\psi DO$) has been defined, the development of a calculus of $A\psi DO$'s paralleling the calculus of $\psi DO$'s is straightforward.

As we were working to understand Connes's work, we realized that a formula for the character of a Fredholm module in which all of the terms were asymptotic operators would be extremely useful. One of the main difficulties Connes needed to address was that $D_t = tD$ is not an $A\psi DO$ so $D_t^{-1}$ is not an $A\psi DO$ either. We then received a preprint of Getzler and Szenes [GS], in which they presented such a formula, which is due to Jaffe, Lesniewski, and Osterwalder [JLO]. This formula has three beautiful attributes:

1. one does not have to cup with the small complex to make $D$ invertible;
2. all terms are manifestly asymptotic operators; and
3. it applies to the larger context of $\theta$-summable Fredholm modules.
Using this formula for the Chern character, we write out a computation of the cyclic cocycle attached to the Dirac operator on a spin manifold.

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1. Review of Clifford Algebras and Spinors.

In this section a brief review of the construction of the Clifford algebra and spinors is presented. (For more details see [Gi], [ABP].) Let \( V \) be an even-dimensional vector space with an inner product, \( (\cdot, \cdot) \). Denote by \( V^n \) the \( n \)-fold tensor product of \( V \) with itself, so

\[
V^n = V \otimes V \otimes \cdots \otimes V \ 	ext{n-factors}
\]

(We set \( V^0 \) to be \( \mathbb{R} \).) The tensor algebra of \( V \) is the graded algebra

\[
T(V) = \bigoplus_{n=0}^{\infty} V^n.
\]

Let \( I \) be the ideal generated by the relations \( v \otimes v - (v, v) \). It should be noted that the ideal is not homogeneous under the \( \mathbb{Z} \) grading but is homogeneous under the \( \mathbb{Z}_2 \) grading. We have the following.

**Definition 1.1.** The Clifford algebra of the pair \( (V, (\cdot, \cdot)) \) is \( \text{Cliff}(V) = T(V)/I \).

Since the ideal \( I \) is homogeneous under the \( \mathbb{Z}_2 \) grading, we have that \( \text{Cliff}(V) \) is a \( \mathbb{Z}_2 \) graded algebra. Let \( e_1, e_2, \ldots, e_n \) be an orthonormal basis for \( V \); then \( \text{Cliff}(V) \) is the algebra generated by the \( e_i \)'s subject to the relations:

\[
e_i e_j + e_j e_i = 0 \quad \text{if } i \neq j
\]

\[
e_i^2 = 1.
\]

Using the above relations we can write down a basis for \( \text{Cliff}(V) \). Let \( I = (i_1 < i_2 < \cdots < i_p) \) be a multi-index and set \( e_I = e_{i_1} e_{i_2} \cdots e_{i_p} \); then \( \text{Cliff}(V) \) has the \( 2^n \) monomials \( e_I \) as a basis. Denote by \( \otimes \) the graded tensor product. By considering the basis of \( \text{Cliff}(V) \) as above we see that:

\[
\text{Cliff}(V \otimes W) = \text{Cliff}(V) \otimes \text{Cliff}(W).
\]

Define the transpose map \( a \rightarrow a^t \) for \( a \in \text{Cliff}(V) \) by defining it on a basis and then extending it by linearity, for \( e_I \) we have:

\[
(e_{i_1} \cdots e_{i_p})^t = e_{i_p} \cdots e_{i_1}.
\]

A simple computation shows that \( (e_{i_1} \cdots e_{i_p})^t = (-1)^{p(p-1)/2} (e_{i_1} \cdots e_{i_p}) \). Next define \( \tau = (i)^{n/2} e_1 \cdots e_n \) where \( e_1, \ldots, e_n \) is an oriented orthonormal basis of \( V \). Then we see that \( \tau^2 = 1. \)
Let $\text{Cliff}(V)_C$ denote the complexified Clifford algebra, $\text{Cliff}(V)_C = \text{Cliff}(V) \otimes_R \mathbb{C}$. If $V$ is a two-dimensional vector space, we can define the well-known isomorphism of $\text{Cliff}(V)_C$ with the two-by-two complex matrix algebra. This isomorphism is implemented by the Pauli matrices:

$$
\begin{align*}
    \epsilon_1 & \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
    \epsilon_2 & \rightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
    \tau & = i \epsilon_1 \epsilon_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
$$

(1.6)

If the dimension of the vector space $V$ is $n = 2m$, then it follows from (1.4) and (1.6) that $\text{Cliff}(V)_C$ is isomorphic to the matrix algebra $M_{2^n}(\mathbb{C})$ of $2^m \times 2^m$ matrices. In particular, it is a simple algebra. Left multiplication of $\text{Cliff}(V)_C$ on itself decomposes into a direct sum of irreducible submodules. After choosing an isomorphism of $\text{Cliff}(V)_C$ with $M_{2^n}(\mathbb{C})$, each column of $M_{2^n}(\mathbb{C})$ yields an irreducible submodule. All of these are isomorphic, and in fact there is only one irreducible $\text{Cliff}(V)_C$ module up to isomorphism.

**Definition 1.2.** Let $\Delta$ be an irreducible submodule for $\text{Cliff}(V)_C$ acting on itself by left multiplication; then $\Delta$ is called the space of spinors.

**Definition 1.3.** Define $\text{Spin}(V)$ by:

$$
\text{Spin}(V) = \{ w \in \text{Cliff}(V) \mid w = v_1 \cdots v_{2j} \text{ and } |v_k| = 1 \}.
$$

(1.7)

Note that $\text{Spin}(V)$ is a group with $w^{-1} = w^t$.

Define a map $\rho : \text{Cliff}(V) \rightarrow \text{End}(\text{Cliff}(V))$ as follows. If $w \in \text{Cliff}(V)$, then:

$$
\rho(w)(v) = wvw^t.
$$

(1.8)

Suppose that $e_1, \ldots, e_n$ is an orthonormal basis for $V$. Then:

$$
\begin{align*}
    \rho(e_1)(e_1) &= e_1 \\
    \rho(e_1)(e_i) &= -e_i
\end{align*}
$$

(1.9)

We see that if $v \in V \subset \text{Cliff}(V)$, then $\rho(v)$ takes $V$ to $V$. Furthermore, if $|v| = 1$, then $\rho(v) \in O(V)$, the orthogonal group of $V$. Consequently, $\rho$ maps $\text{Spin}(V)$ into $SO(V)$, and since every element in $SO(V)$ can be written as a product of an even number of reflections, we see that $\rho$ maps onto $SO(V)$. If $\rho(w) = 1$, then $wvw^t = v$ for all $v \in V$. It is easy to check that this implies that $w$ is in the center of $\text{Cliff}(V)$. Since the center of $\text{Cliff}(V)$ consists of the scalars and $|w| = 1$, we have that $w = \pm 1$. Therefore, we have the short exact sequence of groups:

$$
1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(V) \rightarrow SO(V) \rightarrow 1.
$$

(1.10)
To show that Spin(V) is connected, we only have to produce a path from -1 to 1 in Spin(V). We simply write one down:

\[(1.11)\quad \gamma(\theta) = (\cos(\theta)e_1 - \sin(\theta)e_2)e_1.\]

Thus \(\gamma(0) = 1\), \(\gamma(\pi) = -1\), and we conclude that Spin(V) is a two-fold cover of SO(V).

Once we fix an orthonormal basis of V, \(e_1, \cdots, e_n\), we can identify SO(V) with SO\((n, R)\). The Lie algebra of SO\((n, R)\) and of Spin(V) is then identified with the skew symmetric matrices of \(M_n(R)\), which we denote by \(so(n)\). If \(A = (a_{ij}) \in so(n)\), then the action of \(A\) on \(\Delta\) is given by Clifford multiplication by:

\[(1.12)\quad \frac{1}{4} \sum_{ij} a_{ij} e_i e_j.\]

For the details of this computation, see [G, p.173].

**Definition 1.4.** Let V be an oriented vector space and \(e_1, \cdots, e_n\) be an oriented basis for V. If \(\tau = (i)^{n/2} e_1 \cdots e_n\), then we define the half-spin representations of Spin(V) by \(\Delta^\pm = \{ z \in \Delta | \tau \cdot z = \pm z \}\).

Note that \(\tau\) commutes with every element of Spin(V), so \(\Delta^\pm\) are invariant under the action of Spin(V). If \(v \in V\), let \(c(v)\) be the operator on \(\Delta\) which is Clifford multiplication by \(v\). Then it is clear that:

\[(1.13)\quad c(v) : \Delta^\pm \to \Delta^{\mp}.\]

Let V be an oriented inner-product space with oriented orthonormal basis \(e_1, \cdots, e_n\). Let \(\Lambda(V)\) be the associated exterior algebra of V.

**Definition 1.5.** Define a linear isomorphism \(\sigma\) from Clif(V) to \(\Lambda(V)\) as follows: for each multi-index \(I = (i_1 < \cdots < i_p)\) we have

\[(1.14)\quad \sigma(e_{i_1} \cdots e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}.\]

It is easy to check that this map is independent of the basis and that

\[(1.15)\quad \sigma(a \cdot b) = \sigma(a) \wedge \sigma(b) \mod \Lambda^{\deg(a)+\deg(b)-1}.\]

Recall that a \(Z_2\) graded vector space, \(V = V^+ \oplus V^-\), is called a supervector space. Let \(\epsilon\) be the operator that is +1 on \(V^+\) and -1 on \(V^-\); then \(\epsilon\) is called the grading operator for V. If V is \(Z_2\) graded, then \(End(V)\) is also \(Z_2\) graded with the grading operator \(Ad(\epsilon)\). If V is finite dimensional, we define the supertrace of \(a \in End(V)\) by:

\[(1.16)\quad tr_\epsilon(a) = tr(\epsilon a).\]

The fundamental property of a trace is that \(tr([a, b]) = 0\), where \([a, b] = ab - ba\). There is an analogous property for the supertrace, but one has to replace the
commutator $[a, b]$ by the graded commutator. If $a \in \text{End}(V)$, we say that $\text{deg}(a) = 0$ or 1 when $\text{Ad}(e)(a) = \pm a$, and we write $(-1)^a$ for $(-1)^{\text{deg}(a)}$. For $a, b \in \text{End}(V)$, we define the graded commutator as:

$$
[a, b]_s = ab - (-1)^{\text{deg}(a)\text{deg}(b)}ba.
$$

For the supertrace we have:

$$
\text{tr}_s([a, b]) = 0.
$$

Let $V$ and $W$ be two $\mathbb{Z}_2$ graded vector spaces, and recall that $\hat{\otimes}$ represents the graded tensor product. We have:

$$
\text{End}(V \hat{\otimes} W) = \text{End}(V) \hat{\otimes} \text{End}(W).
$$

Under this isomorphism we have:

$$
(a \hat{\otimes} b)(v \hat{\otimes} w) = (-1)^{\text{deg}(a)\text{deg}(v)}a(v) \hat{\otimes} b(w).
$$

It follows from (1.20) that:

$$
\text{tr}_s(a \hat{\otimes} b) = \text{tr}_s(a)\text{tr}_s(b).
$$

The space of spinors $\Delta$ is naturally a super vector space with the grading operator $\tau$, and $\text{Cliff}(V)_C = \text{End}(\Delta)$ is a super algebra. Equation (1.21) makes it easy to compute the value of the supertrace on elements of the Clifford algebra acting on the spinors. We start off with a two-dimensional vector space spanned by the orthonormal basis $e_1$ and $e_2$. Recall the isomorphism of equation (1.6) implemented by the Pauli matrices; then we see that $\text{tr}_s(e_1) = \text{tr}_s(e_2) = 0$ and $\text{tr}_s(e_1e_2) = \frac{2}{i}$. More generally, for a real vector space of dimension $n = 2m$, $\text{Cliff}(V)_C$ is identified with $M_{2^n}(C) = \otimes^m(M_2(C))$ and the spinors are identified with $2^m$-dimensional column vectors. By equation (1.21) we have:

$$
\text{tr}_s(e_I) = \begin{cases} 
0 & I \neq \{1, \ldots, n\} \\
\left(\frac{2}{i}\right)^m & I = \{1, \ldots, n\}
\end{cases}
$$

2. Pseudodifferential Operators.

This section introduces the main ingredients for the asymptotic pseudodifferential operators. We first discuss the spin bundle and then talk about the standard symbol spaces and the ordinary (that is, non-asymptotic) pseudodifferential operators.
The Spin Bundle.

Let $M$ be a compact, oriented, Riemannian manifold with tangent bundle $TM \to M$. Since $M$ is oriented, we can find an open cover, $\{U_\alpha\}$, of $M$ that trivializes the tangent bundle such that the transition functions $g_{\alpha \beta}$ are in $C^\infty(U_\alpha \cap U_\beta; SO(n))$. Assume that $M$ is equipped with a Riemannian connection $\nabla$. If $e_1, \ldots, e_n$ is a local orthonormal frame, then

$$\nabla_{e_i}(e_j) = \sum_k \Gamma^k_{ij} e_k.$$  \hspace{1cm} (2.1)

Set $f^i$ to be the one-form dual to $e_i$ and define $\omega^k_i = \sum_j \Gamma^k_{ij} f^i$, then

$$\nabla(e_i) = \sum_k \omega^k_i e_k.$$  \hspace{1cm} (2.2)

The $\omega^k_i$ are called the local connection one forms defining the connection. From

$$0 = d(e_i, e_j) = \langle \nabla(e_i), e_j \rangle + \langle e_i, \nabla(e_j) \rangle$$  \hspace{1cm} (2.3)

we see that $\omega = (\omega^k_i)$ is an $so(n)$-valued one-form. The curvature of $\nabla$ is defined by

$$\Omega = d\omega - \omega \wedge \omega.$$  \hspace{1cm} (2.4)

If we evaluate $\Omega(e_i, e_j) \in Hom(TM)$, then

$$\Omega(e_i, e_j) = \sum R^k_{ij} e_k \otimes f^i.$$  \hspace{1cm} (2.5)

The tensor $R^k_{ij}$ is skew in the $ij$ indices (that is, $\Omega$ is a two form) and also skew in the $kl$ indices (that is, $\Omega(e_i, e_j) \in so(n)$) [H].

Recall that $\rho : Spin(n) \to SO(n)$ is a two-fold covering map. If the $U_\alpha$ are suitably chosen (contractible will certainly do), then we can lift the transition functions $g_{\alpha \beta}$ to functions $\tilde{g}_{\alpha \beta} : U_\alpha \cap U_\beta \to Spin(n)$ such that $\rho(\tilde{g}_{\alpha \beta}) = g_{\alpha \beta}$. However, we may not be able to lift them so that the cocycle condition $\tilde{g}_{\alpha \beta} \tilde{g}_{\beta \gamma} = \tilde{g}_{\alpha \gamma}$ holds. If we can, then we say that the manifold is spinable, and a choice of liftings is called a spin structure (see [Gi]). We will assume that a spin structure has been chosen for $M$; then we can form the bundle of spinors over $M$, which we denote by $S$. Let us recall how $S$ is constructed. First, form the disjoint union $S = U_\alpha U_\alpha \times \Delta$. Then define an equivalence relation on $S$ as follows: if $(m, x) \in U_\alpha \times \Delta$ and $(n, y) \in U_\beta \times \Delta$, then $(m, x) \sim (n, y)$ iff $m = n$ and $\tilde{g}_{\alpha \beta}(x) = y$. The bundle of spinors is then equal to $S = S/\sim$. It is clear that $S$ trivializes over $U_\alpha$ and that $\tilde{g}_{\alpha \beta}$ are the transition functions.

Let $e_1, \ldots, e_n$ be an orthonormal frame for $TM$ over $U_\alpha$, and let $\omega$ be the associated connection one-form. We can define a local connection one-form on $S$ restricted to $U_\alpha$ by using the homomorphism from $so(n)$ into $Cliff(R^n)$:

$$\bar{\omega} = \frac{1}{4} \sum_i \omega^k_i e_i e_j \in Cliff(TM|U_\alpha).$$  \hspace{1cm} (2.6)
Denote by $\nabla$ the corresponding connection on $S$. The curvature of this connection is given by:

$$
(2.7) \quad \Omega(e_i, e_j) = \frac{1}{4} \sum_{k,l} R_{ijkl} e_k e_l \in \text{Cliff}(TM|\mathcal{U}_*).
$$

Since $M$ is assumed to be oriented, we have a decomposition of $S = S^+ \oplus S^-$ corresponding to the decomposition of $\Delta$ into $\Delta^+ \oplus \Delta^-$. In fact, $\tau = (i)^{n/2} e_1 \cdots e_n$ defines a global section of $\text{Cliff}(TM)$ and $\Gamma(S^\pm) = \{ s \in \Gamma(S) | rs = \pm s \}$. Since $\omega = \omega_r$ the connection $\nabla$ restricts to a connection on $S^\pm$, which we still denote by $\nabla$.

If $E$ is any vector bundle over $M$ with connection $\nabla_E$, then we will always use the tensor product connection on $E \otimes S$:

$$
(2.8) \quad \nabla_{E \otimes S} = \nabla_E \otimes \text{Id} + \text{Id} \otimes \nabla_S.
$$

**Symbols and Their Calculus.**

Recall the standard notion of a symbol for a vector bundle $E$ over a compact manifold $M$. Let $\pi : T^* M \to M$ be the natural map and let $S = \pi^*(\text{Hom}(E,E))$ be the pull-back of the bundle $\text{Hom}(E,E)$ to a bundle over $T^* M$.

**Definition 2.1.** A section $p \in S$ is called a symbol of order $m$ if for every multi-index $\alpha$ and $\beta$ we have the estimates:

$$
(2.9) \quad \| \partial_{\xi}^\alpha \partial_{\xi}^\beta p(x, \xi) \| \leq C_{\alpha \beta}(1 + |\xi|)^{m-|\beta|}.
$$

We denote by $\Sigma^m(E)$ the symbols of order $m$.

Note that $\Sigma^m(E)$ is a Fréchet space with respect to the semi-norms:

$$
(2.10) \quad \rho_{\alpha \beta}(p) = \inf \{ C_{\alpha \beta} : \| \partial_{\xi}^\alpha \partial_{\xi}^\beta p(x, \xi) \| \leq C_{\alpha \beta}(1 + |\xi|)^{m-\beta} \}.
$$

Following Getzler [G], we will introduce a new filtration on the space of symbols of $E \otimes S$. Recall that we have a linear isomorphism of vector bundles $\sigma : \text{Cliff}(TM) \to \Lambda(TM) \cong \Lambda(T^* M)$, where we use the inner product to implement the isomorphism between $TM$ and $T^* M$. Consequently, we have:

$$
(2.11) \quad \text{Hom}(E \otimes S) \cong \text{Hom}(E) \otimes \text{Hom}(S) \\
\cong \text{Hom}(E) \otimes \text{Cliff}(TM) \cong \text{Hom}(E) \otimes \Lambda(T^* M).
$$

We will denote by $\bar{\sigma}$ the linear isomorphism from $\text{Hom}(E) \otimes \text{Cliff}(TM)$ to $\text{Hom}(E) \otimes \Lambda(T^* M)$ and by $\bar{\sigma}^{-1}$ its inverse. We denote by $\Omega^j(M)$ the smooth forms of degree $j$ on $M$. Let $L$ be the pull-back to $T^* M$ of the bundle $\text{Hom}(E) \otimes \Lambda(T^* M)$ via the map $\pi : T^* M \to M$. 

DEFINITION 2.2. A section \( p \in \Gamma(L) \) will be called an \( s \)-symbol of order \( l \) if

\[
p = \sum_{j=0}^{m} p_j(x, \xi) \otimes \omega_j \quad \text{with} \quad p_j \in \Sigma^{l-j} \quad \text{and} \quad \omega_j \in \Omega^l(M).
\]

We will denote the collection of \( s \)-symbols of order \( l \) by \( \Sigma^l(E) \).

It should be noted that \( \Sigma^l \) is also a Fréchet space when we give it the projective tensor product topology, since \( \Omega(M) \) is a Fréchet space.

**Remark.** This definition of an \( s \)-symbol seems a bit strange and its ultimate justification will be the resulting asymptotic calculus that we develop. However, if one looks at the papers by the physicist Alvarez-Gaumé [AG], and Witten [W], one sees that they were led to this definition in a relatively straightforward manner from the study of supersymmetric quantum field theory.

Given a \( s \)-symbol \( p \in \Sigma^l \), we want to define an operator \( P = \theta(p) \). To do this we follow Bokobza-Haggiag's definition. First, we need to recall the definition of normal coordinates.

Let \( m \) be a point in the manifold \( M \) and \( T_m M \) be the tangent space to \( M \) at \( m \). Since we have a connection on \( TM \), there is an exponential map \( \exp_m : T_m M \rightarrow M \). If \( X \in T_m M \), let \( \gamma_X(t) \) be the unique geodesic through \( m \) such that \( \gamma_X(0) = X \); then \( \exp_m(X) = \gamma_X(1) \). Choose an orthonormal basis, \( X_1, \ldots, X_n \), for \( T_m M \), then \( \exp(x_1X_1 + \cdots + x_nX_n) = (x_1, \ldots, x_n) \) defines a system of normal coordinates in a neighborhood of \( m_0 \). \( \exp \) can also be viewed as a map \( \exp : TM \rightarrow M \times M \) which is a diffeomorphism of a neighborhood of the zero section onto a neighborhood of the diagonal in \( M \times M \):

\[
\exp(m, X_m) = (m, \exp_m(X_m)).
\]

Let \( \alpha \) be a function that is identically one in a neighborhood of the diagonal of \( M \times M \) such that the exponential map is a diffeomorphism on the support of \( \alpha \).

Let \( (m, x) \in \text{Supp}(\alpha) \). There is a unique geodesic from \( m \) to \( x \); if \( x = \exp_m(X) \) then that geodesic is \( \exp_m(tX) \). Let \( \tau(m, x) : (E \otimes S)_m \rightarrow (E \otimes S)_x \) be parallel translation along the unique geodesic from \( m \) to \( x \). If \( s \in \Gamma(E \otimes S) \), then we define:

\[
\hat{s}_m(x) = \alpha(m, x) \tau(x, m)s(x).
\]

Notice that \( \hat{s}_m(x) \) is really a function on \( T_m(M) \), since if \( x \) is not in the image of the exponential map, \( \alpha(x, m) = 0 \) and \( \hat{s}_m(x) = 0 \). We will write \( \hat{s}_m(X) \) for \( \hat{s}_m(\exp_m(X)) \) when there is no chance of confusion.

**DEFINITION 2.3.** Let \( p \in \Sigma^l(E) \) and \( s \in \Gamma(E \otimes S) \); then we define:

\[
\theta(p)(s)(m) = \int_{T_m(M) \times T^*_m(M)} e^{-i(\langle X, \xi \rangle \beta p(m, \xi) \hat{s}_m(X) dX d\xi.
\]
Remark. The existence of a Riemannian structure on \( M \) yields a smooth choice of Lebesque measure on \( T_m(M) \). We choose the measure \( d\xi \) to be dual to \( dX \) in the sense that Fourier inversion holds:

\[
\phi(o) = \int_{T^*_m(M)} \int_{T_m(M)} e^{-i(X,\xi)} \phi(X) \, dX \, d\xi.
\]

Remark. The operator \( \theta(p) \) depends on the choice of the cut-off function \( \alpha \); however, when we introduce the notion of an asymptotic operator, we will see that the asymptotic operator is independent of the choice of \( \alpha \).

The set of all such operators, along with all infinitely smoothing operators, will be denoted by \( Op^a(E) \), and we set \( Op(E) = \bigcup_\alpha Op^a(E) \).

**Definition 2.4.** Given \( s \in \Gamma(E \otimes S) \), define \( \bar{s}_m(x) = \alpha(m, x) \tau(m, x) s(m) \). Let \( P \in Op(E) \) and \( s \in \Gamma(E \otimes S) \); define \( \mu(P) \in End(E)_m \otimes End(S)_m \) by:

\[
\mu(P)(m, \xi)(s(m)) = P_y \left( \exp^{-1}(y, \xi) \bar{s}_m(y) \right)|_{y=m}.
\]

Recall the map \( \tilde{\sigma} : Clif(TM) \to \Omega(M) \) from equation 2.10. We define the \( s \)-symbol of \( P \) by:

\[
\sigma(P)(m, \xi) = \tilde{\sigma}(\mu(P)(m, \xi)).
\]

Thus, \( \sigma(P)(m, \xi) \) is a form-valued endomorphism of \( E_m \).

**Remark.** The two maps \( \theta \) and \( \sigma \) are not inverses of each other; however, at the level of asymptotic operators, they will be inverses.

The following two examples will be key steps in obtaining the formula for the \( s \)-symbol of the composition of two operators.

Fix a point, \( m \in M \), and choose an orthonormal basis, \( X_1, \ldots, X_n \), of \( T_m(M) \); then we have normal coordinates, \( (x_1, \ldots, x_n) \to \exp_m(x_1 X_1 + \cdots + x_n X_n) \), in a neighborhood of \( m \). We also have that \( (x_1, \ldots, x_n) \) provide coordinates on \( T^*_m(M) \). Let \( (\xi_1, \ldots, \xi_n) \) be dual coordinates to \( (x_1, \ldots, x_n) \) on \( T^*_m(M) \), so \( \xi = \sum \xi_i X^*_i \). Let \( s = (s_1, \ldots, s_p) \) be a basis for \( (E \otimes S)_m \) and extend this to a synchronous frame around \( m \). Thus, \( s(x) \) is parallel translation of \( s \) along the unique geodesic from \( m \) to \( x \). If \( s(m) = \sum s_i(m) \cdot f_i(m) \), then \( s_m(x) = \sum s_i(m) \cdot s_i(x) \); that is, the coefficients of \( \bar{s}_m(x) \) with respect to the frame \( s \) are constant.

**Example 1.**

Let \( X = \sum c_i \frac{\partial}{\partial s_i} \), with \( c_i \in \mathbb{R} \). (This is only valid in a neighborhood of \( m \).)

We can then compute:

\[
\sigma(\nabla X)(s)(m) = \nabla_X \left( e^{i\exp_m^{-1}(y, \xi) \bar{s}_m(y)} \right)|_{y=m} = i(X, \xi)_m s(m) + \sum_{i,j} s_i(m) \omega^j(X)_m(s_j)_m.
\]

In a synchronous frame around \( m \), \( \omega^j(X)_m = 0 \); so we have:

\[
\sigma(\nabla X)(m, \xi) = i(X, \xi)_m.
\]
Example 2.
Next, we let $X = \sum c_i \frac{\partial}{\partial x_i}$ and $Y = \sum d_i \frac{\partial}{\partial x_i}$ with $d_i, c_i \in \mathbb{R}$. We have:

\[
\sigma(\nabla_X \nabla_Y)(m, \xi)s(m) = \nabla_X \nabla_Y (\exp^{-1}(y, \xi) \tilde{s}_m(y))|_{y=m} = \\
\nabla_X (i(Y, \xi)c_i \exp^{-1}(y, \xi) \tilde{s}_m(y) + c_i \exp_{-1}(y, \xi)(\nabla_Y \tilde{s}_m)(y))|_{y=m} = \\
(i(Y, \xi)c_i \exp^{-1}(y, \xi) \tilde{s}_m(y) + i(Y, \xi)c_i \exp_{-1}(y, \xi) \nabla_X \tilde{s}_m(y) + \tilde{s}_m(y) + i(Y, \xi)c_i \exp_{-1}(y, \xi)(\nabla_Y \tilde{s}_m)(y))|_{y=m}.
\]

(2.21)

As we saw above, $(\nabla_X \tilde{s}_m(y))|_{y=m} = 0$, and it follows from appendix II of [ABP] that in this coordinate system, $\nabla_X \nabla_Y \tilde{s}_m(y)|_{y=m} = \Omega(X, Y)s(m)$, where $\Omega$ is the curvature of the bundle $E \otimes S$. We can break $\Omega$ into two pieces, $\Omega_E + \Omega_S$.

As we saw before, $\Omega_E(X, Y) = \frac{1}{4} \sum R_{ij}^k c_i d_j f_k \wedge f_i + \Omega_E(X, Y)$.

Note that the first two terms are of order two while the last term, $\Omega_E$, is of order zero.

Given two polynomial $s$-symbols, $p, q$, with associated differential operators, $\theta(p)$ and $\theta(q)$, we want to compute $\sigma(\theta(p) \circ \theta(q))$. The idea behind the computation is very simple. Recall that on the real line $\frac{d}{dx}$ generates a one-parameter group, $e^{t \frac{d}{dx}}$ and $(e^{t \frac{d}{dx}} \cdot f)(x) = f(t + x)$. Clearly, we have:

\[
e^{t \frac{d}{dx}}(f \cdot g) = (e^{t \frac{d}{dx}} f)(e^{t \frac{d}{dx}} g).
\]

There is an analogous construction when we have a vector bundle with a connection over a compact manifold. Suppose that $F \to M$ is a vector bundle over $M$ with connection $\nabla$. If $X \in \Gamma(TM)$, denote by $\psi^X_t$ the one-parameter group of diffeomorphisms generated by $X$. Given a point, $m \in M$, the map $t \to \psi^X_t(m)$ is a curve in $M$. Let $s \in \Gamma(F)$: then we have $e^{t \nabla X} s$ is parallel translation of $s$ along the integral curve $t \to \psi^X_t(m)$. If we have $\phi \in C^\infty(M)$ and $s \in \Gamma(F)$, then:

\[
e^{t \nabla_X} (\phi \cdot s)(m) = \phi(\psi^X_t(m))(e^{t \nabla_X} s)(m).
\]

**Lemma 2.1.** In a normal coordinate system centered at $m$, let $X = \sum c_i \frac{\partial}{\partial x_i}$; then $\sigma(e^{t \nabla_X}(x, \xi)) = e^{t \nabla_X}(m, \xi)(1 + r(t))$, where $r(t) \in \Omega(M)$. If we write $r(t) = \sum t^i r_i, r_i \in \Omega(M)$, then $\deg(r_i) < i$.

**Proof:** This is just an application of Taylor's theorem with remainder as well as the Campbell-Baker-Hausdorff (CBH) formula. For $s \in \Gamma(E \otimes S)$, we have:

\[
\sigma(e^{t \nabla X})(m, \xi)s(m) = e^{t \nabla_X} (\exp^{-1}(y, \xi) \cdot s_m(y))|_{y=m} = \\
e^{t \nabla_X} (\exp^{-1}(y, \xi) \cdot s_m(y))|_{y=m} = \\
e^{t \nabla_X} (\exp^{-1}(y, \xi) \cdot s_m(y))|_{y=m} = \\
e^{t \nabla_X} (\exp^{-1}(y, \xi) \cdot s_m(y))|_{y=m} =
\]

(2.25)
Now we saw before that $\nabla_X \bar{s}_m |_{y = m} = 0$ and $(\nabla^2_X \bar{s}_m) |_{y = m} = \Omega(X, X) s(m)$; but $\Omega$ is skew so we have $\Omega(X, X) = 0$. One should note that $r(t)$ has no $\xi$ dependence, $r(t) \in \Gamma(\wedge^{even}(M) \otimes End(E))$.

Next, we need to recall the CBH formula. This formula says that if $c_1(A, B)$ are defined by:

$$\text{exp}(A) \text{exp}(B) = \exp \left( \sum_{j=1}^{\infty} c_j(A, B) \right),$$

then if $c_1(A, B) = A + B$, the $c_j$’s are uniquely determined by the following recursion formula:

$$\sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1 + \cdots + k_{2p} = n} [c_{k_1}(A, B), \cdots, [c_{k_{2p}}(A, B), A + B] \cdots].$$

Let $s$ be the synchronous frame for $E \otimes S$ associated with our coordinate system. Thus we have $\bar{s}_m(y) = \sum s_i(m) s_i(y)$, the point being that the coefficients in this frame for $\bar{s}_m$ are constants.

In this frame we have $\nabla_X = \partial_X + \omega(X)$, where $\omega(X)$ is the connection one-form. Thus we have:

$$e^{t \nabla_X} = e^{t \partial_X + t \omega(X)} = e^{t(\partial_X + t \omega(X))} e^{-t \partial_X} e^{-t \omega(X)} e^{t \omega(X)} e^{t \partial_X}.$$

(1) Set $A = \partial_X$;

(2) Set $B = \omega(X)$.

If we apply the CBH formula to equation 2.28, we get:

$$e^{t(\nabla_X + \omega(X))} = e^{t \omega + \{tA, tB\}} e^{-t \omega(X)} e^{t \omega(X)} e^{t \partial_X}.$$

The term in the brackets, $\{tA, tB\}$, is of the form $[E_1[E_2, \cdots, [E_l, F], \cdots]]$, where the $E_i$’s are either $tA$ or $tB$. Now $tB$ is in $so(n)$ and $\{tA, tB\} = t^2 \partial_X (B) \in so(n)$. Since $so(n)$ is closed under brackets, we see that $\{tA, tB\}$ is an $so(n)$-valued section of $Cliff$; hence, as a differential operator, it is of degree two. Now assign to $t$ the weight $-1$ and assign to a Clifford operator the weight that is its degree. We see then that the weight of the term $\{tA, tB\}$ is less than or equal to zero. If we apply CBH once more, we have:

$$e^{t(\nabla_X + \omega(X))} = e^{G(tA, tB) t \omega(X)} e^{t \partial_X}.$$

Again, the explicit form of the CBH formula tells us that now the weight of $G(tA, tB)$ is strictly less than zero. Finally, if we evaluate this at the point $m$, the term $e^{t \omega(X)} |_{m} = 1$, since $\omega(X) |_{m} = 0$. Thus we see that:

$$e^{t(\nabla_X - \bar{s}_m)} |_{y = m} = e^{G(tA, tB) \bar{s}(m)}.$$

Since the weight of $G(tA, tB)$ is strictly less than zero, we have that the weight of $e^{G(tA, tB)}$ is also strictly less than zero. If we write $e^{G(tA, tB)} = \sum t^r r_i$, then this says that $deg(r_i) < i$.

One should also note that $\psi^X_t(m) = exp_m(tX)$ for small $t$, since solving the flow equation is purely a local matter. $\blacksquare$
Corollary 2.2. Let \( m \in M \) and \( X \) be the vector field defined in a neighborhood of \( m \) as in Lemma 2.1. Then in that neighborhood, we have:

\[
\sigma(\nabla_X^n) = (i(X, \xi))^n + p(m, \xi)
\]

where \( p(m, \xi) = \sum_{i \geq 1} \omega^i \otimes p_i(m, \xi) \) and \( p_i \in \mathcal{S}^{n-1}(E) \).

**Proof:** One has that \( \sigma(\nabla_X^n) \) is equal to \( n! \) times the coefficient of \( t^n \) in the expansion of \( \sigma(e^{t\nabla_X}) \). This now follows from lemma 2.1. \( \blacksquare \)

Recall that an \( s \)-symbol is a section of the vector bundle over \( T^*(M) \) defined by \( \pi^*(\Omega(M) \otimes \text{Hom}(E)) \) with certain growth properties. In [G] a proof of the following theorem was outlined. The following proof is similar to the one in [G] but relies on Taylor's theorem instead of the algebra of formal symbols.

Let \( L \) be the bundle over \( T^*(M) \) given by:

\[
L = \pi^*(\text{End}(E) \otimes \text{Cliff}(T^*(M)) foreseeable text)
\]

The natural map \( c : \text{Cliff}(T^*(M)) \otimes \text{Cliff}(T^*(M)) \to \text{Cliff}(T^*(M)) \) given by the Clifford product induces a map from \( \text{End}(E) \otimes L \to L \), which we will also call \( c \).

**Theorem 2.1.** Let \( p \in S\Sigma^1(E) \) and \( q \in S\Sigma^k(E) \) be two symbols whose dependence on the \( \xi \) variable is polynomial. There exist differential operators \( a_n : \Gamma(L \otimes L) \to \Gamma(L) \) such that if \( a_n(p, q) = \sigma(a_n(p, q)) \), then:

1. \( a_n(p, q) \in S\Sigma^{k+n}(E) \);
2. \( p \circ q = \sum_{n=0}^{\infty} a_n(p, q) \) (finite sum); and
3. \( a_0(p, q) = \frac{1}{4} R_{ijkl}(p(x, \xi) \wedge q(x, \eta)) \).

We have used the notation:

\[
-\frac{1}{4} R_{ijkl}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(p, q) = -\frac{1}{4} \sum_{ijkl} R^k_{ij} f_k \wedge f_l \wedge \frac{\partial p}{\partial x_i}(x, \xi) \wedge \frac{\partial q}{\partial x_j}(x, \eta).
\]

**Remark.** There are two important parts to this theorem. First, the order of the symbol of the composition of two operators is less than or equal to the sum of the orders of the operators. While this is true for ordinary differential operators, the presence of the Clifford variables makes this far from trivial for supersymmetric differential operators. It will be this fact that is crucial in developing the calculus of asymptotic operators. The second part will be the formula for \( a_0(p, q) \).

**Proof:** We will work locally; thus let \( m \in M \) and let \( X \) and \( Y \) be the vector fields defined in a neighborhood of \( m \) as in example 2. Let \( \phi^X \in \sigma(\nabla_X^n) \); then by corollary 2.2 we have \( \phi^X = (i(X, \xi))^n \) plus lower order terms in \( \xi \). An easy induction shows that if \( p \in S\Sigma^1(E) \) is an \( s \)-symbol with polynomial dependence in \( \xi \), then we can write \( p \) as:

\[
p(x, \xi) = \sum_i h_i(x) \phi^X_i, \quad h_i \in \text{End}(E) \otimes \text{Hom}(M).
\]

Since the map \( (p, q) \to \sigma(\theta(p) \circ \theta(q)) \) is bilinear, it suffices to prove the theorem for symbols of the form \( h(x) \phi^X \).
Let \( h, g \in \text{End}(E) \otimes \Omega(M) \); then we have:

\[
\sigma(\tilde{\theta}(h)(pX)^n(\nabla Y)^m)) = \sigma(\tilde{\theta}(h)(\nabla_X)^n(\nabla_Y)^m) = \\
\sigma(\tilde{\theta}(h)) \left( \sum_k \tilde{\theta}(\nabla_X^{n-k})(\nabla_Y^k) \right) = \\
\sum_k \sigma(\tilde{\theta}(h)) \tilde{\theta}(\nabla_X^{n-k}) \sigma((\nabla_X)^k(\nabla_Y)^m) \mod \text{lower order terms.}
\]

In the last line we have used equation 1.15. Thus, we are reduced to proving the theorem for the operators \((\nabla_X)^n\) and \((\nabla_Y)^m\).

Next, we need to recall the Campbell-Baker-Hausdorff formula again.

Note that it follows from 2.27 that \( c_n(tA, tB) = t^n c_n(A, B) \), so \( c_n(A, B) \) is a homogeneous polynomial in \( A \) and \( B \) of degree \( n \). If we apply 2.26 with \( A = t\nabla_X \) and \( B = s\nabla_Y \), then we have:

\begin{enumerate}
\item \( c_1(t\nabla_X, s\nabla_Y) = t\nabla_X + s\nabla_Y \); and
\item \( c_2(t\nabla_X, s\nabla_Y) = \frac{1}{2} t s [\nabla_X, \nabla_Y] = \frac{1}{2} t s \Omega(X, Y) \).
\end{enumerate}

Recall that \( \Omega = \Omega_s + \Omega_E \), where \( \Omega_s \) is the curvature of the spin bundle and \( \Omega_E \) is the curvature of \( \nabla_E \). As a \( \psi DO \), \( \Omega_s(X, Y) \) is of order two, while \( \Omega_E(X, Y) \) is of order zero.

Now suppose that \( \phi \) is a section of \( \text{Cliff}(T^*(M)) \) such that \( \phi(x) \in so(n) \subset \text{Cliff}(T^*(M)_0) \) for all \( x \in M \); then \( [\nabla_X, \phi] = \nabla_X(\phi) \in so(n) \). Since \( so(n) \) is closed under brackets, we conclude from the Campbell-Baker-Hausdorff formula, (2.32), that \( c_j(\nabla_X, \nabla_Y) \) is of order two as a \( \psi DO \) for \( j \geq 2 \).

**Remark.** It will be important to notice that as a \( \psi DO \), the operators \( c_j(X, Y) \) contain NO differentiation; they are purely Clifford variable operators. This follows from the explicit form of the CBH formula.

In what follows, associate with \( t \) and \( s \) weight -1 and associate with \( \xi \) and \( \epsilon \) weight 1. We have that \( \sigma((t\nabla_X)^n \circ (s\nabla_Y)^m) \) is the coefficient of \( t^a s^b \) in the expression \( \sigma(e^{t\nabla_X} \circ e^{s\nabla_Y}) \). We need to show that the weight of each term of \( t^a s^b \) is less than or equal to zero and to determine the term of weight zero. Applying the CBH formula we have:

\[
\sigma(e^{t\nabla_X} \circ e^{s\nabla_Y}) \equiv \sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2} t s \Omega(X, Y) + C(X, Y)}) \equiv \\
\sigma(e^{t\nabla_X + s\nabla_Y + \frac{1}{2} t s \Omega_s(X, Y) + C(X, Y)}).
\]

Define:

\begin{enumerate}
\item \( A = t\nabla_X + s\nabla_Y + \frac{1}{2} t s \Omega_s(X, Y) \) — this is weight 0;
\item \( B = C(X, Y) \) — this is weight strictly less than 0.
\end{enumerate}

We have:

\[
\epsilon^{A + B} = \epsilon^A \epsilon^{-A} \epsilon^B = (\epsilon^{B + \{A, B\}}) \epsilon^A \text{ by the CBH formula.}
\]

We see from CBH that if the weight of \( B \) is less than zero, then the weight of the term \( \{A, B\} \) is also less than zero; also the operator \( \epsilon^{B + \{A, B\}} \) is purely a Clifford
operator. Since this term is a Clifford operator, we can use equation 1.15 again to conclude that:

\[ (2.39) \quad \sigma(e^{B+(A,B)}e^A) = \sigma(e^{B+(A,B)}) \wedge \sigma(e^A) \mod \text{lower order terms.} \]

Now \( \sigma(e^{B+(A,B)}) \) is of the form \( 1 + f \), where \( f \) has weight strictly less than zero. Thus to finish the theorem, we need to show that \( \sigma(e^A) \) has the desired form and compute the term of weight 0. To do this, we use the CBH formula again.

\[ (2.40) \quad \sigma(e^{i\nabla_x + s\nabla_Y + \frac{1}{2} s\sigma_4(X,Y)}) = \sigma(e^{\frac{1}{2} s\sigma_4(X,Y)}) e^{i\nabla_x + s\nabla_Y} \mod \text{Clifford terms of negative weight.} \]

Finally, using lemma 2.1, we have equation 2.40 becomes:

\[ (2.41) \quad \sigma(e^{i\nabla_x + s\nabla_Y + \frac{1}{2} s\sigma_4(X,Y)}) = e^{\frac{1}{2} s\sigma_4(X,Y)} e^{i\nabla_x + s\nabla_Y} \mod \text{terms of negative weight.} \]

Replacing \( X \) by \( iX \), \( Y \) by \( iY \) and equating the powers of \( t \) and \( s \), we obtain the conclusion of the theorem. \( \Box \)

3. Asymptotic Pseudodifferential Operators and their Calculus.

Widom developed a theory of asymptotics for pseudodifferential operators. If \( p(x, \xi) \in \Sigma^m(E) \) is an ordinary symbol, Widom considered the family of pseudodifferential operators defined by \( P_t = \theta(p(x, t\xi)) \). When one considers this family instead of just the single operator \( P \), one finds that the asymptotic expansions normally associated with heat kernels (as well as other expansions) follow naturally from the calculus that he developed. In our setting, we want to extend Widom's ideas to the class of \( s \)-symbols. The \( s \) stands for supersymmetric, and in supersymmetry Clifford multiplication is viewed as a differential operator.

Let \( p \in \Sigma^m(E) \), so that \( p(x, \xi) = \sum_{i=0}^m p_i(x, \xi) \otimes \omega_i \), with \( p_i \in \Sigma^{m-i}(E) \) and \( \omega_i \in \Omega^i(M) \). We have the following definition, which was introduced in \([G]\).

**Definition 3.1.** If \( p(x, \xi) \in \Sigma^m(E) \) then:

\[ (3.1) \quad p_t(x, \xi) = \sum_{i=0}^m p_i(x, t\xi) \otimes t^i \omega_i. \]

Next we need to recall the classical notion of an asymptotic expansion.

**Definition 3.2.** Let \( p(x, \xi, t) \in \Sigma^m(E) \) be a family of symbols; then \( p(x, \xi, t) \) is called an asymptotic family if there exist symbols \( p_n \in \Sigma^{m-n}(E) \) such that the following asymptotic expansion holds:

\[ (3.2) \quad p(x, \xi, t) \sim \sum_{n=0}^\infty t^n p_n(x, \xi). \]
In other words, given \( N > 0 \), we have (recalling that the symbol spaces are Fréchet spaces):

\[
\lim_{t \to 0} t^{-N} \left( p(x, \xi, t) - \sum_{i=0}^{N} t^i p_i(x, \xi) \right) = 0 \quad \text{in} \Sigma^m \cap \Sigma^{m-n}(E).
\]

We will call the first term, \( p_0(x, \xi) \), the leading symbol of \( p \) (not to be confused with the principal symbol).

As usual, the notion of an asymptotic expansion gives us an equivalence relation on the space of asymptotic families of symbols. This leads to the following definition:

**Definition 3.3.** Given an asymptotic family of symbols, \( p(x, \xi, t) \), we call the equivalence class that \( p_i \) determines an asymptotic symbol. We will call a representative of the equivalence class an asymptotic symbol if no confusion is likely to arise.

Let \( H^s(E) \) be the \( s \)-th Sobolev space formed from the sections of \( E \otimes S \), so \( H^s(E) \) is the completion of \( \Gamma(E \otimes S) \) (see [Gii]). If \( P \) is an infinitely smoothing operator, then \( P : H^s(E) \to H^0(E) \), and we denote by \( \|P\|_s \) the corresponding norm. This turns the space of smoothing operators into a Fréchet space.

**Definition 3.4.** Let \( P_t \) be a family of smoothing operators; then \( P_t \) is called asymptotically zero if \( P_t \to 0 \) in the Fréchet space topology. Thus, given \( N > 0 \), we have for all \( s \):

\[
\lim_{t \to 0} t^{-N} \|P_t\|_s = 0.
\]

**Definition 3.5.** Given two families of operators, \( P_t \) and \( Q_t \), we will say that \( P_t \) is equivalent to \( Q_t \) if the difference \( P_t - Q_t \) is asymptotically zero. If \( P_t \) is equivalent to \( \theta(p(x, \xi, t)) \) for \( p \) an asymptotic symbol, then we will call the equivalence class that \( P_t \) determines an asymptotic pseudodifferential operator, \( \Lambda \psi DO \). As before, we will call a representative of this class an \( \Lambda \psi DO \) if no confusion is likely to arise.

**Remark.** Since the map \( \theta \) depends on the choice of a cut-off function \( \alpha \), we should write \( \Lambda \psi DO^\alpha \) for the corresponding collection of operators. However, the next lemma shows that this collection is independent of the choice of \( \alpha \).

**Lemma 3.6.** Let \( \phi \in C^\infty(M \times M) \) which is identically zero in a neighborhood of the diagonal \( \Delta \subset M \times M \) and whose support is contained in the neighborhood of \( \Delta \) where the exponential map is a diffeomorphism. Given any asymptotic symbol, \( p(x, \xi, t) \), define \( \tilde{P}_t \) by:

\[
\tilde{P}_t s(m) = \int e^{i(X_m \cdot \xi_m)} \delta(p(m, \xi, t)) \phi(m, \text{Exp}_m(X)) \delta_m(X_m) m dX_m d\xi_m.
\]

Then \( \tilde{P}_t \) is an asymptotically zero operator.

**Proof:** This is similar to lemma 1.2.6 of [Gii]. We can assume that \( p(x, \xi, t) \in \Sigma^k(E) \), since \( p \) will be a finite sum of elements of this form times a form. Define
a Laplacian by $\Delta_\xi = \sum_j (i \partial_{\xi_j})^2$; then $\Delta_\xi e^{i(X_m, \xi_m)} = |X_m|^2 e^{i(X_m, \xi_m)}$. Since $|x_m|^{-2l} \phi(m, X_m) \in C^\infty(T(M))$ for all $l$, we can integrate by parts to get:

\begin{equation}
\tilde{P}_t \phi(m) = \int e^{i(X_m, \xi)} |X_m|^{-2l} \phi(m, X_m) \delta(\Delta_\xi p(m, t\xi, t)) \tilde{s}_m(X_m) dX_m d\xi_m.
\end{equation}

Now $\Delta_\xi p(x, t\xi, t) = t^{2l}(\Delta_\xi p)(x, t\xi, t) \in \Sigma^{k-2l}(E)$. Note that we get powers of $t$ coming out of the repeated integration by parts. This is sufficient to prove the lemma.

**Corollary 3.7.** Let $p(x, \xi, t)$ be an asymptotic symbol; then the asymptotic operator $P_t = \theta(p(x, \xi, t))$ is independent of the choice of the cut-off function $\alpha$ that was chosen to define $\theta$.

**Proof:** This follows from lemma 3.6.

**Lemma 3.8.** Let $r(m, \xi_m, y, t) \in C^\infty(M, \Sigma^d(E))$ such that for all $\alpha, \beta, \gamma$ we have:

\[ \| \partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma r(m, \xi_m, y, t) \| \leq C_{\alpha, \beta, \gamma} (1 + |\xi_m|^{d-|\beta|}) \]

where $C_{\alpha, \beta, \gamma}$ is independent of $t$. We denote the space of such functions by $S$, which is a Fréchet space with the semi-norms:

\[ \rho_{\alpha, \beta, \gamma}(r) = \inf \{ C_{\alpha, \beta, \gamma} \| \partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma r(m, \xi_m, y, t) \| \leq C_{\alpha, \beta, \gamma} (1 + |\xi_m|^{d-|\beta|}) \} \]

Furthermore, assume that $r(m, \xi, y, t)$ has an asymptotic expansion:

\[ r(m, \xi, y, t) \sim \sum_{n=0}^{\infty} t^n r_n(m, \xi, y) \]

If we define:

\[ R_t(s)(m) = \int e^{-i(X_m, \xi_m)} r(m, \xi_m, \exp_m(X_m), t) \tilde{s}(X_m) dX_m \]

then $R_t$ is an $A\psi DO$, and if $R_t \sim \theta(p_t)$ we have:

\[ p(m, \xi_m, t) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha r(m, \xi_m, \exp_m(X_m), t) |_{X_m=0} \]

**Proof:** We will write $r(m, \xi_m, X_m, t)$ for $r(m, \xi_m, \exp_m(X_m), t)$. In what follows we will assume that $r$ has no form component. Note that since $\tilde{s}(X_m)$ vanishes off a neighborhood of $\Delta \subset M \times M$, $R_t$ depends only on $r(m, \xi_m, X_m, t)$ for $X_m$ in a neighborhood of zero. Thus we will suppose WLG that $r(m, \xi_m, X_m, t)$ has compact support in the $X_m$ variable. We will write:

\[ \tilde{s}(\xi_m) = \int e^{-i(X_m, \xi_m)} \tilde{s}(X_m) dX_m \quad \text{and} \]

\[ q(m, \xi_m, \eta_m, t) = \int e^{-i(X_m, \eta_m)} \tilde{s}(X_m) dX_m \]
Using the fact that the Fourier transform of the product is the convolution of the Fourier transforms, we get:

\[
(R_t \delta)(m) = \int e^{-i\langle \xi, \xi \rangle} r(m, t \xi_m, X_m, t) \bar{\delta}(X_m) \, dX_m \, d\xi_m = \\
\int q(m, t \xi_m, \xi_m - \eta_m, t) \bar{\delta}(\eta_m) \, d\eta_m \, d\xi_m.
\]  

(3.7)

We estimate \(|q(m, t \xi_m, \xi_m - \eta_m, t)|\) in the usual way. The estimates on \(r\) and the fact that \(q(m, \xi, \eta, t)\) is rapidly decreasing in the \(\eta\) variable yields:

\[
|q(m, t \xi_m, \xi_m - \eta_m, t)| \leq C_k (1 + |\xi|)^d (1 + |\xi - \eta|)^{-k}.
\]  

(3.8)

Now Peetre's inequality says: \((1 + |z + y|)^s \leq (1 + |x|)^s (1 + |y|)^{|s|} \). Using this inequality and the fact that for all \(l \; |\bar{\delta}(\eta)| \leq C_l (1 + |\eta|)^{-l}\), we get that \(|q(m, t \xi_m, \xi_m - \eta_m, t) \bar{\delta}(\eta_m)|\) is integrable; thus we can interchange the order of integration. Define \(p\) by:

\[
p(m, t \eta_m, t) = \int q(m, t \xi_m, \xi_m - \eta_m, t) \, d\xi_m = \\
\int q(m, t \xi_m + t \eta_m, \xi_m, t) \, d\xi_m.
\]  

(3.9)

Then we have:

\[
R_t(\delta)(m) = \int e^{-i\langle \xi, \xi \rangle} p(m, t \eta_m, t) \bar{\delta}(X_m) \, dX_m \, d\xi_m.
\]  

(3.10)

Thus we only need to check that \(p(m, \eta_m, t)\) is a symbol of order \(d\) and that it has the requisite asymptotic expansion.

Define a linear map \(T : \mathcal{S} \to \mathcal{S}(E)\) by \(T(r) = p\), where \(p\) is defined as above. We have that \(T\) is continuous from the Fréchet space \(\mathcal{S}\) to the Fréchet space \(\mathcal{S}(E)\). We have:

\[
|\partial_\xi^{\alpha} \partial_{\eta_m}^{\beta} q(m, t \xi_m + \eta_m, \xi_m, t)| = \\
|\int e^{-i\langle \xi, \xi \rangle} \partial_\xi^{\alpha} \partial_{\eta_m}^{\beta} r(m, t \xi_m + \eta_m, X_m, t) \, dX_m| = \\
|\int e^{-i\langle \xi, \xi \rangle} \xi_m^{\gamma} \partial_\xi^{\alpha} \partial_{\eta_m}^{\beta} \partial_{X_m}^{\gamma} r(m, t \xi_m + \eta_m, \xi_m, t) \, dX_m| \leq \\
KC_{\alpha, \beta, \gamma} (1 + |\xi_m|)^{-|\gamma|} (1 + |t \xi_m + \eta_m|)^{d-|\beta|} \leq \\
KC_{\alpha, \beta, \gamma} (1 + |\xi_m|)^{-|\gamma|} (1 + |t \xi_m|)^{d-|\beta|} (1 + |\eta_m|)^{d-|\beta|}.
\]  

(3.11)
The constant $K$ is independent of $r$ and bounds the volume of the compact set that $r(m, \xi, X_m, t)$ has for its $X_m$ support. If $|\gamma|$ sufficiently large we can integrate the above inequality to get:

$$\rho_{r, \beta}(T(r)) \leq K \rho_{r, \beta, \gamma}(r).$$

This shows that $T(r)$ is a symbol and that if $r(m, \xi, X_m, t)$ has an asymptotic expansion, then, $T(r)$ also has an asymptotic expansion.

Using Taylor's formula we have:

$$q(m, t\xi_m + \eta_m, t) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \frac{1}{t^{[\alpha]}} \partial_{\xi_m}^\alpha q(m, \eta_m, t)(t\xi_m)^\alpha.$$  

Thus we have:

$$p(m, \eta_m, t) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \int \int \int e^{-i(X_m, \xi_m)} s_{X_m} \partial_{\xi_m}^\alpha r(m, \eta_m, X_m, t) \, dX_m \, d\xi_m \sim \int \int \int \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{X_m}^{\alpha} \partial_{\xi_m}^\alpha r(m, \eta_m, X_m, t)|_{X_m=0}.$$  

**Lemma 3.9.** If $P_t \sim \theta(p_t)$ is an $A\psi DO$, then the adjoint, $P_t^*$, is also an $A\psi DO$.

**Proof:** Recall that $a_{\delta}(y)$ is parallel translation of $a(y)$ from $(E \otimes S)_y$ to $(E \otimes S)_m$ times $a_\phi(m, y)$ where $\phi(m, y)$ is zero off of a neighborhood of $\Delta \subset M \times M$, where the exponential map is a diffeomorphism. Let $s, h$ be sections and set $t = \exp_m(X_m)$; then we have:

$$\langle P_t s, h \rangle = \int e^{-i(X_m, \xi_m)} \langle p(m, t\xi_m, t)s(X_m), h(m) \rangle \, dX_m \, d\xi_m \, dvol(m) = \int e^{-i(X_m, \xi_m)} \langle \phi(m, y)\tau(m, y)s(y), p(m, t\xi_m, t)^*h(m) \rangle \, dX_m \, d\xi_m \, dvol(m).$$

Recall that parallel translation is an isometry. Thus we have:

$$\langle \tau(m, y)s_m(y), p(m, t\xi_m, t)^*h(m) \rangle |_{m} = s(y), \tau(y, m)(p(m, t\xi_m, t)^*h(m)) |_{y} = s(y), (\tau(y, m)p(m, t\xi_m, t)^*)(\tau(y, m)h(m)) |_{y} = s(y), (\tau(y, m)p(m, t\xi_m, t)^*)s(\tau(X_y)) |_{y}.$$  

Let $dX_m = J(m, y)dvol(y)$, and $dvol(m) = J^{-1}(y, m) dX_y$; then we have $dX_m dvol(m) = a(m, y)dvol(y) dX_y$, where the function $a(m, y)$ has all of its derivatives bounded on the support of $\phi(m, y)$.

Now $\exp_m(sX_m) = \gamma(t)$ is the unique geodesic with $\gamma(t) = m$ and $\gamma(1) = y$, thus $\gamma(1-t)$ is the unique geodesic from $y$ to $m$. Since $\gamma(t) = -\tau(y, m)(X_m)$, we
see that $\exp_{\mathbf{m}}^{-1}(m) = -\tau(y, m)(X_m)$. Now $(X_m, \xi_m) = (\tau(y, m)X_m, \tau(y, m)\xi_m)$, so 3.15 equals:

$$
\int e^{-i<X_m, \xi_m>} \phi(m, y)a(m, y) \\
(\tilde{s}(y, \tau(y, m)p(m, \tau(y, m)\xi_y, t)*h_y(X_y)) dX_y d\xi_y dvol(y).
$$

(3.17)

If we set $r(y, \xi_y, X_y, t) = \phi(m, y)a(m, y)\tau(y, m)p(m, \tau(y, m)\xi_y, t)^* = \phi(m, y)a(m, y)\tau(y, exp_y(X_y))p(\exp_y(X_y), \tau(y, m)\xi_y, t)^*$, then we see that:

$$
(P^*\hat{h})(y) = \int e^{-i<X_m, \xi_y>} r(y, \xi_y, X_y, t)*h(X_y)dX_y d\xi_y.
$$

(3.18)

Once we show that $r$ satisfies the conditions of lemma 3.8 we are done.  

**Lemma 3.9.** Suppose that $r(m, \xi_m, t)$ is a family of symbols in $\Sigma^\infty(E)$ such that the symbol estimates hold uniformly in $t$. If $R_t = \theta(r_t)$, then $\sigma(R_t)_{t=1}(m, \xi_m) \sim r(m, \xi_m, t)$.

(It should be noted that we are NOT assuming that $r$ is an asymptotic symbol.)

**Proof:** Without loss of generality, we can assume that $r$ has no form component. Now we compute, If $y = \exp_m(X_m)$, then we have:

$$
\sigma(R_t)(\xi_m/t, m)s(m) = R_t (e^{-i(\exp_{\mathbf{m}}^{-1}(y), \xi_m/t)} \tilde{s}(y) |_{y=m} = \\
(\int e^{i<X_m, \eta>} e^{-i<X_m, \xi_m/t>} r(m, t\eta, t) \phi(m, X_m) dX_m d\eta_m) s(m) = \\
(\int e^{i<X_m, \eta>} r(m, t\eta + \xi_m, t) \phi(m, X_m) dX_m d\xi_m) s(m).
$$

(3.19)

Now we use Taylor's theorem on $r(m, t\eta + \xi_m, t)$ to get:

$$
r(m, t\eta + \xi_m, t) = \sum_{|\alpha| < m} \frac{t|\alpha|}{\alpha!} \partial_\eta^\alpha r(m, \xi_m, t) + E
$$

(3.20)

where $E = \sum_{|\alpha| = m} \frac{m}{\alpha!} (t\alpha)^m \int_0^1 u^{m-1} \partial_\eta^\alpha r(m, \xi + (1 - u)t\eta) du$.

Recall that the function $\phi(m, X_m)$ is identically one in a neighborhood of zero; so we have:

$$
\frac{t|\alpha|}{\alpha!} \int e^{i<X_m, \eta>} \eta^\alpha \partial_\eta^\alpha r(m, \xi_m, t) dX_m d\eta_m = \\
\frac{t|\alpha|}{\alpha!} \partial_\eta^\alpha r(m, \xi_m, t) \partial_{X_m} \phi(m, X_m) |_{X_m=0} = \\
\begin{cases} 
    r(m, \xi_m, t) \text{ if } \alpha = 0 \\
    0 \text{ if } \alpha \neq 0
\end{cases}
$$

(3.21)
To finish, we have to estimate the remainder term. First we note:

\begin{equation}
\| \partial_\xi^\ell r(m, \xi + (1 - u)t\eta, t) \| \leq C(1 + |\xi + (1 - u)t\eta|)^{n-m} \leq C(1 + |\xi|)^{n-m}(1 + |(1 - u)t\eta|)^{|n-m|} \leq C(1 + |\xi|)^{n-m}(1 + |\eta|)^{|n-m|}.
\end{equation}

Therefore, integrating this estimate and remembering that \( \int e^{i(X_m, \eta_m)} \phi(m, X_m) \, dX_m \) is rapidly decreasing in \( \eta_m \) yields:

\begin{equation}
\| \int \int e^{i(X_m, \eta_m)} E(m, \xi_m, \eta_m, t) \phi(m, X_m) \, dX_m \, d\eta_m \| \leq C t^m(1 + |\xi_m|)^{n-m} \int \int e^{i(X_m, \eta_m)} \phi(m, X_m) \, dX_m \, d\eta_m \leq C t^m(1 + |\xi_m|)^{n-m}.
\end{equation}

One can similarly estimate the derivatives of the error term. We conclude that for all \( m \):

\begin{equation}
\| r(m, \xi_m, t) - \sigma(R_t)_{\xi_m}(m, \xi_m) \| \leq C t^m(1 + |\xi_m|)^{n-m}.
\end{equation}

\[ \square \]

**Lemma 3.10.** Let \( p(m, \xi, t) \) and \( q(m, \xi, t) \) be two asymptotic symbols and let \( P_t = \theta(p_t) \) and \( Q_t = \theta(q_t) \) be the respective pseudodifferential operators; then \( P_t \circ Q_t = \theta(r_t) \), where \( r \) is also an asymptotic symbol. Furthermore, the leading symbol of \( P_t \circ Q_t \) is given by \( a_0(p_0, q_0) \), where \( p_0 \) and \( q_0 \) are the leading symbols of \( p \) and \( q \) respectively.

**Proof:** We recall how \( \theta(p_t) \) was defined. Let \( \alpha(m, y) \) be a smooth function defined in a neighborhood of the diagonal \( \Delta \subset M \times M \), where the exponential map is a diffeomorphism. Let \( r(m, y) \) be parallel translation along the unique geodesic from \( m \) to \( y \) for \( (m, y) \in supp(\alpha) \). For a section \( s \) we defined \( \hat{s}_m(y) = \alpha(m, y) r(m, y) s(y) \). If \( y = \exp_m(X_m) \), then we wrote \( \hat{s}(X_m) \) for \( \hat{s}_m(y) \). The point to remember is that \( \hat{s}(X_m) \) vanishes outside a neighborhood of zero.

In what follows we will omit the cut-off function \( \alpha \) from most of the calculations to make the notation clearer. Now we compute. Let \( y = \exp_m(X_m) \) and \( z = \exp_y(X_y) \); then:

\begin{equation}
P_t \circ Q_t(s)(m) = \int \int e^{i(X_m, \xi_m)} p(m, t\xi_m) r_m^y \left( \int \int e^{i(X, \eta_y)} \tau_y^s \phi(z) \, dX_y \, d\eta_y \right) \, dX_m \, d\xi_m.
\end{equation}

Let \( Y_m \) be the parallel translate along the geodesic from \( y \) to \( m \) of \( X_y \) and \( \zeta_m \) the parallel translate of \( \eta_y \) along the geodesic from \( y \) to \( m \). Since parallel
translation is an isometry, we have that \((X_y, \eta_y)_y = (Y_m, \zeta_m)_m\), and the measures are also the same. Thus (3.21) becomes:

\[
\int e^{i(Y_m, \zeta_m)} \left( \int e^{i(X_m, \xi_m)} p(m, t\xi_m) \tau^*_m q(y, t\eta^*_m(\zeta_m)) \tau^m_y \tau^*_y \tau^m_z dX_m d\xi_m \right) \\
\tau^*_m s(\exp_m(\phi(Y_m))) dY_m d\zeta_m
\]

In the above equation we have written, \(\phi(Y_m) = \exp_m^{-1}(\exp_m(\tau^*_m(\eta^*_m(Y_m))))\); note that \(\phi(0) = X_m\).

Changing variables, with \(J\) the Jacobian of the transformation, yields:

\[
\int e^{i(\phi^{-1}(Y_m), \zeta_m)} \left( \int e^{i(X_m, \xi_m)} p(m, t\xi_m) \tau^*_m q(y, t\eta^*_m(\zeta_m)) \tau^m_y \tau^*_y \tau^m_z dX_m d\xi_m \right) \\
\tau^*_m s(\exp_m(\phi(Y_m))) J(Y_m) dY_m d\zeta_m.
\]

Next note that \(\phi^{-1}(0) = -X_m\) so by a result in [Gi] (bottom of the pg. 25); \(\phi^{-1}(Y_m) = -X_m + A(Y_m) \cdot Y_m\), where \(A(Y_m)\) is a linear map that is invertible in a neighborhood of zero. Thus equation (3.23) becomes:

\[
\int e^{i(-X_m + A(Y_m) \cdot Y_m, \zeta_m)} \int e^{i(X_m, \xi_m)} p(m, t\xi_m) \tau^*_m q(y, t\eta^*_m(\zeta_m)) \tau^m_y \tau^*_y \tau^m_z dX_m d\xi_m \\
\tau^*_m s(\exp_m(\phi(Y_m))) J(Y_m) dY_m d\zeta_m.
\]

Make another change of variables to get:

\[
\int e^{i(Y_m, \zeta_m)} r(m, Y_m, t\zeta_m) s_m(Y_m) dY_m d\zeta_m \quad \text{where } r(m, t\zeta_m, Y_m) =
\]

\[
\int e^{i(X_m, \xi_m - A(Y_m)^{-1}(\zeta_m))} p(m, t\xi_m) \tau^*_m q(y, t\eta^*_m(A(Y_m)^{-1}(\zeta_m))) \\
\tau^m_y \tau^*_y \tau^m_z K(Y_m) dX_m d\xi_m.
\]

Finally, we have:

\[
r(m, t^{-1}\zeta_m, Y_m)_t =
\]

\[
\int e^{i(X_m, \xi_m)} p(m, t\xi_m + B(Y_m)\zeta_m) \tau^*_m q(y, t\eta^*_m(C(Y_m))\zeta_m) S(m, y, z) \\
K(Y_m) dX_m d\zeta_m.
\]

Now standard estimates show that \(r\) is a symbol (see [T], vol. 1, pp. 23-29 theorem 3.2 and 3.3).

Next we need to show that the symbol of \(P_l \circ Q_l\) has the requisite asymptotic expansion.

To do this, we appeal to Widom's calculation of the symbol of the composition of two pseudodifferential operators. Recall that in definition 2.4, given a pseudodifferential operator \(P\), we defined \(\mu(P) \in \text{End}(E) \otimes \text{End}(S)\), which would be
the symbol that is defined by Widiom. Widiom then shows that:

\[
\mu(P \circ Q) \sim \sum_{m_1, \ldots, m_k \geq 2} \frac{i^{k-\sum_{i} m_i - \sum_{i} m_i}}{\nu^{p_1 + m_1, \ldots, \nu^{p_k + m_k}} / \nu^{p_0 \cdot u}} D \sum_{\mu} \mu(P) D \sum_{\mu} \mu(Q).
\]

We can rephrase this in terms of theorem 2.1. Let \( L \) be the bundle over \( T^*(M) \) defined in equation 2.7; then there exist differential operators \( a_n : \Gamma(L \otimes L) \to \Gamma(L) \), such that if \( a_n = \sigma \circ a_n \):

\[
\sigma(\theta(p_t) \circ \theta(q_t)) \sim \sum t^n a_n(p, q) t.
\]

The degree of \( a_n(p, q) \) is determined by two factors:

1. how many times \( p \) and \( q \) are differentiated in the \( \xi \) variable; and
2. the degree of the coefficients of \( a_n \) (since the coefficients of \( a_n \) will be in \( \text{End}(E) \otimes \text{End}(S) = \text{End}(E) \otimes \text{Cliff}(T(M)) \)).

The \( a_n \) are determined by what they do on symbols that are polynomial in \( \xi \) (that is, on differential operators). Suppose that \( p = \omega \cdot \bar{p} \) with \( \omega \in \Omega^s(M) \) and \( \bar{p} \) a homogeneous polynomial of degree \( k - s \) in \( \xi \). Then \( p \) is a symbol of degree \( k \) and \( p_t = t^k p \). We make the same assumption about \( q \), so that \( q_t = t^l q \). Theorem 2.1 says:

\[
\sigma(\theta(p_t) \circ \theta(q_t))_{t^{-1}} = t^{l+k} \sum_{n=0}^{\infty} a_n(p, q) t_{t^{-1}}.
\]

Now \( a_n(p, q) \in S^{l+k-n}(E) \), thus \( a_n(p, q)_{t^{-1}} = t^{-(l+k-n)} a_n(p, q) \). If we put this into equation 3.33 we have:

\[
\sigma(\theta(p_t) \circ \theta(q_t))_{t^{-1}} = \sum_{n=0}^{\infty} t^n a_n(p, q).
\]

This shows that we have the requisite asymptotic expansion and identifies the first term of the expansion as \( a_0(p, q) \).

**Lemma 3.11.** Let \( p_n \in S^{l-n}(E) \), \( n=1, 2, \ldots \), then there exists an asymptotic symbol \( p(m, \xi, t) \) such that \( p(m, \xi, t) \sim \sum_{n=0}^{\infty} t^n p_n(m, \xi) \).

**Proof:** This is a standard argument (see [W1], lemma 4.2). One chooses a \( C^\infty \) function \( \phi \) such that \( \phi(x) = 0 \) if \( x \leq 1 \) and \( \phi(x) = 1 \) if \( x \geq 2 \). Then choose \( \epsilon_n \), positive numbers tending to zero as \( n \to \infty \). Set \( p(m, \xi, t) = \sum_{n=0}^{\infty} t^n \phi(\epsilon_n (\frac{1}{t} + |\xi|^2)) p_n(m, \xi) \). Then it is easy to show that \( p(m, \xi, t) \sim \sum_{n=0}^{\infty} t^n p_n(m, \xi) \). 

**Definition 3.1.** Let \( P_t \) be an A\( \psi \)DO with symbol \( p(m, \xi, t) \sim \sum t^n p_n(m, \xi) \); then \( P_t \) is called asymptotically elliptic if the map \( q \to a_0(p_0, q) \) is invertible.

**Remark.** The notion of ellipticity for A\( \psi \)DO's is slightly more subtle than it is for ordinary pseudodifferential operators. For example, let \( \epsilon_i \) be a local
frame for $T^*(M)$ in a neighborhood $U$, and consider the operator $C(e_i)$, Clifford multiplication by $e_i$. Then $C(e_i)$ is an invertible operator in the neighborhood $U$. The asymptotic operator $tC(e_i)$, which is of order one, is, however, not invertible as an asymptotic operator, since its inverse would be $t^{-1}C(e_i)$, which is not an $A\psi DO$. It is also clear that the map $p \to a_0(p, e_i)$ is not invertible, since $e_i \wedge e_i = 0$.

**Theorem 3.1.** If $P_t$ is an asymptotically elliptic operator, then there exist $Q_t$, an asymptotic operator, such that $P_t \circ Q_t \sim 1$.

**Proof:** Let $\sigma(P_t)_{t^{-1}} \sim \sum_{n=0}^{\infty} t^n P_n$; then we find a symbol, $q \sim \sum_{n=0}^{\infty} t^n q_n$, by solving for the $q_i$'s recursively. Using:

\begin{equation}
\sigma(\theta((p_1))_i \circ \theta((p_2))_i)_{t^{-1}} \sim \sum_{n=0}^{\infty} t^n a_n(p_1, p_2)
\end{equation}

we have:

\begin{equation}
\sum_{n=0}^{\infty} t^n \left( \sum_{i+s+k = n} a_i(p_s, q_k) \right) - 1 = 0.
\end{equation}

This says that $a_0(p_0, q_0) = 1$, or $q_0 = \{a_0(p_0, \cdot)\}^{-1}(1)$. For $n \geq 0$ we have:

\[ a_0(p_0, q_n) + \sum_{i+k+s = n, k < s} a_i(p_s, q_k) \text{ or:} \]

\begin{equation}
q_n = \left\{ a_0(p_0, \cdot) \right\}^{-1} \left\{ - \sum_{i+k+s = n, k < s} a_i(p_s, q_k) \right\}.
\end{equation}

Lemma 3.11 now says that there is an asymptotic symbol with the desired expansion. If we set $Q_t = \theta(q_i)$, then $Q_t$ will be the desired inverse. \hfill \Box

**Theorem 3.2.** If $P_t$ is an $A\psi DO$ of order less than zero, then $Tr_s(P_t)$ exist and:

\begin{equation}
Tr_s(P_t) = (2\pi)^{-n} \left( \frac{2}{i} \right)^3 \int_{T^*(M)} tr_E(\sigma(P_t))_{t^{-1}} \, d\xi.
\end{equation}

**Proof:** For the most part, the proof is standard and is obtained from computing the trace by integrating the kernel of an integral operator along the diagonal. For some of the details one can consult [G], (theorem 3.7). However, some explanation of notation is called for at this point.

Recall that we had an intermediate map, $\hat{\theta} : \Sigma(E) \to \Sigma(E \otimes \mathbb{S})$, where $\Sigma(F)$ is the space of (ordinary) symbols for the bundle $F$. The map $\hat{\theta}$ sent a form-valued
symbol of $E$ to a regular symbol of $E \otimes S$, where $S$ is the bundle of spinors, by sending the form part to the corresponding operator in $\text{End}(S) \cong \text{Cliff}(T^*(M))$.

Suppose that $P_t = \hat{\theta}(p_t)$. If we denote by $\text{tr}_s$ the supertrace on the bundle $E \otimes S$, then it is standard (\cite{W1}, pp. 46) that:

\begin{equation}
\text{Tr}_s(P_t) = (2\pi)^{-n} \int_{T^*(M)} \text{tr}_s(\hat{\theta}(p_t)) \, dx \, d\xi.
\end{equation}

Let $\text{tr}^E(T)$ denote the trace of the endomorphism $T$ on the bundle $E$; using equation 1.22 we have that $\text{tr}_s(\hat{\theta}(p_t)) = \left(\frac{2}{i}\right)^{\frac{n}{2}} \text{top}[\text{tr}^E(p_t)]$. Here, $\text{top}[\omega]$ means the top dimensional piece of the form $\omega$. Since only the top dimensional piece of a form contributes to the integral over the manifold $M$, equation 3.39 becomes:

\begin{equation}
\text{Tr}_s(P_t) = (2\pi)^{-n} \left(\frac{2}{i}\right)^{\frac{n}{2}} \int_{T^*(M)} \text{tr}^E \sigma(P_t) \, d\xi.
\end{equation}

The symbol has not yet been inverse scaled. If we inverse scale the top piece, we introduce a factor of $t^{-n}$ in front of the integral. However, the cotangent fiber also gets scaled by a factor of $t^{-1}$ which introduces a factor of $t^n$ in front of the integral, and the two factors cancel. Thus, we get exactly the formula 3.38 for the supertrace of $P_t$.

**Corollary 3.3.** If $P_t$ is an $A\psi DO$ with leading symbol $p_t$ of degree less than zero, then $\text{Tr}_s(P_t)$ admits an asymptotic expansion:

\begin{equation}
\text{Tr}(P_t) \sim \sum_{m=0}^{\infty} b_m t^m, \quad \text{with} \quad b_0 = (2\pi)^{-n} \left(\frac{2}{i}\right)^{\frac{n}{2}} \int \text{tr}^E(p_t) \, d\xi.
\end{equation}

**Proof:** By assumption we have $\sigma(P_t)_{k-1} \sim \sum_{m=0}^{\infty} t^m p_m$. Recalling that this expansion holds in the topology of the symbol space and using theorem 3.2 yields the desired expansion.

Now we want to apply the above results to the Dirac operator on a compact spin manifold. Let $E$ be a vector bundle over the compact spin manifold $M$ and let $\nabla_E$ be a connection on $E$. We will denote by $\nabla$ the tensor product connection on $E \otimes S$ (or $E \otimes S^\mathbb{K}$). Let $\alpha : E \otimes S \otimes T^*(M) \to E \otimes S \otimes T(M)$ be defined by the inner product on $T(M)$. Let $C(X)$ denote Clifford multiplication by $X$, and let $c : E \otimes S \otimes T(M) \to E \otimes S$ be defined as follows:

\begin{equation}
c(\epsilon \otimes s \otimes X) = \epsilon \otimes c(X) \cdot s.
\end{equation}

The Dirac operator is defined by:

\begin{equation}
D(s) = (c \circ \alpha \circ \nabla)(s).
\end{equation}

If $e_1, \cdots, e_n$ is a local orthonormal frame for the tangent bundle, then locally we have:

\begin{equation}
D(s) = \sum_{i=0}^{n} c(e_i) \cdot (\nabla e_i, s).
\end{equation}
Remark. Note that the order of $D$ is two, so $tD$ is not an asymptotic operator; however, $t^2D$ is an $A\Psi DO$.

Next we need to know $D^2$. This is a standard computation, so we will only outline it. Fix a point $m \in M$ and choose a synchronous frame in a neighborhood of $z$ ( $\nabla z_i (e_j)(x) = 0$, or $F_{ij}^k(x) = 0$). At the point $z$ one computes:

$$D^2 = \sum_{i=1}^n (\nabla z_i)^2 + \frac{1}{4} \sum_{ij,kl} F_{ijkl} e_i e_j e_k e_l + \frac{1}{2} \sum_{ij} F_{ij} e_i e_j,$$

where $F_{ij}$ is the curvature of the bundle $E$. By using the Bianchi identities and with $K$ the scalar curvature of $M$, this reduces to:

$$D^2 = \sum_{i=1}^n (\nabla z_i)^2 + \frac{1}{2} \sum_{ij} F_{ij} e_i e_j + \frac{1}{4} K.$$  

Lemma 3.12. Let $D_t = tD$, and let $\phi \in C^\infty(M)$; then:

1. $D_t^2$ is an $A\Psi DO$ with leading symbol $(-|\xi|^2 + \frac{1}{2} \sum F_{ij} f_i \wedge f_j)$; and
2. $[D_t, \phi]$ is an $A\Psi DO$ with leading symbol $d\phi$.

Proof: We have:

$$\sigma(t^2D^2) = -t^2|\xi|^2 + \frac{1}{2} t^2 \sum F_{ij} f_i \wedge f_j + \frac{1}{4} t^2 K.$$  

Inverse scale to get:

$$\sigma(t^2D^2)_{t^{-1}} = (-|\xi|^2 + \frac{1}{2} \sum F_{ij} f_i \wedge f_j) + \frac{1}{4} t^2 K.$$  

For the other part we have:

$$[D_t, \phi] = t \sum_{i=0}^n d\phi(e_i) \cdot e_i.$$

Consequently, $\sigma([D_t, \phi]) = t d\phi$; so

$$\sigma([D_t, \phi])_{t^{-1}} = d\phi.$$  

Remark. We noted that $D_t$ is not an $A\Psi DO$; consequently, $D_t^{-1}$ is not an $A\Psi DO$ either. It is precisely this fact that makes Connes's original computation of the Chern character so difficult. One should notice that in the letter to Quillen, Connes isolates the expression $tD_t^{-1}$, the point being that $tD_t^{-1}$ is an $A\Psi DO$. Specifically,

$$\sigma(tD_t^{-1}) = \sum t f_i \frac{1}{t^2}.$$
Thus if we inverse scale we get:

\[(3.52) \quad \sigma(tD^{-1}_t)_{t^{-1}} = \sum_{\xi_i} \frac{1}{\xi_i} f_i.\]

The crucial point here is that \(c(\xi_i)^{-1} = c(\xi_i).\)

Given a connection and a "linear" function \(l \in C^\infty(T^*(M) \times M),\) Widom defines a complete symbol for a pseudodifferential operator on \(M.\) It was shown in [R] that the function \((exp_{m}^{-1}(x), \xi_m) = l(\xi_m, x)\) is such a linear function. If \(P_t\) is an \(A\Psi DO\) then we see that \(P_t\) is a family of pseudodifferential operators in the sense of Widom (the treatment of the Clifford variables as differential operators only adds a dependence on \(t\) that is uniform for \(t \in [0, 1]\), so Widom's theory works for our asymptotic operators.) We see that if we choose for the linear function \((exp_{m}^{-1}(x), \xi_m),\) then Widom's symbol is exactly:

\[(3.53) \quad \sigma(P_t)(m, t^{-1}\xi).\]

Lemma 4.10 of [W1] can be stated as follows:

**Lemma 3.13.** The operator \(e^{t^2D^2}\) is a \(A\Psi DO.\)

**Proof:** Applying the function calculus to \(e^{t^2D^2}\) gives:

\[(3.54) \quad e^{t^2D^2} = \frac{1}{2\pi} \int_{\Gamma} (t^2D^2 - \lambda)^{-1} e^{\lambda} d\lambda.\]

Next, Widom shows that:

\[(3.55) \quad \sigma(e^{t^2D^2}) = \frac{1}{2\pi} \int_{\Gamma} \sigma((t^2D^2 - \lambda)^{-1}) e^{\lambda} d\lambda.\]

The delicate point is that the following expansion is uniform in \(\lambda:\)

\[(3.56) \quad \sigma((t^2D^2 - \lambda)^{-1}) \sim \sum_{n=0}^{\infty} t^n (p_n(\lambda))_t.\]

Thus we can integrate the expansion term by term to obtain the desired asymptotic expansion for \(\sigma(e^{t^2D^2}).\)

**Lemma 3.14.** Let \(p\) be the leading symbol of \(t^2D^2;\) then the leading symbol of \(e^{t^2D^2}\) is \(e^{a_0(p_\cdot, \cdot)}(1).\)

**Proof:** Note that \((t^2D^2 - \lambda)^{-1}\) has the leading symbol \(a_0(p - \lambda, \cdot)^{-1}(1) = (a_0(p_\cdot) - \lambda)^{-1}(1).\) Thus \(e^{t^2D^2}\) has the leading symbol:

\[(3.57) \quad \left(\frac{1}{2\pi} \int_{\Gamma} (a_0(p_\cdot, \cdot) - \lambda)^{-1} e^{\lambda} d\lambda\right)(1) = e^{a_0(p_\cdot)}(1).\]
Corollary 3.15. With notation as above we have:

\[ Tr_s(e^{t^2 D^2}) = (2\pi)^{-\frac{n}{2}} \int_{T^*(M)} e^{a_0(p, \cdot)}(1) \, d\xi + O(t), \quad \text{hence} \]

\[ (3.58) \]

\[ \text{ind}(D) = \lim_{t \to 0} Tr_s(e^{t^2 D^2}) = (2\pi)^{-\frac{n}{2}} \int_{T^*(M)} e^{a_0(p, \cdot)}(1)(x, \xi) \, d\xi. \]

Next we want to compute the integral in equation (3.58). Recall that \( R \) was the curvature of \( T(M) \). Thus if \( X, Y \) are vector fields, \( R(X, Y) \in \text{End}(T(M)) \) and \( R(X, Y) \) is an \( \text{so}(n) \)-valued endomorphism. Choose a local orthonormal frame, \( e_1, \ldots, e_n \), for the tangent bundle and let \( f_1, \ldots, f_n \) be the corresponding dual frame for the cotangent bundle. If the matrix of \( R(X, Y) \) with respect to this frame is \( (R(X, Y))_{ij} \), then the curvature of the spin bundle is given by:

\[ (3.59) \]

\[ \tilde{R}(X, Y) = \frac{1}{2} \sum_{ij} (R(X, Y))_{ij} e_i \wedge e_j \in \text{Cliff}(T(M)). \]

Thus we have:

\[ (3.60) \]

\[ \sigma(\tilde{R}(X, Y)) = \frac{1}{2} \sum_{ij} (R(X, Y))_{ij} f_i \wedge f_j. \]

The frame \( f_1, \ldots, f_n \) determines local coordinates \( (\xi_1, \ldots, \xi_n) \) on \( T^*(M) \), and with respect to these coordinates we have the operator:

\[ (3.61) \]

\[ R\left( \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \eta^j} \right) = 2 \sum_{ij} \sigma(R(e_i, e_j)) \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \eta^j}. \]

In what follows we will simply write: \( \sum_{ij} R(e_i, e_j) \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \eta^j} \), where \( R(e_i, e_j) \) is the two-form \( \sum_{kl} R_{kij} f_k \wedge f_l \). The formula for \( a_0(p, q) \) is:

\[ (3.62) \]

\[ a_0(p, q) = e^{-\frac{i}{4} R(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta})} p(x, \xi) \wedge q(x, \eta)|_{\xi=\eta}. \]

Recall that \( F \) is the curvature of the bundle \( E \), so \( \sigma(F) = \sum F_{ij} f_i \wedge f_j \). In what follows we will write \(-|\xi|^2 + \frac{1}{2} F\) for \(-|\xi|^2 + \frac{1}{2} \sigma(F)\). Given an arbitrary symbol \( q \) we have:

\[ a_0(p, q) = (-|\xi|^2 + \frac{1}{2} F) \wedge q(x, \xi) - \]

\[ (3.63) \]

\[ \frac{1}{4} R\left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right)(-|\xi|^2 + \frac{1}{2} F) \wedge q(x, \eta)|_{\eta=\xi} + \]

\[ \frac{1}{16} \left( R\left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right) \right) \left( R\left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta} \right) \right)(-|\xi|^2 + \frac{1}{2} F) \wedge q(x, \eta)|_{\eta=\xi}. \]
All other terms are zero. First we consider the last two terms. We have:

\[
R\left(\partial_{\xi}, \partial_{\eta}\right)\left(-|\xi|^2 + \frac{1}{2} F\right) \wedge q(\xi, \eta)|_{\eta=\xi} =
\]
\[
-2 \sum_{ij} R(e_i, e_j)\xi_i \frac{\partial}{\partial \xi_i} q(\xi, \xi).
\]

(3.64)

We will denote the operator \(\sum_{ij} R(e_i, e_j)\xi_i \frac{\partial}{\partial \xi_i}\) by \(R(\xi, \frac{\partial}{\partial \xi})\).

For the next term we have:

\[
R\left(\partial_{\xi}, \partial_{\eta}\right)R\left(\partial_{\xi}, \partial_{\eta}\right)\left(-|\xi|^2 + \frac{1}{2} F\right) \wedge q(\xi, \eta) =
\]
\[
\sum_{i,j,k,l} R(e_i, e_j) R(e_k, e_l) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \xi_k} \frac{\partial}{\partial \eta_l} \left(-|\xi|^2 + \frac{1}{2} F\right) \wedge q(\xi, \eta).
\]

(3.65)

Now \(-|\xi|^2 = -\left(\xi_1^2 + \cdots + \xi_n^2\right)\) so \(\frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_i} (-|\xi|^2) = 0\) unless \(i = k\) in which case it equals \(-2\). Also \(R(e_k, e_i) = -R(e_i, e_k)\); hence equation 3.65 becomes:

\[
2 \sum_{ij} R(e_i, e_k) R(e_k, e_j) \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} q(\xi, \xi).
\]

(3.66)

We will write this as:

\[
2 \left( R \wedge R \right) \left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}\right)(q(\xi, \xi)).
\]

(3.67)

Thus equation 3.67 becomes:

\[
a_0(p, q) = \left( -|\xi|^2 + \frac{1}{2} F + \frac{1}{2} R(\xi, \frac{\partial}{\partial \xi}) + \frac{1}{16} R \wedge R\left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}\right) \right) (q(\xi, \xi)).
\]

(3.68)

Recall that we are writing \(1\) for the constant symbol \(1(x, \xi) = 1\), and we need to compute \(e^{a_0(p, \cdot)}(1)\). Note that \(\frac{1}{2} R(\xi, \frac{\partial}{\partial \xi})(-|\xi|^2) = -|\xi|^2\), so we have:

\[
e^{a_0(p, \cdot)}(1) = e^{-|\xi|^2 \frac{1}{2} F + \frac{1}{4} R(\xi, \xi) + \frac{1}{16} R \wedge R(\xi, \xi)}(1) =
\]
\[
e^{\frac{1}{4} F} e^{-|\xi|^2 + \frac{1}{4} R \wedge R(\xi, \xi)}(1).
\]

(3.69)

**Remark.** Of course, 3.69 is only true for the constant symbol \(1\).

Next, we appeal to Mehler’s formula, [GJ]. The operator \(e^{-\frac{1}{4}(\xi^2-a^2(\xi^2))}\) is an integral operator with kernel given by:

\[
k_a(\xi, \eta) =
\]
\[
(2\pi a \sinh(a))^{-\frac{1}{2}} \exp \left( -\frac{1}{2a \sinh(a)} (\cosh(a)(\xi^2 + \eta^2) - 2\xi \eta) \right).
\]

(3.70)
If \( A \) is a skew-symmetric matrix with eigenvalues \( \pm i a_j \), then we can form the corresponding operator, \( H = (z^2 - A^2 \frac{\partial^2}{\partial \xi^2} + \frac{2}{i} \frac{\partial}{\partial \xi}) \). The kernel of \( e^{-H} \) satisfies:

\[
\int k_H(\xi, \eta) \, d\xi \, d\eta = \prod_{j=1}^{n} \frac{2\pi a_j}{\sinh(a_j)} = \left( \det\frac{-2\pi i A}{\sinh(-i A)} \right)^{\frac{1}{2}}.
\]

If we apply this to equation 3.58 and 3.69 we have:

\[
\text{Ind}(D) = \lim_{t \to 0} \text{Tr}_* \left( e^{\frac{1}{2}t^2B^2} \right) = (2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{3}{2}} \int_M e^{F/4} \left( \det\frac{-2\pi i R/4}{\sinh(-i R/4)} \right)^{\frac{1}{2}}.
\]

Given a matrix, \( B \), recall that, ([G] pp. 99):

\[
\tilde{A}(B) = \left( \det\frac{B/4\pi}{\sinh(B/2\pi)} \right)^{\frac{1}{2}}.
\]

Therefore, we have:

\[
(2\pi)^{\frac{3}{2}} \tilde{A}(-2\pi i (R/4)) = (2\pi)^{\frac{3}{2}} \left( \frac{2}{i} \right)^{\frac{3}{2}} \left( \det\frac{-i (R/4)/2}{\sinh(-i (R/4))} \right)^{\frac{1}{2}}.
\]

Next, recall that the Chern character, \( Ch(B) \), is equal to \( tr(e^{-B/2\pi}) \), so we have \( tr(e^{F/4}) = Ch(-2\pi i (F/4)) \). Thus:

\[
\text{Ind}(D) = (2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{3}{2}} (2\pi)^{\frac{3}{2}} \left( \frac{2}{i} \right)^{\frac{3}{2}} \int_M \text{Ch}(-2\pi i (F/4)) \wedge \tilde{A}(-2\pi i (R/4)).
\]

Recalling that the top degree piece of the integrand is homogeneous of degree \( \frac{n}{2} \) we have:

\[
\text{Ind}(D) = (2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{3}{2}} \left( 2\pi \right)^{\frac{3}{2}} \left( \frac{-\pi i}{2} \right)^{\frac{n}{2}} \int_M \text{Ch}(F) \wedge \tilde{A}(R) = (-1)^{\frac{n}{2}} \int_M \text{Ch}(F) \wedge \tilde{A}(R).
\]


In this section we will apply the asymptotic calculus to the calculation of the cyclic cocycle corresponding to the Dirac operator on a compact spin manifold \( M \). See [C3] and [JLO] for details on theta-summable fredholm modules. Let \( H = H^+ \oplus H^- \) denote the Hilbert space of square-integrable spinors over \( M \), and let \( \epsilon \) be the corresponding grading operator, so \( \epsilon|_{H^+} = 1 \) and \( \epsilon|_{H^-} = -1 \). Let \( D \) be the Dirac operator on the space of spinors; thus:

\[
D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.
\]
Here $D^+: H^+ \rightarrow H^-$ and $D^-: H^- \rightarrow H^+$. Let $\mathfrak{A} = C^1(M)$; then $\mathfrak{A}$ is a Banach algebra (it is the completion of $C^\infty(M)$ with respect to the norm $\|f\| = \|f\|_\infty + \|df\|_\infty$). Let $\mathfrak{A} = \mathfrak{A}/\mathbb{C}$ and define $C_n(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{A}^{\otimes n}$ where $\otimes_n$ denotes the projective tensor product. The bar complex is $\sum_{n=0}^{\infty} C_n(\mathfrak{A})$ with a suitable topology. The cobar complex is the topological dual and the Chern character of a theta-summable Fredholm module is a cochain in the cobar complex, closed with respect to the coboundary operator $\delta = b + B$. Thus, for appropriate $B_i \in C_i(\mathfrak{A})^*$:

\[
Ch((D, H)) = \sum_{i=0}^{\infty} B_i.
\]

The Chern character of the module $(H, D)$ as an entire cyclic cocycle is given by:

\[
(Ch((D, H)), (f_0, \ldots, f_{2k})) = B_{2k}(f_0, \ldots, f_{2k}) = 
\int_{\Delta_n} Tr_s \left( f_0 e^{-t_1 D^2} [D, f_1] e^{-t_2 (1-t_1) D^2} \cdots [D, f_{2k}] e^{-(1-t_{2k}) D^2} \right) dt_1 \cdots dt_{2k}.
\]

Let $D_\alpha = \alpha D$. As noted before, $D_\alpha$ is not an asymptotic operator; however, $D_\alpha^2 = \alpha^2 D^2$ and $[D_\alpha, f] = \alpha C(df)$ are asymptotic operators. (Recall that $C(df)$ is Clifford multiplication by $df$.)

**Theorem 4.1.**

1. $(Ch((D_\alpha, H)), (f_0, \ldots, f_{2k}))$ is convergent for $\alpha \rightarrow 0$ and for $f_i \in \mathfrak{A}$.
2. The limit $\tau_k(f_0, \ldots, f_{2k}) = \lim_{\alpha \rightarrow 0} (Ch((D_\alpha, H)), (f_0, \ldots, f_{2k})$ gives a cocycle (considering all of the $\tau_k$’s together) in the cobar complex, and:

\[
\tau_k(f_0, \ldots, f_{2k}) = \text{vol}(\Delta_{2k})(2\pi)^{-n} \left( \frac{1}{2} \right)^{n} \int_M \hat{A}(-2\pi R/4)f_0 df_1 \wedge \cdots \wedge df_{2k}.
\]

**Proof:** That the collection of $\tau_k$’s form a cyclic cocycle follows from the existence of the limit and the corresponding fact for $Ch((D, H))$, which can be found in [GS]. Consider the operator:

\[
P_\alpha = f_0 e^{-t_1 D^2} [D_\alpha, f_1] e^{-(1-t_1)D^2} \cdots [D_\alpha, f_{2k}] e^{-(1-t_{2k}) D^2}.
\]

By the above remarks, each of the individual terms is an asymptotic operator; hence, so is their product. To compute $\lim_{\alpha \rightarrow 0} Tr_s(P_\alpha)$, it suffices to compute the leading symbol. Recall that the leading symbols of $e^{\frac{1}{2} L^2}$ and $[D_\alpha, f]$ are:

\[
(a) \quad e^{-t(|f|^2 - \frac{1}{8\pi} RA(R(\partial_c, \partial_c))} (1)
\]

\[
(b) \quad df.
\]
By the formula for the composition of symbols, 4.6a and 4.6b commute, since $df$ contains no differentiation and 4.6a is a form of even order. Thus we have:

$$(4.7) \quad \lim_{\alpha \to 0} Tr_{\alpha}(P_{\alpha}) = \lim_{\alpha \to 0} (2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{n}{2}} \int (\sigma(P_{\alpha}))_{\alpha} d\xi =$$

$$(2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{n}{2}} \int \sigma(\partial_{0} e^{-t_{1}D_{\alpha}^{2}} [D_{\alpha}, f_{1}] e^{-t_{2}D_{\alpha}^{2}} \cdots [D_{\alpha}, f_{2k}] e^{-(1-t_{2k})D_{\alpha}^{2}})_{\alpha} =$$

$$(2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{n}{2}} \int \partial_{0} df_{1} \wedge \cdots \wedge df_{2k} \wedge e^{-t_{1}-(t_{2}-(t_{2}-(t_{2k}-(1-t_{2k}))))(1)} =$$

$$(2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{n}{2}} \int_{M} d\wedge df_{2k} e_{p}(\cdot)(1) =$$

$$(2\pi)^{-n} \left( \frac{2}{i} \right)^{\frac{n}{2}} \int_{M} \hat{A} (2\pi R/4) f_{0} df_{1} \wedge \cdots \wedge df_{2k}.$$

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