A Riemann Hilbert correspondence for infinity local systems

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Abstract

We descibe a dg-equivalence of dg-categories between Block's $\mathcal{P}_{\mathsf{A}}$, corresponding to the de Rham dga A of a compact manifold M and the dg-category of ∞-local systems on M. We understand this as a generalization of the Riemann-Hilbert correspondence to $\mathbb{Z}$-graded connections (superconnections in some formulations). An ∞-local system is an $(\infty, 1)$ functor between the $(\infty, 1)$-categories $\pi_\infty M$ and the linear simplicial nerve of the dg-category of cochain complexes. This theory makes crucial use of Igusa's notion of higher holonomy transport for $\mathbb{Z}$-graded connections which is a derivative of Chen's main idea of generalized holonomy. In the appendix we describe the linear simplicial nerve construction.

1 Introduction and Summary

Given a compact manifold M, the classical Riemann-Hilbert correspondence gives an equivalence of categories between $Rep(\pi_1(M))$ and the category $Loc(M)$ of local systems on M. While beautiful, this correspondence has the primary drawback that it concerns the truncated object π_1, which in most cases contains only a small part of the data which comprises the homotopy type of M. From the perspective of (smooth) homtopy theory the manifold M can be replaced by its infinity-groupoid $Sing_\bullet^\infty M := \pi_\infty M$ of smooth simplices. Considering the correct notion of a representation of this object will allow us to produce an untruncated Riemann-Hilbert theory. More specifically, we define an infinity local system to be a map of simplicial sets which to each simplex of $\pi_\infty M$ assigns a homotopy coherence in the category of chain complexes over $\mathbb{R}$. Our main theorem is a stable, tensor, dg-quasi-equivalence

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}(\mathbb{R})}(\pi_\infty M) \tag{1}$$

Where, $\mathcal{P}_{\mathsf{A}}$ is the dg-category of graded bundles on M with flat $\mathbb{Z}$-graded connection and $\mathsf{Loc}^{\mathcal{C}(\mathbb{R})}(\pi_\infty M)$ is the dg-category of infinity local systems on M.

The classical Riemann-Hilbert equivalence

$$Loc(M) \to Rep(\pi_1(M)) \tag{2}$$

is proven by calculating the holonomy of a flat conection. The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness. The other direction

$$Rep(\pi_1(M)) \to Loc(M) \tag{3}$$

is acheived by the obvious associated bundle construction.

In the first case our correspondence proceeds analogously by a calculation of the holonomy of a flat $\mathbb{Z}$-graded connection. The technology of iterated integrals suggests a precise and rather natural notion of such holonomy. Given a vector bundle V over M with connection, the usual parallel transport can be understood as a form of degree 0 on the path space PM taking values in the bundle $\mathrm{Hom}(ev_1^*V, ev_0^*V)$. The higher holonomy is then a string of forms of total degree 0 on the path space of M taking values in the same bundle. Such a form can be integrated over cycles in PM, and the flatness of the connection implies that such a pairing induces a representation of π_∞ as desired. This is the functor

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}(\mathbb{R})}(\pi_\infty M). \tag{4}$$

It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction. However we chose instead to prove quasi-essential surjectivity of the above functor. Given an ∞-local system (F, f) one can form a complex of sheaves over X by considering $\mathsf{Loc}^{\mathcal{C}(\mathbb{R})}(\pi_\infty U)(\mathbb{R}, F)$. This complex is quasi-isomorphic to the sheaf obtained by extending by the sheaf of C^∞ functions and then tensoring with the de Rham sheaf. Making use of a theorem of Illusie we construct from this data a perfect complex of $\mathcal{A}^0$-modules quasi-isomorphic to the zero-component of the connection in $\mathcal{RH}(F)$. Finally we follow an argument of [B1] to complete this to an element of $\mathcal{P}_{\mathsf{A}}$ which is quasi isomorphic to $\mathcal{RH}(F)$.

We reserve the appendix to work out some of the more conceptual aspects of the theory as it intersects with our understanding of homotopical/derived algebraic geometry (in the parlance of Lurie, Toen-Vezzosi, et. al.). One straightforward extension of this theory is to take representations in any linear ∞-category. In fact, considering representations of $\pi_\infty M$ in the category of A_∞-categories leads to a fruitful generalization of recent work of Emma Smith-Zbarsky [SZ] who has considered the action of a group G on families of A_∞ algebras over a $K(G, 1)$.

2 Infinity Local Systems

First we develop an $(\infty, 1)$ version of a local system. These objects will be almost the same as the A_∞-functors of [Ig02], but tailored to suit our equivalence result. We want to emphasize the analogy with classical local systems. Throughout this paper ∞-category will mean $(\infty, 1)$-category. There is a theory of stability for such ∞ categories which subsumes the theory of pre-triangulated dg-categories, however we may use the terms "pre-triangulated" interchangeably with "stable" when discussing dg-categories. Let $\mathcal{C}$ be a pre-triangulated dg-category over k a characteristic 0 field (which we are implicitly regarding at $\mathbb{R}$ in this paper), and K a simplicial set which is a ∞-category (a weak Kan complex, see [Lu2]). Fix a map $F : K_0 \to Ob\,\mathcal{C}$. Then define:

$$\mathcal{C}_F^k(K) := \bigoplus_{i+j=k, i\geq 0} \mathcal{C}_F^{i,j}, \tag{5}$$

with,

$$\mathcal{C}_F^{i,j} := \{\text{maps } f : K_i \to \mathcal{C}^j |\ f(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), F(\sigma_{(0)}))\} \tag{6}$$

There are some obvious gradings to keep track of. For $f \in \mathcal{C}_F^{p,q}$ define

$$T(f) := (-1)^{p+q} f =: (-1)^{|f|} f, \quad K(f) := (-1)^q f, \quad J(f) := (-1)^p f \tag{7}$$

With respect to the simplicial degree in $\mathcal{C}_F^k$, we write

$$f = f^1 + f^2 + \dots \tag{8}$$

$f^i \in \mathcal{C}_F^{i,\bullet}$.

We define some operations on these maps:

$$(df^i)(\sigma_i) := d(f^i(\sigma_i)) \tag{9}$$

$$(\delta f^i)(\sigma_{i+1}) := \sum_{l=1}^{i} (-1)^l f^i(\partial_l(\sigma)) \tag{10}$$

$$\hat{\delta} = \delta \circ T(\bullet) \tag{11}$$

and for $g^p \in \mathcal{C}_F^{p,q}$,

$$(f^i \cup g^p)(\sigma \in K_{i+p}) := (-1)^{i(p+q)} f^i(\sigma_{(0\ldots i)}) g^p(\sigma_{(i\ldots p+i)}). \tag{12}$$

Extend by linearity to sums in $\oplus \mathcal{C}^s(K)$ so that the cup product is defined as the sum of the cups across all internal pairs of faces:

$$(f \cup g)(\sigma_k) := \sum_{t=1}^{k-1} (-1)^{t|g^{k-t}|} f^t(\sigma_{(0\ldots t)}) g^{k-t}(\sigma_{(t\ldots k)}) \tag{13}$$

More transparently we could write $f \cup g := \mu \circ (f \otimes g) \circ \Delta$ were Δ is the usual comultiplication which splits a simplex into a sum over all possible splittings into two faces:

$$\Delta(\sigma_k) = \sum_{p+q=k, p,q \geq 1} \sigma_p \otimes \sigma_q \tag{14}$$

and μ is the composition $\mathcal{C}^\bullet \otimes \mathcal{C}^\bullet \to \mathcal{C}^\bullet$. The sign above appears because an i-simplex passes an element of total degree $p+q$.

Definition 2.1. *A pair (F, f) with $f \in \mathcal{C}_F^1(K)$ wuch that $0 = \hat{\delta} f + df + f \cup f$ is called an ∞-local system. The set of ∞-local systems valued in $\mathcal{C}$ is denoted $\mathsf{Loc}_\infty^{k,\mathcal{C}}(K)$*

We will often denote an ∞-local system (F, f) by just F if no confusion will arise.

Example 2.2. *If F denotes an ordinary local system, then it naturally defines an ∞-local system.*

Proof. Exercise. $\square$

Often it will be the case that the differential in $\mathcal{C}$ will be given by commutation with some family of degree-1 elements $d_x \in \mathcal{C}^1(x, x)$ i.e.,

$$df(\sigma_k) = d_{F(\sigma_{(0)})} \circ f_k(\sigma_{(0\ldots k)}) - (-1)^{|f_k|} f_k(\sigma_{(0\ldots k)}) \circ d_{F(\sigma_{(k)})} \tag{15}$$

A more conceptual description is the following

Definition 2.3. *An ∞-local system on K valued in $\mathcal{C}$ is an element of the functor category $s\mathcal{S}et(K, \mathcal{C}_\infty)$.*

Here $\mathcal{C}_\infty$ is the linear simplicial nerve construction applied to the dg-category $\mathcal{C}$. We work out this prespective in the appendix, but prefer in our paper to give more explicit constructions of these objects.

We define a morphism between two ∞-local systems F,G:

$$\mathsf{Loc}_\infty^{k,\mathcal{C}}(K)(F, G) := \bigoplus_{i+j=k} \{\phi : K_i \to \mathcal{C}^j | \phi(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), G(\sigma_{(0)}))\} \tag{16}$$

Proposition 2.4. $\mathsf{Loc}_\infty^{\mathcal{C}}(K)$ *is a dg-category.*

Proof. We define the differential D:

$$D\phi = \hat{\delta}\phi + d\phi + G \cup \phi - (-1)^{|\phi|}\phi \cup F. \tag{17}$$

In the above, $\phi = \phi^0 + \phi^1 + \dots$ is of total degree $|\phi| = p$, and

$$(\hat{\delta}\phi)(\sigma_k) := \delta \circ T = \sum_{j=1}^{k-1} (-1)^{j+|\phi|}\phi^{k-1}(\partial_j(\sigma_k)) \tag{18}$$

$$d\phi(\sigma_k) := d_{G(\sigma_{(0)})} \circ \phi(\sigma_{(0\dots k)}) - \phi(\sigma_{(0\dots k)}) \circ d_{F(\sigma_{(k)})} \tag{19}$$

$D^2 = 0$ follows from the following observations:

$$\hat{\delta}[F, G] = [\hat{\delta}F, G] + (-1)^{|F|}[F, \hat{\delta}G] \tag{20}$$

where $[,]$ is the graded commutator $[A, B] = A \cup B + (-1)^{|A||B|}B \cup A$,

$$[F, [G, H]] = [[F, G], H] + (-1)^{|F||G|}[G, [F, H]], \tag{21}$$

and the fact F and G are local systems:

$$\hat{\delta}F + dF + F \cup F = \hat{\delta}G + dG + G \cup G = 0 \tag{22}$$

□

We can define a shift functor in $\mathsf{Loc}_\infty^{\mathcal{C}}(K)$. Given $F \in \mathsf{Loc}_\infty^{\mathcal{C}}(K)$, define $F[q]$ via, $F[q](x \in K_0) := F(x)[q]$ and $F[q](\sigma_k) := (-1)^q F(\sigma_k)$. For a morphism ϕ the shift is inconsequential: $\phi[q] = \phi$.

And a cone construction: Given a morphism $\phi \in \mathsf{Loc}_\infty^{\mathcal{C}}(K)(F, G)$ of total degree q, define the map $C(\phi) : K_0 \to Ob\mathcal{C}$ by the assignment $x \mapsto F[1-q](x) \oplus G(x)$. And define the element $c(\phi)$ of $\mathcal{C}^1_{C(\phi)}$ via,

$$c(\phi) = \begin{pmatrix} f[1-q] & 0 \\ \phi[1-q] & g \end{pmatrix} \tag{23}$$

In general this will not be an element of $\mathsf{Loc}_\infty^{\mathcal{C}}(K)$, but this useful construction will appear in our calculations later.

Evidently the Maurer-Cartan equation is preserved under this shift, and the resulting category is a pre-triangulated dg-category[Bondal-Kapranov].

A degree 0 closed morphism ϕ between two ∞-local systems F, G over K is a *homotopy equivalence* if it induces an isomorphism in $\mathrm{Ho}\,\mathsf{Loc}_\infty^{\mathcal{C}}(K)$. We want to give a simple criterion for ϕ to define such a homotopy equivalence. On the complex $\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F, G)$ define a decreasing filtration by

$$F^k\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F, G) = \{\phi \in \mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F, G))| \ \phi^i = 0 \text{ for } i < k\}$$

Proposition 2.5. *There is a spectral sequence*

$$E_0^{pq} \Rightarrow H^{p+q}(\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F, G) \tag{24}$$

where

$$E_0^{pq} = gr\ (\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F, G)) = \{\phi : K_p \to \mathcal{C}^q | \phi(\sigma) \in \mathcal{C}^q(F(\sigma_{(i)}), G(\sigma_{(0)}))\}$$

with differential

$$d_0(\phi^p) = d_G \circ \phi^p - (-1)^{p+q}\phi^p \circ d_F$$

Corollary 2.6. *For two ∞-local systems F and G, the E_1-term of the spectral sequence is a local system in the ordinary sense.*

Proposition 2.7. *A closed morphism $\phi \in \mathsf{Loc}_\infty^{0,\mathcal{C}}(K)(F, G)$ is a homotopy equivalence if and only if $\phi^0 : (F_x, d_F) \to (G_x, d_G)$ is a quasi-isomorphism of complexes for all $x \in K_0$.*

Proof. The proof follows as in the proof of Proposition 2.5.2 in [B1]. □

2.1 A_∞ Local Systems

We also point out that a local system can take values in an A_∞-category. We are not particularly interested in developing this theory here but point out the corresponding definitions for the sake of presentation and interest. We will use almost entirely the same notation. Let $\mathcal{C}$ be an A_∞-category, with multiplications denoted μ_i, and K a simplicial set which is an ∞-category.

As before, an object F consists of a choice of a map $F : K_0 \to Ob\mathcal{C}$ along with an element f of total degree 1 from the set

$$f \in \mathcal{C}^1_F(K) := \underset{i+j=1, i\geq 0}{\oplus} \mathcal{C}^{i,j}_F, \tag{25}$$

with,

$$\mathcal{C}^{i,j}_F := \{\text{k-linear maps } f : K_i \to \mathcal{C}^j | F(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), F(\sigma_{(0)}))\} \tag{26}$$

and which satisfies a generalized Maurer-Cartan equation.

Morphisms are also as before:

$$\mathsf{Loc}_{A_\infty}(K)^q(F, G) = \{\text{k-linear maps } K_i \to \mathcal{C}^j(F(i), G(0)\} \tag{27}$$

We define a series of multiplications on composible tuples of morphisms. Consider an $n+1$-tuple of objects $(F_n, \dots, F_0)$ and a corresponding tuple of composible morphisms $(\phi_n \otimes \dots \otimes \phi_0)$.

$$m_n : \otimes_{n \geq i \geq 0} \mathsf{Loc}_{A_\infty}(K)^\bullet(F_{i+1}, F_i) \to \mathsf{Loc}_{A_\infty}(K)(F_0, F_n)^\bullet[2-n] \tag{28}$$

For $n = 1$,

$$m_1 : \phi \mapsto \mu_1 \circ (\phi) - (-1)^{|\phi|} \phi \circ \mu_1 + (-1)^{|\phi|} (\sum_l (-1)^l \phi \circ \partial_l) \tag{29}$$

and for $n \geq 1$,

$$m_n : (\phi_n \otimes \dots \otimes \phi_0) \mapsto \mu_n \circ (\phi_n \otimes \dots \otimes \phi_0) \circ \Delta^{(n)} \tag{30}$$

Definition 2.8. *A pair (F, f) with $f \in \mathcal{C}^1_F(K)$ such that $0 = \sum_{i=1}^\infty m_i(f^{\otimes i})$ is called an A_∞-local system. The set of A_∞-local systems is denoted $\mathsf{Loc}^{\mathcal{C}}_{A_\infty}(K)$.*

It is important to note that the Maurer-Cartan equation above is not finite, but has a finite number of terms when evaluated on any simplex due to the fact that $\Delta^n(\sigma) = 0$ for $n >> 0$.

If the multiplications defined above satisfy the constraint that for any tuple of composible morphisms, $(\phi_0, \dots, \phi_N)$, the graded vector space $\oplus_{i,j} \mathsf{Loc}_{A_\infty}(K)^\bullet(F_i, F_j)$ equipped with the direct sums of the above multiplications becomes an A_∞-algebra, then $\mathsf{Loc}_{A_\infty}(K)$ is an A_∞-category. In particular the one-point A_∞-category is just an A_∞-algebra.[K-S]

Proposition 2.9. $\mathsf{Loc}^{\mathcal{C}}_{A_\infty}(K)$ *is an A_∞-category.*

In a recent pre-print[SZ] Emma Smith-Zbarsky develops a closely related theory. She considers a $\mathbb{Z}$-graded bundle V over a smooth manifold M and then passes to the bundle which is fiberwise

$$\mathfrak{g}^\bullet_x = \prod_{n=1}^\infty \text{Hom}^\bullet(V_x^{\otimes n}, V_x[1-n]) \tag{31}$$

If one chooses a usual local system ∇, a total degree 1 form α valued in this bundle which satisfies the relevant Maurer-Cartan equation gives the graded bundle V extra structure. In particular the 0-form part of α turns V into a bundle of A_∞ algebras. In her paper Smith-Zbarsky calculates the first couple terms of the holonomy of such an M-C element to show that the MC equation implies that the fiber of V over each point is an A_∞ algebra and that parallel transport is an A_∞ morphism. For a smooth, pointed manifold (M, x) which is a $K(G, 1)$, loops at x act on the fiber of V over x by A_∞ morphisms and a homotopy of paths yields an A_∞

homotopy between the corresponding parallel transports morphisms. Thus Zbarsky-Smith calls this type of action of $\pi_1(M,x)$ a homotopy group action of A_∞ algebras. In forthcoming work Zbarsky-Smith will further develop this A_∞ point of view with applications to Lagrangian Floer theory.

This setup can be fruitfully understood in terms of our formalism. The bundle $\mathfrak{g}$ is the bundle whose fiber is

$$\mathrm{Hom}_{\mathrm{CoAlg}}(B(V_x), B(V_x)) \tag{32}$$

where $B(\bullet)$ denotes the bar construction. Given a usual local system ∇, a $\mathfrak{g}$-valued form α satisfying

$$d_\nabla \alpha + \alpha \circ \alpha = 0 \tag{33}$$

yields a flat $\mathbb{Z}$-graded connection via

$$d_\nabla + [\alpha, \ \ ] : \mathfrak{g}^\bullet \otimes_{\mathcal{A}^0} A^\bullet \to \mathfrak{g}^\bullet \otimes_{\mathcal{A}^0} A^\bullet \tag{34}$$

whose holonomy yields an ∞-local system F satisfying

$$dF + \hat{\delta} F + F \cup F = 0. \tag{35}$$

(A useful guide is [Me]). This holonomy breaks further according to the components in $\mathfrak{g}$ to give the sum-over-trees formulas that one usually sees in such A_∞ applicaitions. In fact, this construction yields a local system valued in the category of A_∞ algebras, and can be worked out in this case over any manifold. However, one should be careful to note that the starting bundle in this case is not of finite rank, so developing this theory technically involves an extension of our results which we expect to go through without serious hitches.

3 Iterated Integrals and Holonomy of $\mathbb{Z}$-graded Connections

Now let $\mathsf{A} = (\mathcal{A}^\bullet(M), d)$ be the deRham DGA of a compact, C^∞-manifold M. We define $\pi_\infty M$ to be $\mathrm{Sing}^\infty_\bullet M$, the simplicial set over $k = \mathbb{R}$ of C^∞-simplicies. By $\mathcal{C}$ we denote the dg-category of complexes over $\mathbb{R}$. Our goal is to derive an equivalence of categories between $\mathcal{P}_{\mathsf{A}}$ and $\mathsf{Loc}^{\mathcal{C}}_\infty(\pi_\infty M)$, where the latter should be thought of as representations of $\pi_\infty M$. Specifically, to any dg-category $\mathcal{C}$ over k we can associate to it the so-called k-linear simplicial nerve $\mathcal{C}_\infty$ which is a simplicial set satisfying the adjunction

$$s\mathcal{S}et(\Delta^n, \mathcal{C}_\infty) = dg\mathcal{C}at_k(I \circ DK \circ k \circ \mathfrak{C}[\Delta^n], \mathcal{C}) \tag{36}$$

See the appendix for the relevant definitions. In the case we are considering the target category for representations is $\mathcal{C}_\infty$ where $\mathcal{C}$ is the category of cochain complexes over $\mathbb{R}$. Sussinctly an ∞-local system is a functor in $s\mathcal{S}et(\pi_\infty M, \mathcal{C}_\infty)$ or likewise $dg\mathcal{C}at_k(I \circ DK \circ k \circ \mathfrak{C}[\pi_\infty M], \mathcal{C})$. One could conceivably then descibe the Riemann-Hilbert correspondence as the representability of the functor which sends a space to its category of "homotopy-locally constant" sheaves of vectorspaces. In fact Toen and Vezzosi present such a notion in [TV] and [Toen]. Their primary concern is a Segal Tannakian theory, and they show that for a CW complex X, $\pi_\infty X$ can be recovered from the category of simplicial local systems. Our Reimann-Hilbert correspondence concerns smooth manifolds (and uses iterated integrals) and so we have not developed a theorem which applies to CW complexes.

An object of $\mathcal{P}_{\mathsf{A}}$ is a pair $(E^\bullet, \mathbb{E})$ where $E^\bullet$ is a $\mathbb{Z}$-graded (bounded), finitely-generated, projective, right $\mathcal{A}^0$-module and $\mathbb{E}$ is a $\mathbb{Z}$-connection with the flatness condition $\mathbb{E} \circ \mathbb{E} = 0$. Such a module corresponds to the smooth sections of a $\mathbb{Z}$-graded vector bundle $V^\bullet$ over M with the given $\mathbb{Z}$-connection.

In [I] Kiyoshi Igusa presents from scratch a notion of higher parallel transport for a $\mathbb{Z}$-connection. This is a tweaked example of Chen's higher transport outlined in [Ch] which makes crucial use of his theory of iterated integrals. We slightly reformulate and extend this idea to produce a functor from $\mathcal{P}_{\mathsf{A}}$ to $\mathsf{Loc}^{\mathcal{C}}_{\infty}(K)$ which is a dg-quasi-equivalence. To start we present a version of iterated integrals valued in a graded endomorphism bundle.

3.1 Sign Conventions

Let V be a graded vector bundle on M, then End(V) is a graded algebra bundle on M. The symbols T,J,K will be used to denote an alternating sign with respect to the total degree, form degree, and bundle-grading degree of a form valued in a graded bundle. For instance if $\omega \in V^q \otimes_{\mathcal{A}^0} \mathcal{A}^p$, then $T\omega = (-1)^{p+q}\omega, K\omega = (-1)^q\omega$, and $J\omega = (-1)^p\omega$. The similar convention carries over for forms valued in the $End(V)$ which has an obvious grading.

Given a form $A = (f \otimes \eta) \in End^k(V) \otimes_{\mathcal{A}^0} \mathcal{A}^p$ it is understood as a homomorphism

$$V^\bullet \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \to V^\bullet \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \tag{37}$$

via

$$(f \otimes \eta)(v \otimes \alpha) \mapsto (-1)^{|v|}(f(v) \otimes \eta \wedge \alpha) \tag{38}$$

Hence composition in $End^\bullet V \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet$ is calculated,

$$(f \otimes \eta) \circ (g \otimes \rho) := (f\Xi g \otimes \eta \wedge \rho) \tag{39}$$

Here Ξ is the alternating identity section $(-1)^{k+1}Id^k_x : V_x \to V_x$ for any $x \in M$. Note that with respect to these conventions $I \otimes 1$ acts as Ξ.

We will use $\circ$ to denote the above composition of homomorphisms and $\wedge$ used to denote the coordinatewise product. Left-contraction doesn't see the bundle degree:

$$\iota : \Gamma(M, TM^{\otimes k}) \to Hom(End^\bullet V \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet, End^\bullet V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet - k}) \tag{40}$$

via

$$\xi = \xi_1 \otimes \ldots \otimes \xi_k \mapsto (f \otimes \alpha \mapsto f \otimes \iota_\xi \alpha) \tag{41}$$

With local coordinates in a trivializing patch, we have a differential d which acts on V-valued forms via:

$$d(v \otimes \alpha) = (dv) \cdot \alpha + ((-1)^{|v|} v \otimes d\alpha) \tag{42}$$

This implies that on endomorphism-valued forms (which are matrix-valued forms locally) we have

$$d(\phi \otimes \eta) = d \circ A - TA \circ d = d\phi \cdot \eta + (-1)^{|\phi|}\phi\Xi \otimes d\eta \tag{43}$$

and it follows

$$d(A \circ B) = dA \circ B + TA \circ dB \tag{44}$$

Note that this convention for the local description of a superconnection agrees with the formula for the shift in $\mathcal{P}_{\mathsf{A}}$, namely, $\mathbb{E}[q] = (-1)^q\mathbb{E}$.

3.2 Path Space Calculus

In [Ch] and earlier works, Chen defined a notion of a differentiable space –the archetypical differentiable space being PM for some smooth manifold M. This is a space whose topological structure is defined in terms of an atlas of plots –maps of convex neighborhoods of 0 in $\mathbb{R}^n$ into the space which cohere with composition by smooth maps– and the relevant analytic and topological constructs are defined in terms of how they pull back onto the plots. In particular one can construct a reasonable definition of vector bundles over a differentiable space as well as differential forms. One can likewise define an exterior differential, and subsequently a so-called Chen deRham complex [H]. We will try to make transparent use of these constructions, but we defer the reader to the existing discussions of these matters in [Ch],[H],[I],[B-H].

The primary reason that path-space calculus is relevant to our discussion is that the holonomy of a $\mathbb{Z}$-graded connection on V can be defined as a sequence of smooth forms on PM with values in the bundle $Hom(p_1^*V, p_0^*V)$. The usual parallel transport will be the 0-form part of the holonomy. The higher terms will constitute the so-called higher holonomy.

3.3 Iterated Integrals

Now define

$$\eta : PM \mapsto PM : \gamma \mapsto c_{\gamma(0)} \tag{45}$$

$$F : PM \times I \to PM : F_\tau(\gamma)(t) = \gamma(\tau t) \tag{46}$$

where c_x is the constant path at x, and F as the "spaghetti" homotopy between η and the identity map. Let V be a graded bundle on PM, and in a trivializing patch we identify End(V) as a graded matrix bundle $Mat^\bullet(V)$. And denote the Chen DGA of PM by $P\mathcal{A}$ and more generally on any Chen space X by $\mathcal{A}(X)$. Then F induces a so-called Poincare operator:

$$\int_F : Mat^\bullet(V) \otimes_{P\mathcal{A}^0} P\mathcal{A} \mapsto Mat^\bullet(V) \otimes_{P\mathcal{A}^0} P\mathcal{A} \tag{47}$$

given by $\omega \mapsto \int_0^1 (\iota_{\frac{\partial}{\partial t}} F^*\omega)\Xi dt$. In this notation, iterated integrals are defined recursively:

$$\int \omega_1 \ldots \omega_r := \int_F T(\int \omega_1 \ldots \omega_{r-1}) \circ p_1^* \omega_r \Xi \tag{48}$$

Here p_i is the projection from $PM \to M$ given by evaluation at i.

We can write this process differently and perhaps more transparently:

Let us parametrize the k-simplex by k-tuples $t = (1 \geq t_1 \geq t_2 \geq \ldots \geq t_k \geq 0)$. Then we define the obvious evaluation and projection maps:

$$ev_k : PM \times \Delta^k \to M^k : (\gamma, (t_1, \ldots, t_k)) \mapsto (\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_k)) \tag{49}$$

$$\pi : PM \times \Delta^k \to PM \tag{50}$$

Let $E = Mat^\bullet(V)$ be the trivial bundle of matrices as above, we can embed $i : (E \otimes_{\mathcal{A}^0} \mathcal{A})^{\otimes k} \to E^{\boxtimes k} \otimes_{\mathcal{A}^0} \mathcal{A}$. Given the space of forms $ev_k^* E^{\boxtimes k} \otimes_{\mathcal{A}(PM \times \Delta^k)^0} \mathcal{A}(PM \times \Delta^k,)$ we can use the multiplication in E to define

$$\mu : ev_k^* E^{\boxtimes k} \otimes_{\mathcal{A}(PM \times \Delta^k,)^0} \mathcal{A}(PM \times \Delta^k) \to p_0^* E \otimes_{\mathcal{A}(PM \times \Delta^k,)^0} \mathcal{A}(PM \times \Delta^k) \tag{51}$$

by multiplication in the fiber. The iterated integral map is the composition:

$$(-1)^{\spadesuit} \pi_*(\mu(ev_k^*(i(a_1 \otimes \ldots \otimes a_k))))\Xi \tag{52}$$

Since E is graded, the elements $\{a_i\}$ are bi-graded as usual. Calculate

$$\spadesuit = \sum_{1 \leq i < k} (T(a_i) - 1)(k - i). \tag{53}$$

3.4 $\mathbb{Z}$-graded Connection Holonomy

Suppose V has a $\mathbb{Z}$-connection $\mathbb{E}$. Locally $\mathbb{E}$ is of the form $d - [A^0 + A^1 + \ldots + A^m]$. (With the above conventions, $(-1)^k d$ is locally the trivial connection on E^k) Let $\omega = A^0 + A^1 + \ldots + A^m$. This is a form of total degree 1, i.e. in $\oplus End^{1-i}(V) \otimes_{\mathcal{A}^0} \mathcal{A}^i$. To any such form we can associate its holonomy/parallel transport

$$\Psi := I + \int \omega + \int \omega\omega + \int \omega\omega\omega + \ldots \tag{54}$$

which breaks further into its components with respect to the "form-grading". For instance,

$$\Psi_k = \int A^{k+1} + \sum_{i+j=k+2} \int A^i A^j + \sum_{i_1+i_2+i_3=k+3} \int A^{i_1} A^{i_2} A^{i_3} + \ldots \tag{55}$$

Chen calculated the differential of a holonomy form (whithout the graded changes we have worked into our definition)

$$\begin{aligned} d\Psi = -\int \kappa + (-\int \kappa\omega + \int J\omega\kappa) + \ldots \\ + \sum_{i+j=r-1} (-1)^i \int (J\omega)^i \kappa\omega^j + \ldots + -p_0^*\omega \wedge \Psi + J\Psi \wedge p_1^*\omega. \end{aligned} \tag{56}$$

where $\kappa = d\omega - J\omega \wedge \omega$ defines the curvature of ω, and the integral itself is defined with different sign conventions.

A modified form of the above calculation can be proven with two basic lemmas which are slight modifications of Chen's.

Proposition 3.1. *Let $w = h \otimes \alpha$ be a form in $Mat^k \otimes_{P\mathcal{A}^0} P\mathcal{A}^p$. Then*

$$d_{PM} \int_F w + \int_F d_{PM} w = F^*_{t=1} w\Xi - F^*_{t=0} w\Xi \tag{57}$$

Proof. Let $F^*w = v = f \otimes dt \wedge v' + g \otimes v''$, broken into its components with respect to t-dependence in the form part. Then,

$$\begin{aligned} \int_F d_{PM} w = \int_0^1 \iota_{\frac{\partial}{\partial t}} (d_{PM\times I} f \cdot dt \wedge v' - (-1)^{|f|} f\Xi \otimes dt \wedge dv' + \\ + (-1)^{|g|} g\Xi \otimes dv'' + d_{PM\times I} g \cdot v'')dt = \\ = \int_0^1 (-d(f \otimes v') + (-1)^{|g|} g\Xi \otimes \frac{\partial v''}{\partial t} + \frac{\partial g}{\partial t} \otimes v'')dt = \\ = \int_0^1 (-d_{PM}(f \otimes v') + \frac{\partial (g\Xi \otimes v'')}{\partial t})dt \end{aligned} \tag{58}$$

And,

$$d_{PM} \int_F w = d_{PM} \int_0^1 (f \otimes v')dt = \int_0^1 d_{PM}(f \otimes v')dt \tag{59}$$

So, summing the terms yields,

$$d_{PM} \int_F w + \int_F d_{PM} w = g\Xi \otimes v''(1) - g\Xi \otimes v''(0) = F_1^* w\Xi - F_0^* w\Xi. \tag{60}$$

the last equality because F is a homotopy between the identity map and the trivial one. ∎

In most cases we will be concerned with such forms restricted to a fixed-endpoint path space such as $PM(x_0, x_1)$ in which case the above result becomes

$$d_{PM} \int_F w + \int_F d_{PM} w = w\Xi - F_0^* w\Xi \tag{61}$$

Proposition 3.2.

$$\begin{aligned} d\int \omega_1 \dots \omega_r = \\ = \sum_{i=1}^{r} (-1)^i \int T\omega_1 \dots d\omega_i \omega_{i+1} \dots \omega_r - \sum_{i=1}^{r-1} (-1)^i \int T\omega_1 \dots (T\omega_i \circ \omega_{i+1})\omega_{i+2} \dots \omega_r - \\ - p_0^* \omega_1 \circ \int \omega_2 \dots \omega_r + T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \end{aligned} \tag{62}$$

Proof. Using the previous proposition, and the definition of iterated integrals, we have

$$\begin{aligned} d\int \omega_1 \dots \omega_r = d\int_F T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \Xi = \\ = -\int_F d(T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \Xi) + F_1^* T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \Xi\Xi - \\ - F_0^* T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \Xi\Xi. \end{aligned} \tag{63}$$

which expands to

$$\begin{aligned} \int_F T(d\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \Xi + (-1)^r \int_F T(\int T\omega_1 \dots T\omega_{r-1}) \circ p_1^* d\omega_r \Xi + \\ + T(\int \omega_1 \dots \omega_{r-1}) \circ p_1^* \omega_r \end{aligned} \tag{64}$$

It is easy to see that he F_0^* term disappears because evaluating $\int_F \omega$ at a constant path yields 0. The final result then follows by induction after calculating a low dimensional case such as:

$$d\int \omega_1\omega_2 = -\int d\omega_1\omega_2 + \int T\omega_1 d\omega_2 + \int (T\omega_1 \circ \omega_2) - p_0^*\omega_1 \circ \int \omega_2 + T(\int \omega_1) \circ p_1^* \omega_2 \tag{65}$$

□

If ω has total degree 1, then

$$\begin{aligned} d\int (\omega)^r = -\sum_{i+j+1=r} \int (\omega)^i d\omega (\omega)^j - \sum_{i+j+2=r} \int (\omega)^i (\omega \circ \omega)(\omega)^j + \\ - p_0^* \omega \circ \int (\omega)^{r-1} + \int (\omega)^{r-1} \circ p_1^* \omega \end{aligned} \tag{66}$$

So for ω in $\oplus End^{1-k}(V) \otimes_{\mathcal{A}^0} \mathcal{A}^k$,

$$\begin{aligned} d\Psi = \\ [-\int \varkappa + (-\int \varkappa\omega - \int \omega\varkappa) - \dots - \sum_{i+j=r-1} \int (\omega)^i \varkappa \omega^j + \dots] + -p_0^*\omega \circ \Psi + \Psi \circ p_1^* \omega. \end{aligned} \tag{67}$$

where here $\varkappa := d\omega - T\omega \circ \omega = d\omega + \omega \circ \omega$ is the curvature.

Note that if $\varkappa = 0$ then we have

$$d\Psi = -p_0^*\omega \circ \Psi + \Psi \circ p_1^*\omega \tag{68}$$

and on $PM(x_0, x_1)$ this reduces further to

$$d\Psi = -p_0^* A^0 \circ \Psi + \Psi \circ p_1^* A^0 \tag{69}$$

The condition $\varkappa = 0$ locally amounts to the series of equations

$$\begin{gathered} A^0 \circ A^0 = 0 \\ A^0 \circ A^1 + A^1 \circ A^0 + (dA^0) = 0 \\ \cdots \\ \sum_{i=0}^{q+1} A^i \circ A^{q-i+1} + (dA^q) = 0 \\ \cdots \end{gathered} \tag{70}$$

which is identical to the flatness condition $\mathbb{E} \circ \mathbb{E} = 0$ in $\mathcal{P}_{\mathsf{A}}$. With the help of the Stokes formula the equation (69) is equivalent to the integral form

$$-A^0_{x_0} \circ \int_{I^q} h^*\Psi_q + (-1)^q (\int_{I^q} h^*\Psi_q) \circ A^0_{x_1} = \int_{\partial I^q} h^*\Psi_{q-1} \tag{71}$$

for any q-family of paths $h : I^q \to P(M, x_0, x_1)$ inside a trivializing patch. The sign $(-1)^q$ appears when the q-simplex passes the degree 1 element A^0.

3.5 Holonomy With Respect to the Stable Structure

3.5.1 Holonomy with Respect to the Shift

Let $(E^\bullet, \mathbb{E})$ be an element of $\mathcal{P}_{\mathsf{A}}$, $d - A$ a local coordinate description, and Ψ its associated holonomy transport. It is not hard to see that holonomy commutes with the shift functor. Each iterated integral of length k has k copies of Ξ embedded so a shift on the forms is exactly cancelled. But the overall holonomy understood as a homomorphism via our conventions alternates with respect to the bundle degree. Hence a shift by q gives an overall sign of $(-1)^q$

3.5.2 Holonomy of a Cone

Suppose we have a morphism in $\mathcal{P}_{\mathsf{A}}$, i.e. an element ϕ of total degree q of

$$\mathrm{Hom}^q_{\mathcal{P}_{\mathsf{A}}}(E_1, E_2) = \{\phi : E_1 \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \to E_2 \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet | \phi(ea) = (-1)^{q|a|}\phi(e)a\} \tag{72}$$

The differential is defined

$$d\phi := E_2 \circ \phi - (-1)^{|\phi|}\phi \circ E_1 \tag{73}$$

We can construct the cone complex associated to ϕ, $C(\phi)$.

$$C(\phi)^k = (E_1[1-q]^k \oplus E_2^k) \tag{74}$$

with the differential (this has total degree 1):

$$D = \begin{pmatrix} \mathbb{E}_1[1-q] & 0 \\ \phi[1-q] & \mathbb{E}_2 \end{pmatrix} \tag{75}$$

Note that $\mathbb{E}_i[1-q] = (-1)^{1-q}\mathbb{E}_i$, and $\phi[1-q] = \phi$, and that that D is flat iff ϕ is a closed morphism.

In a trivializing coordinate patch write $D^\phi = d - \omega$ and denote the corresponding $\mathbb{Z}$-connection holonomy by Ψ^ϕ. Then applying Chen's formula for $d\Psi$ on $PM(x_0, x_1)$, we calculate,

$$d\Psi^\phi = -\int \varkappa + (-\int \varkappa\omega - \int \omega\varkappa) - \dots \\ - \sum_{i+j=r-1} \int \omega^i \varkappa \omega^j + \dots] - p_0^*\omega^0 \circ \Psi^\phi + \Psi^\phi \circ p_1^*\omega^0 \quad (76)$$

And since D_{11}^ϕ, and D_{22}^ϕ are flat, it is evident that

$$\varkappa = d\omega + \omega \circ \omega = \left(\begin{smallmatrix} 0 & 0 \\ d\phi & 0 \end{smallmatrix}\right) \quad (77)$$

Then the 21-component is:

$$d\Psi_{21}^\phi = [-\int d\phi - \dots - \sum_{i+j+2=r} \int (B)^i d\phi[1-q](A[1-q])^j - \dots]- \\ - p_0^* B^0 \circ \Psi_{21}^\phi + \Psi_{21}^\phi \circ p_1^* A^0[1-q] \quad (78)$$

Alternately, if we first take $d\phi$ in $\mathcal{P}_{\mathsf{A}}$ and take the holonomy of its cone $(C(d\phi), D^{d\phi})$. We already showed that since $D^{d\phi}$ is flat, $d\Psi^{d\phi} = -p_0^*\omega^{d\phi,0} \circ \Psi^{d\phi} + \Psi^{d\phi} \circ p_1^*\omega^{d\phi,0}$. And we have,

$$\Psi_{21}^{d\phi} = \int d\phi + \dots \sum_{i+j+1=r} \int (B)^i d\phi[q](A[q])^j + \dots = \\ = \int d\phi[q] + \dots \sum_{i+j+1=r} (-1)^j \int (B)^i d\phi[q](A[1-q])^j + \dots \quad (79)$$

But according to our definition of iterated integrals each term in each integral has a hidden factor of Ξ so the shift gives a factor of $(-1)^j$ again. Thus, the above formula becomes,

$$\Psi_{21}^{d\phi} = -\int d\phi - \dots \sum_{i+j+1=r} \int (B)^i d\phi(A[1-q])^j + \dots \quad (80)$$

And consequently,

$$d\Psi_{21}^\phi = -\Psi_{21}^{d\phi} - p_0^* B^0 \circ \Psi_{21}^\phi + \Psi_{21}^\phi \circ p_1^* A^0 \quad (81)$$

3.6 Cubes to Simplices

Now we want to integrate over simplices rather than cubes, which will involve realizing any simplex as a family of paths with fixed endpoints. This construction is described technically in [Chen77],[I] and elsewhere and only outlined here. P is the path space functor.

Given a geometric k-simplex, $\sigma : \Delta^k \to M$, we want to realize this as a factor of a (k-1)-family of paths into M. That is, we produce a map $\theta : I^k \to \Delta^k$ which then can be viewed as a family of paths $\theta_{(k-1)} : I^{k-1} \to P\Delta^k$. This map is factored into two parts: $I^k \xrightarrow{\lambda} I^k \xrightarrow{\pi_k} \Delta^k$. Here π_k is an order-preserving retraction. λ is given by the map $\lambda_w : I \to I^k$ parametrized by $w \in I^{k-1}$. The result, is an I^{k-1}-family of paths in I^k (we call this $\lambda_{(k-1)} : I^{k-1} \to PI^k$) each starting at $(w_1, w_2, \dots, w_{k-1}, 1)$ and ending at $(0, 0, \dots, 0)$. When post-composed with π_k we get a $(k-1)$-family of paths in Δ^k which start at σ_k and end at σ_0. Define $\theta_{(k-1)} : I^{k-1} \to P\Delta^k$

by $P\pi_k \circ \lambda_{(k-1)}$

We restate the characteristic properties of such a factorization c/o [I]:

- If $x \le X'$ in the sense that $x_i \le x'_i$ for all i, then $\pi_k(x) \le \pi_k(x')$. Furthermore, $\pi_k(x) \ge x$.
- π_k sends $\partial_i^+ I^k = \{x \in I^k | x_i = 1\}$ to the back $k-i$ face of Δ^k spanned by $\{v_i, \dots, v_k\}$ and given bt the equation $y \ge v_i$.
- π_k sends $\partial_i^- I^k = \{x \in I^k | x_i = 0\}$ onto $\partial_i \Delta^k = \{y \in \Delta^k | y_i = y_{i+1}\}$.

and

- The adjoint of $\theta_{(k)}$ is a piecewise-linear epimorphism $I^k \twoheadrightarrow \Delta^k$
- For each $w \in I^{k-1}$, θ_w is a path from $\theta_w(0) = v_k$ to $\theta_w(1) = v_0$.
- θ_w passes through the vertex v_i iff $w_i = 1$.
- $\theta_{(k)}$ takes each of the 2^{k-1} vertices of I^{k-1} to the shortest path from v_k to v_0 passing through the corresponding subset $\{v_1 \dots v_{k-1}\}$.

4 A dg-quasi-equivalence of categories

In this section we establish our Riemann-Hilbert correspondence for ∞-local systems. Recall that $\mathcal{C}$ is the category of cochain complexes over $\mathbb{R}$.

Theorem 4.1. *There is a dg-functor*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$$

which is a dg-quasi-equivalence.

4.1 The functor $\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$

On objects the functor $\mathcal{RH} : Ob(\mathcal{P}_{\mathsf{A}}) \to Ob(\mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M))$ is described as follows. Given an element $(E^\bullet, \mathbb{E}) \in \mathcal{P}_{\mathsf{A}}$ take the corresponding graded bundle V over M with a $\mathbb{Z}$-graded connection $\mathbb{E}$. Define an ∞-local system by the following assignment:

$$\mathcal{RH}((E^\bullet, \mathbb{E}))(x) = (V_x, \mathbb{E}_x^0) \tag{82}$$

$$\mathcal{RH}((E^\bullet, \mathbb{E}))_k(\sigma_k) := \int_{I^{k-1}} \theta^*_{(k-1)} (P\sigma)^* \Psi \tag{83}$$

i.e. assign to each k-simplex the integral of the higher holonomy integrated over that simplex (understood as a k-1 family of paths), which is a degree (1-k) homomorphism from the fiber over the endpoint to the fiber over the starting point of the simplex. To a 0-simplex this yields a degree 1 map in the fiber over that point which we shall see will be a differential as a result of the flatness of the $\mathbb{Z}$-graded connection. To a 1-simplex (a path) we get the usual parallel transport of the underlying graded connection. Flatness will imply that this is a cochain map with respect to the differentials on the fibers over the endpoints of the path.

So far we only have a simplicial set map $\pi_\infty M \to \mathcal{C}_\infty$. Call it F. Since we are integrating a flat $\mathbb{Z}$-graded connection,

$$d\Psi = -p_0^* A^0 \circ \Psi + \Psi \circ p_1^* A^0. \tag{84}$$

So via Stokes' Theorem, F satisfies the local system condition:

$$\mathbb{E}^0 \circ F_k(\sigma) - (-1)^k F_k(\sigma) \circ \mathbb{E}^0 = \sum_{i=1}^{k-1} (-1)^i F_{k-1}(\sigma_0, \dots, \hat{\sigma_i}, \dots, \sigma_k) + \\ - \sum_{i=1}^{k-1} (-1)^i F_i(\sigma_0, \dots, \sigma_i) \circ F_{k-i}(\sigma_i, \dots, \sigma_k) \tag{85}$$

which is the Maurer-Cartan equation

$$dF + \hat{\delta} F + F \cup F = 0 \tag{86}$$

Proving this abmounts to the task of figuring out what $\int_{\partial I^{q-1}} h^* \Psi$ is in the case that h is the map constructed above which factors through σ. That is we must relate ∂I^{q-1} to $\partial \Delta^{q-1}$. Igusa works this out elegantly in [I] and obtains (If we write $I := \int \theta^*(P[\bullet])^*(\Psi)$)

$$\int \theta^*(P[\bullet])^*(d\Psi) = -\hat{\delta} I - I \cup I \tag{87}$$

On morphisms $\mathcal{RH}$ is described as follows. Given a morphism $\phi \in \mathcal{P}_{\mathsf{A}}{}^q(E_1, E_2)$ we can form the homological cone and the holonomy transport of the cone Ψ^ϕ. Applying $\mathcal{RH}$ get $I = \mathcal{RH}(C(\phi)) : \pi_\infty \to \mathcal{C}_\infty$ a simplicial set morphism that isn't necessarily a local system. However, if $d\phi = 0$ then we have already shown that I is a local system, i.e. it satisfies:

$$dI + \hat{\delta} I + I \cup I = 0 \tag{88}$$

Let $F := \mathcal{RH}(E_1), G := \mathcal{RH}(E_2)$. If we represent I as a 2x2 matrix it is not hard to see that $I_{11} = \mathcal{RH}(E_1[1-q]) = F[1-q]$ and $I_{22} = F_2 := \mathcal{RH}(E_2)$.

Most importantly I_{21} can be regarded as a morphism of local systems between $F_1[1-q]$ and F_2 after we specify that $I_{21}(\text{point}) = \phi^0(\text{point})$

Theorem 4.2. *The assignment $\phi \mapsto I_{21} = \mathcal{RH}(C(\phi))_{21}$ gives a map of complexes,*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}}{}^q(E_1, E_2) \to \mathsf{Loc}_\infty^{q;\mathcal{C}}(\pi_\infty)(F_1, F_2). \tag{89}$$

And in fact is a dg-functor $\mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}_\infty^{q;\mathcal{C}}(\pi_\infty)$.

Proof. Given a morphism $\phi \in \mathcal{P}_{\mathsf{A}}{}^q(E_1, E_2)$, denote the holonomy transport associated to the homological cone $(C(\phi), D^\phi)$ by Ψ^ϕ, and the transport associated to the cone $(C(d\phi), D^{d\phi})$ by $\Psi^{d\phi}$. Locally write $D^\phi = d - D^{\phi,0} - \dots$ and similarly for $D^{d\phi}$.

We already calculated that (on $PM(x_0, x_1)$),

$$-(d\Psi^\phi)_{21} + (p_0^* D^{\phi,0} \circ \Psi^\phi - \Psi^\phi \circ p_1^* D^{\phi,0})_{21} = \Psi^{d\phi} \tag{90}$$

Thus, applying $\int \theta^*(P[\bullet])^*(\Psi)$ to both sides yields,

$$(I \cup I + \hat{\delta} I + dI)_{21} = \mathcal{RH}(d\phi) \tag{91}$$

and the left hand side is by definition $D\mathcal{RH}(\phi)$.

Proposition 4.3. *The functor $\mathcal{RH}$ is dg-quasi-fully faithful.*

Proof. Consider two objects $E_i = (E_i^\bullet, \mathbb{E}_i) \in \mathcal{P}_{\mathsf{A}}$, $i = 1, 2$. The chain map

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}}(E_1, E_2) \to \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)(\mathcal{RH}(E_1), \mathcal{RH}(E_2))$$

induces a map on spectral sequences (24) and [B1], Theorem 2.5.1. At the E_1-level on the $\mathcal{P}_{\mathsf{A}}$ side, we have that $H^*((E_i, \mathbb{E}_i^0))$ are both vector bundles with flat connection, while according to 2.6, we have $H^*((\mathcal{RH}(E_i), \mathbb{E}_i^0))$ are local systems on M. At the E_2-term the map is

$$H^*(M; Hom(H^*(E_1, \mathbb{E}_1^0), H^*(E_2, \mathbb{E}_2^0))) \to H^*(M; H^*((\mathcal{RH}(E_1), \mathbb{E}_1^0)), H^*((\mathcal{RH}(E_2), \mathbb{E}_2^0)))$$

which is an isomorphism by the ordinary De Rham theorem for local systems. □

4.2 $\mathcal{RH}$ is dg-essentially surjective

We must prove that for any $(F, f) \in \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$, that there is an object $E = (E^\bullet, \mathbb{E}) \in \mathcal{P}_{\mathsf{A}}$ such that $\mathcal{RH}(E)$ is quasi-isomorphic to (F, f). We first define a complex of sheaves on M. Let $\underline{\mathbb{R}}$ denote the constant local system, and thus an ∞-local system. We also view $\underline{\mathbb{R}}$ as a sheaf of rings with which $(M, \underline{\mathbb{R}})$ becomes a ringed space. For an open subset $U \subset M$, let $(C_F(U), D) = (\mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty U)(\underline{\mathbb{R}}|_U, F|_U), D)$. Let $(\underline{C}_F, D)$ denote the associated complex of sheaves. Then $\underline{C}_F$ is soft; see the proof of Theorem 3.15, [W]. By corollary 2.6, $\underline{C}_F$ is a perfect complex of sheaves over $\underline{\mathbb{R}}$. Let $\underline{\mathcal{A}}_M$ denote the sheaf of C^∞ functions and $(\underline{\mathcal{A}}^\bullet, d)$ denote the dg sheaf of C^∞ forms on M. Set $\underline{C}_F^\infty = \underline{C}_F \otimes_{\underline{\mathbb{R}}} \underline{\mathcal{A}}_M$. By the flatness of $\underline{\mathcal{A}}_M$ over $\underline{\mathbb{R}}$, $\underline{C}_F^\infty$ is perfect as a sheaf of $\underline{\mathcal{A}}_M$-modules. Now the map

$$(\underline{C}_F^\bullet, D) \to (\underline{C}_F^\infty \otimes_{\underline{\mathcal{A}}_M} \underline{\mathcal{A}}_M^\bullet, D \otimes 1 + 1 \otimes d)$$

is a quasiisomorphism of sheaves of $\underline{\mathbb{R}}$-modules by the flatness of $\underline{\mathcal{A}}_M$ over $\underline{\mathbb{R}}$.
We need the following

Proposition 4.4. *Suppose $(X, \underline{\mathcal{S}}_X)$ is a ringed space, where X is compact and $\underline{\mathcal{S}}_X$ is a soft sheaf of rings. Then*

1. *The global sections functor*

$$\Gamma : Mod\text{-}\underline{\mathcal{S}}_X \to Mod\text{-}\underline{\mathcal{S}}_X(X)$$

is exact and establishes an equivalence of categories between the category of sheaves of right $\underline{\mathcal{S}}_X$-modules and the category of right modules over the global sections $\underline{\mathcal{S}}_X(X)$.

2. *If $\underline{M} \in Mod\text{-}\underline{\mathcal{S}}_X$ locally has finite resolutions by finitely generated free $\underline{\mathcal{S}}_X$-modules, then $\Gamma(X; \underline{M})$ has a finite resolution by finitely generated projectives.*
3. *The derived category of perfect complexes of sheaves $D_{perf}(Mod\text{-}\underline{\mathcal{S}}_X)$ is equivalent the derived category of perfect complexes of modules $D_{perf}(Mod\text{-}\underline{\mathcal{S}}_X(X))$.*

Proof. See Proposition 2.3.2, Exposé II, SGA6, [SGA6]. □

Theorem 4.5. *The functor*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$$

is dg-essentially surjective.

Proof. By the Proposition , there is a (strictly) perfect complex $(E^\bullet, \mathbb{E}^0)$ of $\mathcal{A}$-modules and quasiisomorphism $e^0 : (E^\bullet, \mathbb{E}^0) \to (X^\bullet, \mathbb{X}^0) := (\Gamma(M, \underline{C}_F^\infty), D)$. Following the argument of Theorem 3.2.7 of [B1], which in turn is based on arguments from [OTT], we construct the higher components $\mathbb{E}^i$ of a $\mathbb{Z}$-graded connection along with the higher components of a morphism e^i at the same time.

We have a $\mathbb{Z}$-graded connection on $X^\bullet$ by

$$\mathbb{X} := D \otimes 1 + 1 \otimes d : X^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

Then we have an induced connection

$$\mathbb{H} : H^k(X^\bullet, \mathbb{X}^0) \to H^k(X^\bullet, \mathbb{X}^0) \otimes_{\mathcal{A}} \mathcal{A}^1$$

for each k. We use the quasi-morphism e^0 to transport this connection to a connection, also denoted by $\mathbb{H}$ on $H^k(E^\bullet; \mathbb{E}^0)$

$$\begin{array}{ccc} H^k(E^\bullet; \mathbb{E}^0) & \stackrel{\mathbb{H}}{\to} & H^k(E^\bullet, \mathbb{E}^0) \otimes_{\mathcal{A}} \mathcal{A}^1 \\ \downarrow e^0 & & \downarrow e^0 \otimes 1 \\ H^k(X^\bullet, \mathbb{X}^0) & \stackrel{\mathbb{H}}{\longrightarrow} & H^k(X^\bullet, \mathbb{X}^0) \otimes_{\mathcal{A}} \mathcal{A}^1 \end{array} \tag{92}$$

The right vertical arrow above $e^0 \otimes 1$ is a quasi-isomorphism because $\mathcal{A}^\bullet$ is flat over $\mathcal{A}$. The first step is handled by the following lemma.

Lemma 4.6. *Given a bounded complex of finitely generated projective $\mathcal{A}$ modules $(E^\bullet, \mathbb{E}^0)$ with connections $\mathbb{H} : H^k(E^\bullet; \mathbb{E}^0) \to H^k(E^\bullet, \mathbb{E}^0) \otimes_{\mathcal{A}} \mathcal{A}^1$, for each k, there exist connections*

$$\widetilde{\mathbb{H}} : E^k \to E^k \otimes_{\mathcal{A}} \mathcal{A}^1$$

lifting $\mathbb{H}$. That is,

$$\widetilde{\mathbb{H}}\mathbb{E}^0 = (\mathbb{E}^0 \otimes 1)\widetilde{\mathbb{H}}$$

and the connection induced on the cohomology is $\mathbb{H}$.

Proof. (of lemma) Since $E^\bullet$ is a bounded complex of $\mathcal{A}$-modules it lives in some bounded range of degrees $k \in [N, M]$. Pick an arbitrary connection on E^M, ∇. Consider the diagram with exact rows

$$\begin{array}{ccccc} E^M & \stackrel{j}{\to} & H^M(E^\bullet, \mathbb{E}^0) & \to 0 \\ \nabla \downarrow & \stackrel{\theta}{\searrow} & \mathbb{H} \downarrow & \\ E^M \otimes_{\mathcal{A}} \mathcal{A}^1 & \stackrel{j \otimes 1}{\to} & H^M(E^\bullet, \mathbb{E}^0) \otimes_{\mathcal{A}} \mathcal{A}^1 & \to 0 \end{array} \tag{93}$$

In the diagram, $\theta = \mathbb{H} \circ j - (j \otimes 1) \circ \nabla$ is easily checked to be $\mathcal{A}$-linear and $j \otimes 1$ is surjective by the right exactness of tensor product. By the projectivity of E^M, θ lifts to

$$\widetilde{\theta} : E^M \to E^M \otimes_{\mathcal{A}} \mathcal{A}^1$$

so that $(j \otimes 1)\widetilde{\theta} - \theta$. Set $\widetilde{\mathbb{H}} = \nabla + \widetilde{\theta}$. With $\widetilde{\mathbb{H}}$ in place of ∇, the diagram above commutes.

Now choose on E^{M-1} any connection ∇_{M-1}. But ∇_{M-1} does not necessarily satisfy $\mathbb{E}^0 \nabla_{M-1} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. So we correct it as follows. Set $\mu = \widetilde{\mathbb{H}}\mathbb{E}^0 - (\mathbb{E}^0 \otimes 1)\nabla_{M-1}$. Then μ is $\mathcal{A}$-linear. Furthermore, $\operatorname{Im} \mu \subset \operatorname{Im} \mathbb{E}^0 \otimes 1$; this is because $\widetilde{\mathbb{H}}\mathbb{E} \in \operatorname{Im} \mathbb{E} \otimes 1$ since $\widetilde{\mathbb{H}}$ lifts $\mathbb{H}$. So by projectivity it lifts to $\widetilde{\theta} : E^{M-1} \to E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ such that $(\mathbb{E}^0 \otimes 1) \circ \widetilde{\theta} = \theta$. Set $\widetilde{\mathbb{H}} : E^{M-1} \to E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ to be $\nabla_{M-1} + \widetilde{\theta}$. Then $\mathbb{E}^0 \widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0$ in the right most square below.

$$\begin{array}{ccccccccc} E^N & \stackrel{\mathbb{E}^0}{\longrightarrow} & E^{N+1} & \stackrel{\mathbb{E}^0}{\longrightarrow} \cdots \stackrel{\mathbb{E}^0}{\longrightarrow} & E^{M-1} & \stackrel{\mathbb{E}^0}{\longrightarrow} & E^M & \to 0 \\ & & & & \nabla_{M-1} \downarrow & \stackrel{\mu}{\searrow} & \widetilde{\mathbb{H}} \downarrow & \\ E^N \otimes_{\mathcal{A}} \mathcal{A}^1 & \stackrel{\mathbb{E}^0 \otimes 1}{\longrightarrow} & E^{N+1} \otimes_{\mathcal{A}} \mathcal{A}^1 & \stackrel{\mathbb{E}^0 \otimes 1}{\longrightarrow} \cdots \stackrel{\mathbb{E}^0 \otimes 1}{\longrightarrow} & E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1 & \stackrel{\mathbb{E}^0 \otimes 1}{\longrightarrow} & E^M \otimes_{\mathcal{A}} \mathcal{A}^1 & \to 0 \end{array} \tag{94}$$

Now we continue backwards to construct all $\widetilde{\mathbb{H}} : E^\bullet \to E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1$ satisfying $(\mathbb{E}^0 \otimes 1)\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. This completes the proof of the lemma. ∎

(Proof of the theorem, continued.) Set $\widetilde{\mathbb{E}}^1 = (-1)^k \widetilde{\mathbb{H}}$ on E^k. Then

$$\mathbb{E}^0\widetilde{\mathbb{E}}^1 + \widetilde{\mathbb{E}}^1\mathbb{E}^0 = 0$$

but it is not necessarily true that $e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0 = 0$. We correct this as follows. Consider $\psi = e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0 : E^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1$. Check that ψ is $\mathcal{A}$-linear and a map of complexes.

$$\begin{array}{ccc} & & (E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1, \mathbb{E}^0 \otimes 1) \\ & \overset{\widetilde{\psi}}{\nearrow} & \downarrow e^0 \otimes 1 \\ E^\bullet & \overset{\psi}{\longrightarrow} & (X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1, \mathbb{X}^0 \otimes 1) \end{array} \tag{95}$$

In the above diagram, $e^0 \otimes 1$ is a quasi-isomorphism e^0 is a homotopy equivalence. So by Lemma 1.2.5 of [OTT] there is a lift $\widetilde{\psi}$ of ψ and a homotopy $e^1 : E^\bullet \to X^{\bullet-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ between $(e^0 \otimes 1)\widetilde{\psi}$ and ψ,

$$\psi - (e^0 \otimes 1)\widetilde{\psi} = (e^1\mathbb{E}^0 + \mathbb{X}^0 e^1)$$

So let $\mathbb{E}^1 = \widetilde{\mathbb{E}}^1 - \widetilde{\psi}$. Then

$$\mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = 0 \text{ and } e^0\mathbb{E}^1 - \mathbb{X}^1 e^0 = e^1\mathbb{E}^0 + \mathbb{X}^0 e^1. \tag{96}$$

So we have constructed the first two components $\mathbb{E}^0$ and $\mathbb{E}^1$ of the $\mathbb{Z}$-graded connection and the first components e^0 and e^1 of the quasi-isomorphism $E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$.

To construct the rest, consider the mapping cone $L^\bullet$ of e^0. Thus

$$L^\bullet = E[1]^\bullet \oplus X^\bullet$$

Let $\mathbb{L}^0$ be defined as the matrix

$$\mathbb{L}^0 = \begin{pmatrix} \mathbb{E}^0[1] & 0 \\ e^0[1] & \mathbb{X}^0 \end{pmatrix} \tag{97}$$

Define $\mathbb{L}^1$ as the matrix

$$\mathbb{L}^1 = \begin{pmatrix} \mathbb{E}^1[1] & 0 \\ e^1[1] & \mathbb{X}^1 \end{pmatrix} \tag{98}$$

Now $\mathbb{L}^0\mathbb{L}^0 = 0$ and $[\mathbb{L}^0, \mathbb{L}^1] = 0$ express the identities (96). Let

$$D = \mathbb{L}^1\mathbb{L}^1 + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^2 e^0 & [\mathbb{X}^0, \mathbb{X}^2] \end{pmatrix}. \tag{99}$$

Then, as is easily checked, D is $\mathcal{A}$-linear and

1. $[\mathbb{L}^0, D] = 0$ and
2. $D|_{0 \oplus X^\bullet} = 0$.

Since $(L^\bullet, \mathbb{L}^0)$ is the mapping cone of a quasi-isomorphism, it is acyclic and since $\mathcal{A}^\bullet$ is flat over $\mathcal{A}$, $(L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0 \otimes 1)$ is acyclic too. Since $E^\bullet$ is projective, we have that

$$\mathrm{Hom}^\bullet_{\mathcal{A}}((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0))$$

is acyclic. Moreover

$$\mathrm{Hom}^\bullet_{\mathcal{A}}((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0)) \subset \mathrm{Hom}^\bullet_{\mathcal{A}}(L^\bullet, (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, [\mathbb{L}^0, \cdot]))$$

is a subcomplex. Now we have $D \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ is a cycle and so there is $\widetilde{\mathbb{L}}^2 \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ such that $-D = [\mathbb{L}^0, \widetilde{\mathbb{L}}^2]$. Define $\mathbb{L}^2$ on $L^\bullet$ by

$$\mathbb{L}^2 = \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix} \tag{100}$$

Then

$$\begin{aligned}[\mathbb{L}^0,\mathbb{L}^2] =&[\mathbb{L}^0,\widetilde{\mathbb{L}}^2+\begin{pmatrix} 0 & 0\\ 0 & \mathbb{X}^2\end{pmatrix}]\\ =&-D+[\mathbb{L}^0,\widetilde{\mathbb{L}}^2+\begin{pmatrix} 0 & 0\\ 0 & \mathbb{X}^2\end{pmatrix}]\\ =&-\mathbb{L}^1\mathbb{L}^1\end{aligned} \tag{101}$$

So

$$\mathbb{L}^0\mathbb{L}^2+\mathbb{L}^1\mathbb{L}^1+\mathbb{L}^2\mathbb{L}^0=0.$$

We continue by setting

$$D=\mathbb{L}^1\mathbb{L}^2+\mathbb{L}^2\mathbb{L}^1+\begin{pmatrix} 0 & 0\\ \mathbb{X}^3e^0 & [\mathbb{X}^0,\mathbb{X}^3]\end{pmatrix} \tag{102}$$

Then $D: L^\bullet\to L^\bullet\otimes_{\mathcal{A}}\mathcal{A}^3$ is $\mathcal{A}$-linear, $D|_{0\oplus X^\bullet}=0$ and

$$[\mathbb{L}^0,D]=0.$$

Hence, by the same reasoning as above, there is $\widetilde{\mathbb{L}}^3\in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet,L^\bullet\otimes_{\mathcal{A}}\mathcal{A}^3)$ such that $-D=[\mathbb{L}^0,\widetilde{\mathbb{L}}^3]$. Define

$$\mathbb{L}^3=\widetilde{\mathbb{L}}^3+\begin{pmatrix} 0 & 0\\ 0 & \mathbb{X}^3\end{pmatrix} \tag{103}$$

Then one can compute that $\sum_{i=0}^3\mathbb{L}^i\mathbb{L}^{3-i}=0$.

Now suppose we have defined $\mathbb{L}^0,\dots,\mathbb{L}^n$ satisfying for $k=0,1,\dots,n$

$$\sum_{i=0}^k\mathbb{L}^i\mathbb{L}^{k-i}=0 \qquad\text{for } k\neq 2$$

and

$$\sum_{i=0}^2\mathbb{L}^i\mathbb{L}^{2-i}=0 \qquad\text{for } k=2$$

Then define

$$D=\sum_{i=1}^n\mathbb{L}^i\mathbb{L}^{n+1-i}+\begin{pmatrix} 0 & 0\\ \mathbb{X}^{n+1}e^0 & [\mathbb{X}^0,\mathbb{X}^{n+1}]\end{pmatrix} \tag{104}$$

$D|_{0\oplus X^\bullet}=0$ and we may continue the inductive construction of $\mathbb{L}$ to finally arrive at a $\mathbb{Z}$-graded connection satisfying $\mathbb{L}\mathbb{L}=0$. The components of $\mathbb{L}$ construct both the $\mathbb{Z}$-graded connection on $E^\bullet$ as well as the morphism from $(E^\bullet,\mathbb{E})$ to $(X^\bullet,\mathbb{X})$.

It follows from Proposition 2.7 that $\mathcal{RH}((E^\bullet,\mathbb{E}))\xrightarrow{e}(F,f)$ is a quasi-isomorphism. ☐

5 Appendix

5.1 The Linear Simplicial Nerve Construction

We describe here a variant of a construction called the simplicial nerve functor which was originally introduced by Cordier [Co] and appears in Lurie's book on higher Topoi [Lu1]. The simplicial nerve is a functor

$$N: s\mathcal{C}at\to s\mathcal{S}et \tag{105}$$

which is defined by the adunction property:

$$sSet(\Delta^n, N(\mathcal{C})) = s\mathcal{C}at(\mathfrak{C}[\Delta^n], \mathcal{C}) \tag{106}$$

Where $\mathfrak{C}[\bullet]$ is a kind of free-functor which in a sense constructs the free simplicial category generated by a simplicial set. The k-simplices of $N(\mathcal{C})$ can be understood as homotopy coherences in the category $\mathcal{C}$. This will become apparent as we further describe the this functor. This exposition is taken almost verbatim from [Lu1] but we include it for completeness.

Let k be a field.

Definition 5.1. *Let $[n]$ denote the linearly ordered, finite set of n+1 elements $\{0, 1, 2, \ldots, n\}$, as well as the corresponding category, and let Δ^n denote the simplicial set which is the combinatorial n-simplex. $\mathfrak{C}[\Delta^n]$ is the element of $s\mathcal{C}at$ given by the following assignments:*

$$Ob\mathfrak{C}[\Delta^n] = [n] \tag{107}$$

for $i, j \in [n]$,

$$\mathfrak{C}[\Delta^n](i,j) = \begin{cases} \emptyset & \text{for } j > i \\ N(P_{i,j}) & \text{for } i \leq j \end{cases} \tag{108}$$

Where $N[\bullet]$ the usual nerve functor, and $P_{i,j}$ the partially ordered set of subsets $I \subset \{i \leq \ldots \leq j\}$ such that both $i \in I$, and $j \in I$. For the ordered tuple, $(i_0 \leq \ldots \leq i_l)$, the compositions in $\mathfrak{C}[\Delta^n]$

$$\mathfrak{C}[\Delta^n](i_0, i_1) \times \ldots \times \mathfrak{C}[\Delta^n](i_{l-1}, i_l) \to \mathfrak{C}[\Delta^n](i_0, i_l) \tag{109}$$

are induced by the poset maps given by the union:

$$P_{i_0,i_1} \times \ldots \times P_{i_{l-1},i_l} \to P_{i_0,i_l} \tag{110}$$

$$I_0 \times \ldots \times I_l \mapsto I_0 \cup \ldots \cup I_l \tag{111}$$

And $\mathfrak{C}$ is functorial:

Definition 5.2. *For $f : [n] \to [m]$ a monotone map of linearly ordered finite sets, we get a morphism*

$$\mathfrak{C}[\Delta^n](f) : \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^m] \tag{112}$$

$$v \in Ob\mathfrak{C}[\Delta^n] \mapsto f(v) \in Ob\mathfrak{C}[\Delta^m] \tag{113}$$

For $i \leq j$ in $[n]$, the map

$$\mathfrak{C}[\Delta^n](i,j) \to \mathfrak{C}[\Delta^m](f(i), f(j)) \tag{114}$$

is induced by

$$f : P_{i,j} \to P_{f(i),f(j)}, \; I \mapsto f(I) \tag{115}$$

by applying $N[\bullet]$.

$\mathfrak{C}$ can be extended uniquely to a functor which preserves small colimits, and so it is easily verified that $\mathfrak{C}[\bullet]$ is a functor

$$\mathfrak{C} : sSet \to s\mathcal{C}at \tag{116}$$

Definition 5.3. *The truncation functor*

$$T : dg\mathcal{C}at_k \to dg\mathcal{C}at_{k,\leq 0} \tag{117}$$

from the category of small dg-categories over k to the category of small dg-categories over k which are supported in non-positive degree is given by setting $ObT(\mathcal{C}) = Ob\mathcal{C}$, and for $X, Y \in Ob\mathcal{C}$,

$$T\mathcal{C}^\bullet(X,Y) = \begin{cases} 0 & \text{for } \bullet > 0 \\ Z(\mathcal{C}^0(X,Y)) & \text{for } \bullet = 0 \\ \mathcal{C}^\bullet(X,Y) & \text{for } \bullet > 0 \end{cases} \tag{118}$$

with the appropriate restriction of the differentials of course.

The adjoint of T is the inclusion $I : dg\mathcal{C}at_{k,\leq 0} \to dg\mathcal{C}at_k$.

We denote by $Ch_{\leq 0}(k)$ the category of cochain complexes over k supported in non-positive degree, and by

$$DK : s\mathcal{V}ect_k \leftrightarrows Ch_{\leq 0}(k) \tag{119}$$

the Dold-Kan adjunction. $s\mathcal{V}ect_k$ is the category of simplicial vector spaces, and $s\mathcal{C}at_k$ the category of small categories enriched on sVk. This adjunction can be applied on homsets to yield an adjunction of enriched categories

$$DK : s\mathcal{C}at_k \leftrightarrows dg\mathcal{C}at_{k,\leq 0} \tag{120}$$

Finally denote by S the forgetful functor

$$S : s\mathcal{C}at_k \to s\mathcal{C}at \tag{121}$$

Definition 5.4. *By composition with T, DK, and S, we define the functor $N(k)$:*

$$N(k) := N \circ S \circ DK \circ T : dg\mathcal{C}at_k \to s\mathcal{S}et \tag{122}$$

Given $\mathcal{C} \in dg\mathcal{C}at_k$ it is instructive to describe the objects of $N(k)[\mathcal{C}]$ explicitly. We describe this as a stand-alone construction and then show it satisfies the adjunction

$$s\mathcal{S}et(\Delta^n, N(k)[\mathcal{C}]) = dg\mathcal{C}at_k(I \circ DK \circ k \circ \mathfrak{C}[\Delta^n], \mathcal{C}) \tag{123}$$

Definition 5.5. *We write $\mathcal{C}_\infty$ in place of $N(k)[\mathcal{C}]$ in order to reduce clutter. The name $N(k)$ is only presented in the appendix. In addition, we may want to mimic the $(k[G]-mod)-Rep_k(G)$ adjunction by abusing k and writing*

$$s\mathcal{S}et(\Delta^n, \mathcal{C}_\infty) = dg\mathcal{C}at_k(k[\Delta^n], \mathcal{C}) \tag{124}$$

Given a dg-category $\mathcal{C}$ in $dg\mathcal{C}at_{k,\leq 0}$, we produce a simplicial set $\mathcal{C}_\infty$ which is an ∞-category, and demonstrate that this is in fact an explicit construction of the linear simplicial nerve above.

Denote by $Y_i([n])$ the set of length-i (ordered) subsets of $[n]$. We denote an element of $Y_j([n])$

as an ordered tuple $(i_0 < i_1 < \ldots < i_j)$ and make use of this notation below.

$$\begin{aligned}
\mathcal{C}_{\infty\,0} = \{F = (F_0, f)| \\
f : Y_1([0]) \to k\mathrm{Ob}\mathcal{C}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{\infty\,1} = \{F = (F_0 + F_1, f)| \\
f : Y_1([1]) \to k\mathrm{Ob}\mathcal{C} \\
F_1 : Y_2([1]) \to \mathcal{C}^0(f(i_1), f(i_0))\}
\end{aligned}$$

$$\ldots$$

$$\begin{aligned}
\mathcal{C}_{\infty\,l} = \{F = (\sum_{i=0}^{l} F_i, f)| \\
f : Y_1([1]) \to k\mathrm{Ob}\mathcal{C} \\
\ldots \\
F_j : Y_{j+1}([l]) \to \mathcal{C}^{1-j}(f(i_j), f(i_0)), \\
\ldots \\
F_l : Y_{l+1}([l]) \to \mathcal{C}^{1-l}(f(i_l), f(i_0))\}
\end{aligned}$$

such that $dF + \hat{\delta}F + F \cup F = 0$, where

$$dF(i_1 < \ldots < i_j) := d(F(i_1 < \ldots < i_j)) \tag{125}$$

$$\hat{\delta}F_j(i_0 < \ldots < i_{j+1}) := -\sum_{q=1}^{k-1}(-1)^q F_j(i_0 < \ldots < \hat{i_q} < \ldots < i_{j+1}) \tag{126}$$

and

$$(F \cup F)_j(i_0 < \ldots < i_j) = \sum_{q=1}^{j-1}(-1)^q F_q(i_0 < \ldots < i_q) \circ F_{j-q}(i_q < \ldots < i_j) \tag{127}$$

The face maps are defined as follows:

$$\partial_q F_j(i_0 < \ldots < i_{j+1}) = F_j(i_0 < \ldots < \hat{i_q} < \ldots < i_{j+1}) \tag{128}$$

and degeneracies,

$$s_q F_j(i_0 < \ldots < i_{j-1}) = F_j(i_0 < \ldots < i_q < i_q < \ldots < i_{j-1}). \tag{129}$$

$\mathcal{C}_\infty$ is a simplicial set, and we need to show that it satisfies the weak Kan extension property.

The q^{th} k-horn Λ^k_q is the simplicial set which consists of the combinatorial k-simplex Δ^k with the unique k-face and codimension-1 face $\partial_q \Delta^k$ removed. If $q \neq 0, k$ we call Λ^k_q an inner horn.

Definition 5.6. *A weak Kan complex is a simplicial set K which satisfies the property that for any sSet map T: $\Lambda^k_q \to S$ from an inner horn to S, can be extended to a map from the entire simplex $\hat{T} : \Delta^k \to S$.*

Proposition 5.7. *$\mathcal{C}_\infty$ is an ∞-category.*

Proof. Let $0 < q < k$, and S be a simplicial set map $S : \Lambda^k_q \to \mathcal{C}_\infty$. We show that this extends to $S : \Delta^k \to \mathcal{C}_\infty$. We first define S on the missing $k-1$-face:

$$\begin{gathered} S_{k-1}(01\ldots\hat{q}\ldots k) := \\ (-1)^{q+1}\sum_{j=1,j\neq q}^{k-1}(-1)^j S_{k-1}(0\ldots\hat{j}\ldots k)+ \\ (-1)^q\sum_{j=1}^{k-1}(-1)^j S_j(0\ldots j)S_{k-j}(j\ldots k) \end{gathered} \tag{130}$$

We then define the new k-face:

$$S_k(01\ldots k) := 0(\text{any closed morphism will work}) \tag{131}$$

Then by this assignment,

$$\begin{gathered} (\hat{\delta}S)_k(0\ldots k) + (S\cup S)_k(01\ldots k) = \\ = -(-1)^q S_{k-1}(0\ldots\hat{q}\ldots k) + \sum_{j=1,j\neq q}^{k-1}(-1)^j S(0\ldots\hat{j}\ldots k)+ \\ +(S\cup S)_k(01\ldots k) = 0. \\ \text{hence, } dS_k + (\hat{\delta}S)_k + (S\cup S)_k = 0. \end{gathered} \tag{132}$$

It remains to show that the new $k-1$-face satisfies the corresponding equation:

$$dS_{k-1}(0\ldots\hat{q}\ldots k) = -(\hat{\delta}S)_{k-1}(0\ldots\hat{q}\ldots k) - (S\cup S)_{k-1}(0\ldots\hat{q}\ldots k) \tag{133}$$

Expanding the LHS we have:

$$\begin{aligned} &(-1)^{q+1}\sum_{j=1,j\neq q}(-1)^j d(S_{k-1}(0\ldots\hat{j}\ldots k))+ \\ &\qquad + (-1)^q\sum_{j=1}^{k-1}(-1)^j d(S_j(0\ldots j))\circ S_{k-j}(j\ldots k)+ \\ &\qquad\qquad + (-1)^q\sum_{j=1}^{k-1}(-1)^j(-1)^{1-j}S_j(0\ldots j)\circ d(S_{k-j}(j\ldots k)) \end{aligned} \tag{134}$$

Expanding just the first of the three terms above we get:

$$\begin{aligned} &(-1)^{q+1}\sum_{j=1,j\neq q}^{k-1}(-1)^{j+1}(\hat{\delta}S)_{k-1}(0\ldots\hat{j}\ldots k)+ \\ &\qquad\qquad (-1)^{q+1}\sum_{j=1,j\neq q}^{k-1}(-1)^{j+1}(S\cup S)_{k-1}(0\ldots\hat{j}\ldots k) \end{aligned} \tag{135}$$

And the first of the two sums above yelds:

$$(-1)^{q+1}\sum_{j=1}^{q-1}(-1)^{j+1}[\sum_{t=1}^{j-1}(-1)^t S_{k-1}(0\dots\hat{t}\dots\hat{j}\dots k)+ \\ +\sum_{t=j+1}^{k-1}(-1)^{t+1}S_{k-2}(0\dots\hat{j}\dots\hat{t}\dots k)]+ \\ +(-1)^{q+1}\sum_{j=q+1}^{k-1}(-1)^j[\sum_{t=1}^{j-1}(-1)^t S_{k-1}(0\dots\hat{t}\dots\hat{j}\dots k)+ \\ +\sum_{t=j+1}^{k-1}(-1)^{t+1}S_{k-1}(0\dots\hat{j}\dots\hat{t}\dots k)] \quad (136)$$

After some inspection it can be seen that all of the terms here appear in cencelling pairs except for when t=q. So this reduces to

$$\sum_{j=1}^{q-1}(-1)^j S_{k-2}(0\dots\hat{j}\dots\hat{q}\dots k) \qquad + \qquad \sum_{j=q+1}^{k-1}(-1)^{j+1}S_{k-2}(0\dots\hat{q}\dots\hat{j}\dots k), \quad (137)$$

which is exactly equal to $(-\hat{\delta}S)_{k-2}(0\dots\hat{q}\dots k)$ on the RHS.

We now expand the second sum of (135):

$$(-1)^{q+1}\sum_{j=1,j\neq q}^{k-1}(-1)^{j+1}[\sum_{t=1}^{j-1}(-1)^t S_t(0\dots t)\circ S_{k-t-1}(t\dots\hat{j}\dots k)+ \\ +\sum_{t=j+1}^{k-1}(-1)^{t-1}S_{t-1}(0\dots\hat{j}\dots t)\circ S_{k-t}(t\dots k)] \quad (138)$$

And likewise we expand the second two sums of (134):

$$(-1)^q\sum_{j=1}^{k-1}(-1)^j(-(\hat{\delta}S)_j(0\dots j)-(S\cup S)_{k-j}(j\dots k))\circ S_{k-j}(j\dots k)+ \\ +(-1)^q\sum_{j=1}^{k-1}S_j(0\dots j)\circ(-(\hat{\delta}S)_{k-j}(j\dots k)-(S\cup S)_{k-j}(j\dots k)) \quad (139)$$

Expanding the above sums without the triple composition terms, we get

$$(-1)^q\sum_{j=1}^{k-1}\sum_{t=1}^{j-1}(-1)^{t+j}S_{j-1}(0\dots\hat{t}\dots j)\circ S_{k-j}(j\dots k)+ \\ (-1)^q\sum_{j=1}^{k-1}\sum_{t=j+1}^{k-1}(-1)^{t+j}S_j(0\dots j)\circ S_{k-j-1}(j\dots\hat{t}\dots k) \quad (140)$$

It can be seen that these pair with the terms in (138) to cancel all but the terms which give $-(S\cup S)_{k-1}(0\dots\hat{q}\dots k)$ on the RHS. It just remains to analyze the triple composition terms of (139):

$$(-1)^q \sum_{j=1}^{k-1} (-1)^j (S \cup S)_j (01 \dots j) \circ S_{k-j}(j \dots k) =$$

$$= (-1)^q \sum_{j=1}^{k-1} \sum_{t=1}^{j-1} (-1)^{j+t+1} S_t(0 \dots j) \circ S_{j-t}(t \dots j) \circ S_{k-j}(j \dots k) \quad (141)$$

$$(-1)^q \sum_{j=1}^{k-1} (-1)^{j+1} (-1)^{1-j} (-1) S_j(0 \dots j) \circ (S \cup S)_{k-j}(j \dots k) =$$

$$= (-1)^q \sum_{j=1}^{k-1} \sum_{t=j+1}^{k-1} (-1)^{t+j} S_j(0 \dots j) \circ S_{t-j}(j \dots t) \circ S_{k-t}(t \dots k). \quad (142)$$

The sum of these above terms can be seen to vanish since each term appears twice with opposite signs. □

Now we loosely demonstrate the adjunction by considering the details of the Dold-Kan construction in the relevant direction (See [G-J] for more details).

Define $G : sAb \to Ch_{\leq 0}$ via

$$G(A)_\bullet = (A_\bullet / DA_\bullet, d) \quad (143)$$

where $d := \sum (-1)^i d^i$ and DA is the subcomplex of A spanned by degenerate simplices.

Let $\mathcal{C}$ be an element of $dg\mathcal{C}at$. A functor H from $\mathcal{D} := I \circ G \circ k \circ \mathfrak{C}[\Delta]$ to $\mathcal{C}$ is specified by its object map and its values on a basis for the hom spaces in $\mathfrak{C}[\Delta]$. To each object $[n]$, H assigns an object of $\mathcal{C}$. To each pair of objects $m < n$ H gives a map

$$H(m, n) : \mathcal{D}(m, n) \to \mathcal{C}(H(m), H(n)) \quad (144)$$

A basis for $\mathcal{D}^{-k}(m, n)$ is the set of length-k compositions in $P_{m,n}$. For the simplicity suppose $m = 0$. One can think of the 0-simplices of $P_{0,n}$ as the vertices of an $(n-1)$-cube, the coordinates indexing the linearly ordered subsets of $[n]$ in the obvious way with $(0, \dots, 0)$ corresponding to $(0, n)$ and $(1, \dots, 1)$ corresponding to $(0, 1, 2, \dots, n)$. Then the simplices of $NP_{0,n}$ give a triangulation of this cube. In particular, the non-degenerate paths are ones which are compositions entirely of proper inclusions. The paths which are not in the image of a face map are the non-degenerate, length-n, paths which follow the edges of the cube.

The functor H must commute with the differentials:

$$H(\sum (-1)^i d_i \sigma) = dH(\sigma) \quad (145)$$

In particular, suppose σ is the $(n-1)$-cube above:

$$\sigma = \sum_{I \in S_{n-1}} (-1)^{sgn(I)} ((0n) \hookrightarrow (0 I_1 n) \hookrightarrow \dots \hookrightarrow (0 I_{12,\dots(n-1)} n)) \quad (146)$$

Where I_1 denotes the image of 1, I_{12} the pair either $I_1 I_2$ or $I_2 I_1$ ordered linearly depending on whether $I_1 \leq I_2$ or not, and so forth.

Then it is evident that the boundary breaks into two parts with the middle terms cancelling:

$$\sum (-1)^i d_i \sigma = d_0 \sigma + (-1)^n d_n \sigma \quad (147)$$

Because of the composition rules in $P_{i,j}$ one can write

$$((0I_1n) \hookrightarrow \ldots \hookrightarrow (0I_{12\ldots(n-1)}n)) = \\ ((0I_1) \hookrightarrow \ldots \hookrightarrow (0\ldots I^-_{1\ldots(n-1)})) \circ ((I_1n) \hookrightarrow \ldots \hookrightarrow (I^+_{1\ldots(n-1)}\ldots n)) \quad (148)$$

Where I^-_{12} represents either I_1 or I_2I_1 depending on whether or not $I_2 \leq I_1$ and so forth. H respects these factorizations by functoriality. Because the notation is cumbersome it is best to see this in some examples:

$$H(013 \hookrightarrow 0123) = H(01)H(13 \hookrightarrow 123) \quad (149)$$

$$H(024 \hookrightarrow 0124 \hookrightarrow 01234) = H(02 \hookrightarrow 012)H(24 \hookrightarrow 234) \quad (150)$$

The terms of $d_0\sigma$ can always be split in this manner between I_1.

Now we define an element F on the left hand side of the adjunction from the data of H. For an object of Δ^n, F assigns an object of $\mathcal{C}_\infty$, which is just an object of $\mathcal{C}$. We think of these as objects assigned to the $n+1$ vertices of an n simplex. Set $F(j) := H(j)$. To any edge (ij) we assign $F(ij) = H((ij))$. And any subset of $[n]$ is a composition of such assignments: eg.,

$$F(01)F(12)F(25) := H(0125) \quad (151)$$

The value on a k-simplex is given by

$$F(01\ldots k) := H(\sum_{\{I \in S_{k-1}\}} ((0k) \hookrightarrow (0I_1k) \hookrightarrow \ldots \hookrightarrow (0I_{1\ldots(k-1)}k))) \quad (152)$$

In the above considerations, it is not hard to see that $H(d_0\sigma)$ computes $-F \cup F$ and that $H((-1)^n d_n\sigma)$ computes $-\hat{\delta}F$ so that

$$dF + \hat{\delta}F + F \cup F = 0 \quad (153)$$

indeed yields an ∞-local system.

References

[B1] J. Block, Duality and Equivalence of module categories in noncommutative geometry, to appear.

[B-H] John Baez, Alexander Hoffnung, Convenient Categories of Smooth Spaces, arXiv: 0807.1704v2

[B-K] Bondal A. I., Kapranov M. M., Enhanced Triangulated Categories, *Math USSR Sbornik*, Vol 70. (1991), (No. 1) 93–107.

[Ch] Kuo-Tsai Chen, Iterated Path Integrals. *Bulletin of the AMS* 83 (1977), no. 5, 831–879.

[Co] Cordier, J. M., Sur la notion de diagramme homotopiquement coherent. Cahiers Top. et Geom. Diff. XXIII 1, 1982, 93-112.

[G-J] Paul Goerss, John F. Jardine, Simplicial Homotopy Theory (Progress in Mathematics), Birkhauser, Basel, 1999.

[H] Richard Hain, Iterated Integrals and Algebraic Cycles: Examples and Prospects, arXiv: 0109204v2.

[Ig02] K. Igusa, Twisting Cochains and Higher Torsion, arXiv: 0303047v1

[I] K. Igusa, Iterated Integrals of Superconnections, in preparation.

[K-S] M. Kontsevich, Y. Soibelman, Homological mirror symmetry and torus fibrations, *Symplectic geometry and mirror symmetry* (Seoul, 2000), 203–263, World Sci. Publ., River Edge, NJ, 2001

[Lu1] J. Lurie, Higher Topos Theory, arXiv: 0608040v4

[Lu2] J. Lurie, Stable Infinity Categories, arXiv: 0608228v5

[Me] Sergei Merkulov, De Rham model for string topology, arXiv: 0309038v3

[OTT] O'Brian, N., Toledo, D., Tong, Y.L., Grothendieck-Riemann-Roch for complex manifolds. *Math. Ann.* 271 (1985), no. 4, 493–526.

[SGA6] Berthelot, P., Grothendieck, A., Illusie, L., Théorie des Intersections et Théorème de Riemann-Roch, Springer LMS, Vol. 225, 1971.

[SZ] Emma Smith-Zbarsky, Families of A_∞-algebras and Homotopy Group Actions, arXiv: 0905.4768v2

[Toen] Toen Bertrand, Homotopical and Higher Categorical Structures in Algebraic Geometry, Habilitation Thesis, arXiv: 0312262v2

[TV] B. Toen, G. Vezzosi, Segal Topoi and Stacks over Segal Categories, arXiv: 0212330v2

[TT1] D. Toledo, Y. L. Tong, A parametrix for $\overline{\partial}$ and Riemann-Roch in Čech theory. *Topology* 15 (1976), no. 4, 273–301.

[TT2] Toledo, D., Tong, Y.L., Duality and intersection theory in complex manifolds. I. *Math. Ann.* 237 (1978), no. 1, 41–77.

[W] R. Wells, Differential Analysis on Complex Manifolds, Prentice Hall Series in Modern analysis. 1973.

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