PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3757–3762
S 0002-9939(97)04154-3

HOMOTOPY INVARIANCE OF NOVIKOV-SHUBIN INVARIANTS AND L^2 BETTI NUMBERS

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(Communicated by Peter Li)

Abstract. We give short proofs of the Gromov-Shubin theorem on the homotopy invariance of the Novikov-Shubin invariants and of the Dodziuk theorem on the homotopy invariance of the L^2 Betti numbers of the universal covering of a closed manifold in this paper. We show that the homotopy invariance of these invariants is no more difficult to prove than the homotopy invariance of ordinary homology theory.

Introduction

In this paper, we study the von Neumann spectral density function of the Laplacian acting on L^2 j-forms on the universal cover $\widetilde{M}$ of a compact Riemannian manifold M, which we call $N_j(\lambda, \widetilde{M})$ (see section 1), where $\lambda \in \mathbb{R}$. There are some important homotopy invariants of M which are associated to $N_j(\lambda, \widetilde{M})$ and which we now recall.

First of all, the L^2 Betti numbers which were defined by Atiyah, [A] are given by

$$b^j_{(2)}(\widetilde{M}) \equiv \lim_{\lambda \to 0+} N_j(\lambda, \widetilde{M})$$

and which were proved to be homotopy invariants of M by Dodziuk, [D].

Next Novikov and Shubin, [ES] proved that the dilatation class (see section 1) of $N_j(\lambda, \widetilde{M})$ was a differential invariant of M for all j. These were later proved to be homotopy invariants of M by Gromov and Shubin, [GS].

We give short proofs of the Gromov-Shubin theorem and of the Dodziuk theorem in this paper, showing that the homotopy invariance of these invariants is no more difficult to prove than the homotopy invariance of ordinary homology theory.

Gromov and Shubin, [GS] prove a basic abstract theorem about the bounded chain homotopy invariance of the small λ asymptotics of $N_j(\lambda, \mathcal{M})$ for Laplacians of abstract $L^2(\Gamma)$ chain complexes $\mathcal{M}$. Their application of this abstract theorem to the geometric case is somewhat complicated, and involves the detailed study of manifolds with boundary. Our proof has the advantage of staying entirely within the category of closed manifolds, and uses stabilization by sphere bundles and simple homotopy equivalence.

Received by the editors July 30, 1996.
1991 *Mathematics Subject Classification.* Primary 58G11, 58G18, 58G25.
Key words and phrases. L^2 Betti numbers, Novikov-Shubin invariants, homotopy invariance, von Neumann algebras.

1. Preliminaries

In this section we establish the main notation of the paper, and recall some basic facts about $L^2(\Gamma)$ modules. For more information on $L^2(\Gamma)$ modules, see [C], [GS].

Let Γ be a finitely generated discrete group. We recall the definition of $L^2(\Gamma)$ modules. A *free* $L^2(\Gamma)$ *module* is of the form $L^2(\Gamma)\otimes\mathcal{H}$ where $\mathcal{H}$ is a complex Hilbert space and Γ acts on $L^2(\Gamma)$ via the left regular representation ρ and trivially on $\mathcal{H}$. The von Neumann algebra generated by $\{\rho(\gamma) : \gamma \in \Gamma\}$ is a finite von Neumann algebra denoted by $\mathcal{U}(\Gamma)$. It has a natural finite normalized trace τ called the von Neumann trace. By a theorem of Dixmier, [DI], the von Neumann algebra of bounded linear operators on $L^2(\Gamma)$ which commute with the Γ action (that is, the commutant $\mathcal{U}(\Gamma)'$) is anti isomorphic to $\mathcal{U}(\Gamma)$. The trace τ can be described as follows: let δ_e denote the function on Γ which is one at the identity and zero elsewhere. If $A \in \mathcal{U}(\Gamma)'$, then $\tau(A) \equiv \langle A\delta_e, \delta_e\rangle_{L^2(\Gamma)}$. The trace τ extends to a trace on the von Neumann algebra of Γ invariant operators on $L^2(\Gamma)\otimes\mathcal{H}$, which is isomorphic to $\mathcal{U}(\Gamma)'\otimes\mathcal{B}(\mathcal{H})$, where it is given by $\tau\otimes Tr$ where Tr denotes the standard trace on $\mathcal{B}(H)$. We denote this trace by τ also. *An* $L^2(\Gamma)$ *module* $\mathcal{M}$ is a closed Γ invariant subspace of a free $L^2(\Gamma)$ module. There is an associated *dimension function*, $\dim_\Gamma$, which is defined on the set of $L^2(\Gamma)$ modules, and is given by the von Neumann trace of the orthogonal projection defining the $L^2(\Gamma)$ module. $\dim_\Gamma$ is independent of the choices made, and takes values in $[0, +\infty]$.

An $L^2(\Gamma)$ *complex* is a complex

$$\mathcal{M} : 0 \to \mathcal{M}_0 \overset{d_0}{\to} \mathcal{M}_1 ... \overset{d_{n-1}}{\to} \mathcal{M}_n \to 0$$

where $\mathcal{M}_k$ are $L^2(\Gamma)$ modules and d_k are closed and densely defined operators commuting with the Γ action satisfying $d_{k+1}\circ d_k = 0$ on domain d_k.

If $\mathcal{N}$ is another $L^2(\Gamma)$ complex, then a *morphism* of $L^2(\Gamma)$ complexes $f : \mathcal{M} \to \mathcal{N}$ is a sequence $f_k : \mathcal{M}_k \to \mathcal{N}_k$ of bounded linear operators commuting with the Γ action such that $f_{k+1}d_k w = d_k f_k w$ for all $w \in$ domain (d_k). A homotopy between two morphisms $f, g : \mathcal{M} \to \mathcal{N}$ is a sequence of bounded linear operators $T_k : \mathcal{M}_k \to \mathcal{N}_{k-1}$ commuting with the Γ-action such that $f_k - g_k = T_{k+1}d_k + d_{k-1}T_k$ on domain (d_k). Homotopy is an equivalence relation.

Two $L^2(\Gamma)$ complexes $\mathcal{M}$ and $\mathcal{N}$ are said to be *homotopy equivalent* if there are morphisms $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{M}$ such that fg and gf are homotopic to the identity morphisms of $\mathcal{N}$ and $\mathcal{M}$ respectively. Homotopy equivalence is an equivalence relation.

If $\mathcal{M}$ is an $L^2(\Gamma)$ complex, then define the functions $F_k(\lambda, \mathcal{M})$ as

$$F_k(\lambda, \mathcal{M}) \equiv \sup\left(\dim_\Gamma L : L \in \mathcal{S}_\lambda^{(k)}(\mathcal{M})\right)$$

where $\mathcal{S}_\lambda^{(k)}(\mathcal{M})$ denotes the set of all closed Γ-invariant subspaces of $\mathcal{M}_k/\ker d_k$ such that $L \subset \operatorname{dom}(d_k)/\ker d_k$ and $\|d_k w\| \le \sqrt{\lambda}\|w\|$ for $w \in L$. (The norm on the right-hand side is the quotient norm.) $F_k(\lambda)$ is an increasing function on $\mathbb{R}$ with values in $[0, \infty]$ and $F_k(\lambda) = 0$ if $\lambda < 0$. Also observe that if $\Delta_k \equiv d_{k-1}\delta_{k-1} + \delta_k d_k$, where δ_k denotes the L^2 adjoint of d_k, has the spectral decomposition

$$\Delta_k = \int_0^\infty \lambda dE_\lambda$$

then the *von Neumann spectral density function* is given by

$$N_k(\lambda, \mathcal{M}) \equiv \int_0^\lambda \tau(dE_\mu) = F_{k-1}(\lambda, \mathcal{M}) + F_k(\lambda, \mathcal{M}) + b^k_{(2)}(\mathcal{M}),$$

where $b^k_{(2)}(\mathcal{M}) = \dim_\Gamma \ker \Delta_k$ are the L^2 *Betti numbers*.

Two functions $F(\lambda)$ and $G(\lambda)$ on $\mathbb{R}$ satisfy $F \lll G$ if there is a positive constant $C > 0$ such that for all small λ, $F(\lambda) \leq G(C\lambda)$.

If $F \lll G$ and $G \lll F$, then F and G are said to be *dilatationally equivalent* and we write $F \sim G$. In other words, the small λ asymptotics of $F(\lambda)$ and $G(\lambda)$ are the same. The following basic abstract theorem is due to Gromov and Shubin, [GS]:

Theorem 1.1. *Let $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{M}$ be morphisms of $L^2(\Gamma)$ complexes such that gf is homotopic to the identity of $\mathcal{M}$. Then $F_k(\lambda, \mathcal{M}) \lll F_k(\lambda, \mathcal{N})$ and $b^k_{(2)}(\mathcal{M}) \leq b^k_{(2)}(\mathcal{N})$ for all k. Hence if $\mathcal{M}$ and $\mathcal{N}$ are homotopy equivalent, then $F_k(\lambda, \mathcal{M}) \sim F_k(\lambda, \mathcal{N})$ and $b^k_{(2)}(\mathcal{M}) = b^k_{(2)}(\mathcal{N})$.*

Although the statement about L^2 Betti numbers is not explicitly stated in their abstract theorem in [GS], it follows easily from their techniques. The de Rham complex $(\Omega^\bullet_{(2)}(\widetilde{M}), d)$ of L^2 differential forms on $\widetilde{M}$ is an example of an $L^2(\Gamma)$ complex where Γ denotes the fundamental group of the closed manifold M. Another example of an $L^2(\Gamma)$ complex which we are interested in is the complex $(C^\bullet_{(2)}(\widetilde{K}), d)$ of L^2 cochains on $\widetilde{K}$, the induced triangulation of the universal cover $\widetilde{M}$, from a triangulation K of M.

2. The topological theorem

In this section we prove the main topological theorem which we use to prove the homotopy invariance of the Novikov-Shubin invariants and the L^2 Betti numbers.

Theorem 2.1. *If M and N are two manifolds that are homotopy equivalent, then there is a sphere bundle $S(N \times S^{2n+1})$ over $N \times S^{2n+1}$ which is diffeomorphic to the product of $M \times S^{2n+1}$ and a sphere.*

Proof. If $h : N \to M$ is a homotopy equivalence, then by crossing with an odd sphere

$$h \times \mathrm{Id} : N \times S^{2n+1} \to M \times S^{2n+1}$$

we calculate that the Whitehead torsion $\tau(h \times \mathrm{Id}) = \tau(h)\chi(S^{2n+1}) + \tau(\mathrm{Id})\chi(M) = 0$ and thus $h \times \mathrm{Id}$ is a simple homotopy equivalence, [RS], Appendix B.5. Hence the theorem follows from the following lemma. $\square$

Lemma 2.2. *If M and N are two manifolds that are simply homotopy equivalent, then there is a sphere bundle $S(N)$ over N diffeomorphic to a product of M and a sphere.*

Proof. This is implicit in Whitehead, [W]. For the convenience of the modern reader we sketch a proof. By definition of simple homotopy equivalence, we can obtain N from M by elementary expansions and collapses, [RS], Chapter 3; that is, we can attach cells along faces to N and M to obtain PL homeomorphic polyhedra. Embed N and M in a large dimensional manifold W so that all the cell attachments can be accomplished inside W. Consider regular neighborhoods of N and M, say P and Q. It is a result in PL topology proved by a technique called shelling that the homeomorphism type of a regular neighborhood is unchanged after an elementary collapse or expansion, [RS], Theorem 3.26. Therefore, P and Q are PL homeomorphic, and so are their boundaries, ∂P and ∂Q which are sphere bundles over N and M. It can be checked that all their arguments apply to diffeomorphisms if W is itself a differentiable manifold. To ensure that the sphere bundle over M is trivial, we can embed both manifolds into $M \times \mathbb{R}^n$ for n large. □

3. The main theorem

In this section we present our short proofs of the following theorem, part (1) of which was first proved by Gromov and Shubin, [GS] and part (2) of which was first proved by Dodziuk, [D].

Theorem 3.1. *Let M and N be closed manifolds which are homotopy equivalent. Then*

(1) $F_k(\lambda, \widetilde{M}) \sim F_k(\lambda, \widetilde{N})$,
(2) $b^k_{(2)}(\widetilde{M}) = b^k_{(2)}(\widetilde{N})$.

Proof. As observed in our topological theorem in section 2, $N \times S^{2n+1}$ is simple homotopy equivalent to $M \times S^{2n+1}$. Since we can choose $n > \dim M$, if we choose the product metrics on $N \times S^{2n+1}$ and on $M \times S^{2n+1}$, then it follows easily that

$$F_k(\lambda, \widetilde{N}) \sim F_k(\lambda, \widetilde{N} \times S^{2n+1})$$

for $k = 0, 1, ..., \dim N$ and

$$F_k(\lambda, \widetilde{M}) \sim F_k(\lambda, \widetilde{M} \times S^{2n+1})$$

for $k = 0, 1, ..., \dim M$.

Hence it is enough to prove the theorem when M and N are simple homotopy equivalent. By Lemma 2.2, there is a sphere bundle $\pi : S(N) \to N$ over N which is diffeomorphic to $M \times S^p$ for some large p. Since we can choose $p > \dim M$, if we choose the product metric on $M \times S^p$, then as before we see that

$$F_k(\lambda, \widetilde{M}) \sim F_k(\lambda, \widetilde{M} \times S^p)$$

for $k = 0, 1, ..., \dim M$. Since $M \times S^p$ and $S(N)$ are diffeomorphic, it follows from [ES] that

$$F_k(\lambda, \widetilde{M} \times S^p) \sim F_k(\lambda, \widetilde{S(N)})$$

for $k = 0, 1, ..., \dim M + p$.

Since π is a submersion, so is the induced map $\widetilde{\pi} : \widetilde{S(N)} \to \widetilde{N}$, which also has fibre S^p. Therefore it induces a morphism of $L^2(\Gamma)$ modules

$$\widetilde{\pi}^* : \Omega^\bullet_{(2)}(\widetilde{N}) \to \Omega^\bullet_{(2)}(\widetilde{S(N)})$$

on L^2 differential forms. Since $p > \dim N = \dim M$, it follows that there is a section s of $\pi : S(N) \to N$. Let $\phi \in \Omega^p(S(N))$ denote the Poincaré dual (cf. chapter 1 in [BT]) to the image of s, $s(N) \subset S(N)$. That is, ϕ $(= s_*1)$ has the property that it is *closed* and that $\pi_*(\phi) = 1$, where π_* denotes integration along the fibres of the submersion π (cf. chapter 1 in [BT]). Then the lift of ϕ to the universal cover, $\widetilde{\phi}$, is a *bounded* differential form and induces a linear map

$$A : \Omega^k_{(2)}(\widetilde{S(N)}) \to \Omega^k_{(2)}(\widetilde{N})$$

for $k = 0, 1, ..., \dim N$, which is defined as

$$A(\eta) = \widetilde{\pi}_*(\eta \wedge \widetilde{\phi}).$$

Note that $\eta \wedge \widetilde{\phi}$ is a degree $p + k$ form on $\widetilde{S(N)}$ and integration along the fibres of $\widetilde{\pi}$ reduces the degree by p. Also one has the estimate

$$||\eta \wedge \widetilde{\phi}||_{L^2} \leq ||\phi||_{L^\infty} ||\eta||_{L^2},$$

and since the fibre of $\widetilde{\pi}$ is compact, it shows that $A(\eta)$ is in L^2. More precisely, one has

$$||A(\eta)||_{L^2} \leq ||\eta \wedge \widetilde{\phi}||_{L^2} \leq ||\phi||_{L^\infty} ||\eta||_{L^2}$$

so that A is a bounded operator which commutes with the action of Γ, i.e. it is a morphism of $L^2(\Gamma)$ modules. Since $\widetilde{\phi}$ is a closed differential form, it follows that A commutes with the de Rham differential, and so A is a morphism of $L^2(\Gamma)$ complexes. Let $\eta \in \Omega^k_{(2)}(\widetilde{N})$; then

$$\begin{aligned} A \circ \widetilde{\pi}^*(\eta) &= \widetilde{\pi}_*(\widetilde{\pi}^*\eta \wedge \widetilde{\phi}) \\ &= \eta \wedge \pi_*\phi = \eta \end{aligned}$$

That is, the composition $A \circ \widetilde{\pi}^*$ is the identity operator on $\Omega^k_{(2)}(\widetilde{N})$. By Theorem 1.1,

$$F_k(\lambda, \widetilde{N}) \prec\!\!\prec F_k(\lambda, \widetilde{S(N)})$$

for $k = 0, 1, ..., \dim N$.

Therefore we see that for $k = 0, 1, ..., \dim N = \dim M$, one has

$$F_k(\lambda, \widetilde{N}) \prec\!\!\prec F_k(\lambda, \widetilde{M}).$$

By symmetry, we deduce part (1) of the theorem.

An identical argument proves part (2) of the theorem where we now refer to [A] instead of [ES]. □

Remarks. (1) Let K be a triangulation of M and $\widetilde{K}$ the induced triangulation of the universal cover $\widetilde{M}$. By a theorem of Efremov, [E], $F_k(\lambda, \widetilde{K}) \sim F_k(\lambda, \widetilde{M})$ for all k. Now let $f : M \to N$ be a homotopy equivalence and K and L be triangulations of M and N respectively. Let $\widetilde{K}$ and $\widetilde{L}$ be the induced triangulations of the universal covers of $\widetilde{M}$ and $\widetilde{N}$ respectively. Then by standard topology [S], there is a simplicial map $g : K \to L$ which is a homotopy equivalence, and which induces a morphism of $L^2(\Gamma)$ complexes, $g^* : C^\bullet_{(2)}(\widetilde{L}) \to C^\bullet_{(2)}(\widetilde{K})$ (Γ denoting the fundamental group of M or N). Now we can apply Theorem 1.1 and deduce part (1) of the theorem.

(2) If $F_k(\lambda, \widetilde{M}) \sim \lambda^{\beta_k}$ then it follows from the above theorem that β_k is a homotopy invariant of M. It then follows that $N_k(\lambda, \widetilde{M}) \sim \lambda^{\alpha_k}$ where $\alpha_k = min(\beta_{k-1}, \beta_k)$ is a homotopy invariant of M. Via the Laplace transform, this is equivalent to a statement about the large time decay of the heat kernel of the Laplacian on L^2 k-forms on $\widetilde{M}$. More precisely, this is equivalent to the statement that if $\theta_k(t) - b^k_{(2)}(\widetilde{M})$ is bounded above and below by constant multiples of $t^{-\frac{\alpha_k}{2}}$,

then α_k is a homotopy invariant of M. Here $\theta_k(t)$ denotes the von Neumann trace [A] of the heat kernel of the Laplacian acting on $L^2 k$-forms on $\widetilde{M}$.

(3) The condition that the bottom of the spectrum of the Laplacian minus the projection to the L^2 harmonic j-forms on the universal cover $\widetilde{M}$ is zero can similarly be shown to be a homotopy invariant of M. The Novikov-Shubin invariants are interesting only when this condition holds.

Acknowledgments

The authors would like to thank Mel Rothenberg and Mikhail Shubin for reading the manuscript and for their helpful comments.

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