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Non-commutative tori and Fourier–Mukai duality

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Abstract

The classical Fourier–Mukai duality establishes an equivalence of categories between the derived categories of sheaves on dual complex tori. In this article we show that this equivalence extends to an equivalence between two dual objects. Both of these are generalized deformations of the complex tori. In one case, a complex torus is deformed formally in a non-commutative direction specified by a holomorphic Poisson structure. In the other, the dual complex torus is deformed in a B-field direction to a formal gerbe. We show that these two deformations are Fourier–Mukai equivalent.

Contents

1. Introduction

It is commonly believed that the most general deformations of a complex algebraic space X are captured by the deformations of some version of the category of coherent sheaves on X. Among the popular choices are the abelian category of coherent sheaves $\mathrm{Coh}(X)$, its derived category $D^b(X)$

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(see [Bon92, Kon91, Kon03]), or a dg-enhancement of the latter. In this paper we look at the deformations of the derived categories and the equivalences between them.

A particular family of infinitesimal deformations of $D^b(X)$ comes from deforming the identity functor on $D^b(X)$. This family is naturally parameterized by the second Hochschild cohomology $HH^2(X)$ of X (see [Bon92, Kel99]). By definition $HH^i(X)$ is the cohomology of the complex $R\operatorname{Hom}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$. If X is a manifold, the geometric version of the Hochschild–Kostant–Rosenberg theorem [GS87, Swa96, Kon03] identifies $HH^i(X)$ with the coherent cohomology of the holomorphic polyvector fields on X. In particular,

$$HH^2(X) \cong H^0\Big(X, \bigwedge^2 T_X\Big) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X). \tag{1.1}$$

Viewing $HH^2(X)$ as infinitesimal deformations of $D^b(X)$ we can interpret the pieces in (1.1) as follows. Elements in $H^1(X, T_X)$ correspond to deformations of X as a complex manifold. Elements in $H^0(X, \bigwedge^2 T_X)$ correspond to deforming the multiplication on $\mathcal{O}_X$ to a $\star$-product. Finally, elements in $H^2(X, \mathcal{O}_X)$ correspond to deforming the trivial $\mathcal{O}^\times$-gerbe on X.

Given two complex manifolds X and Y and an equivalence $\varphi : D^b(X) \to D^b(Y)$ one obtains a natural isomorphism $\widetilde{\varphi} : HH^2(X) \widetilde{\to} HH^2(Y)$. In particular, to every deformation direction $\xi \in HH^2(X)$ for $D^b(X)$ we can associate a deformation direction $\widetilde{\varphi}(\xi) \in HH^2(Y)$ for $D^b(Y)$. The problem we would like to investigate in general is whether the equivalence $\widetilde{\varphi}$ deforms along with $D^b(X)$ and $D^b(Y)$ in the directions ξ and $\widetilde{\varphi}(\xi)$.

In this paper we concentrate on the special case when X is a complex torus, $Y = X^\vee$ is the dual torus, and φ is the classical Fourier–Mukai equivalence. An interesting feature of this case is that $\widetilde{\varphi}$ exchanges the non-commutative deformations of X with the gerby deformations of Y and *vice versa*. Thus, the corresponding deformation of φ, if it exists, will have to exchange sheaves of different geometric origin. We carry out this program to show that φ deforms to an equivalence of the derived category of a formal non-commutative deformation of X and the derived category of a formal gerby deformation of $X^\vee$. Along the way we have to extend some of the classical theory of complex tori to non-commutative tori. In an attempt to make the exposition less cumbersome we have collected the required technical results and some generalizations in the appendices.

The fact that a gerby complex torus should be Fourier–Mukai equivalent to a non-commutative torus was originally conjectured by Orlov based on the behavior of the map $\widetilde{\varphi}$ and on physical considerations. Independently, Kapustin [Kap04] analyzed this setup as a duality transformation between D-branes in type II string theory. He studied branes on a complex manifold X in the presence of a B-field $\boldsymbol{B}$ which is a $\overline{\partial}$-closed $(0,2)$ form. In this case the branes admit two different interpretations. On the one hand they can be viewed as complexes of C^∞ complex vector bundles equipped with Hermitian connections D satisfying $F_D^{0,2} = \boldsymbol{B} \cdot \mathrm{id}$ and, on the other hand, they can be viewed as complexes of coherent sheaves on the topologically trivial holomorphic gerbe classified by $\exp(\boldsymbol{B}) \in H^2(X, \mathcal{O}^\times)$. On a complex torus Kapustin investigated how such branes will transform under T-duality. It is natural in this case to look for a Fourier transform that uses the Poincaré line bundle on the product of the torus and its dual. In order to set up such a transform for the first interpretation of $\boldsymbol{B}$-twisted branes, one needs to exhibit a Hermitian connection on the Poincaré line bundle whose $(0,2)$ curvature is $-\boldsymbol{B}$. Kapustin searched for conditions that would ensure the existence of such a connection. His calculations showed that if a connection exists, then the variables on the dual torus can no longer commute. He then gave a physical identification of the bundles on the $\boldsymbol{B}$-twisted torus and the bundles on the dual non-commutative torus. This led him to conjecture that the Fourier–Mukai transform deforms to a full equivalence of derived categories.

We are carrying out one interpretation of this conjecture. While Kapustin works in a differential geometric context, involving vector bundles and connections, we use the second interpretation of $\boldsymbol{B}$-twisted branes and hence work in an algebraic/complex geometric context, where our tools are sheaves of algebras and modules. In order to do this we are forced to work formally, that is, our non-commutative torus is a formal deformation quantization of a classical complex torus, and our gerbe is a formal deformation of the trivial $\mathcal{O}^{\times}$-gerbe. The reason we are forced to work formally is that by a theorem of Kontsevich [Kon01], the complex torus has no algebraic, or even semi-algebraic deformations. In order to obtain a duality statement which is not only formal but analytic, one needs a different point of view on the category of coherent sheaves on a torus. An appropriate formalism was recently developed by Block [Blo05, Blo06]. He gave an interpretation of the categories of coherent sheaves and their deformations as categories of twisted complexes over differential graded algebras. Using this formalism, he proved a duality statement compatible with both Kapustin's analysis and our formal duality.

As far as we know, ours is the first work on the derived categories of modules over a deformation quantization. This is one reason for the length of this article. We have to make sure that much of the standard yoga of $\mathcal{O}$-modules extends to deformations quantizations.

Relation to other works

Traditionally non-commutative tori are the fundamental testing ground for phenomena in non-commutative geometry. Their differential geometry and instanton theory has been studied extensively by Connes and Rieffel [Con94, CR87, Rie88].

Căldăraru [Căl02], Donagi and Pantev [DP03], and more recently Ben-Bassat [Ben06] extended the classical Fourier Mukai equivalence to families of tori (including some singular fibers). Gerbes also arise naturally here. In the case when the family has a section, the Fourier–Mukai duality is easily extended. When there are no sections, the natural dual family must be interpreted as a gerbe. Our situation is much the same. Our non-commutative torus can be viewed as a family over the formal disk. The well-known fact that the non-commutative torus has no points (that is, no quasi-coherent sheaves supported at points) manifests itself in the appearance of a gerbe on the dual side. In fact, by Gabber's theorem [Gab81], the support of a coherent sheaf of modules on the non-commutative torus must be coisotropic for the complex Poisson structure (and thus has a lower bound on the dimension of its support). On the dual side, the support of a coherent sheaf on the gerbe must be isotropic and thus has an upper bound on the dimension of its support.

Polishchuk and Schwarz investigated the geometry of holomorphic structures on non-commutative real two tori [PS03, Pol04, Pol05]. They also studied the categories of sheaves on the resulting non-commutative complex spaces. Our setup differs from theirs in that our Poisson structures are holomorphic whereas theirs are of type $(1,1)$. In particular, in their case the derived category does not deform in the non-commutative direction (the abelian category of sheaves does deform, however). A unifying approach to the most general holomorphic structures on non-commutative deformations is provided by [Blo05, Blo06].

Ours is not the first paper where non-commutative tori appear as duals of gerbes. In [MR05a, MR05b], Mathai and Rosenberg find that in some cases families of non-commutative tori appear as the duals of families of tori with a gerbe on the total space. Their context is topological and their main result is an isomorphism of topological K-theories of the two dual objects. In contrast, we work holomorphically and our result is an equivalence of the full derived categories of the two dual objects.

Recently, in a beautiful paper [Tod05], Toda proved a very general result, related to our work. He constructed for any smooth projective variety X and a Hochschild class as above, a first-order deformation of the derived category of coherent sheaves on X. He then showed that if there is an equivalence of derived categories between X and Y, then it deforms to the corresponding first-order deformations. It is not at all clear how to extend his results to infinite-order deformations in general. The main result of this paper can be viewed as such an extension in the case of complex tori.

Notation and terminology

We use the following notation and terminology in this article.

- $\mathscr{A}_X$, a sheaf of associative flat $\mathbb{C}[[\hbar]]$ algebras on X satisfying $\mathscr{A}/\hbar \cong \mathcal{O}_X$.
- $\mathbb{C}[[\hbar]]$, the complete local algebra of formal power series in $\hbar$.
- D^*, the derived category of $*$-complexes of $\mathcal{O}$-modules. The decoration $*$ is in the set $\{\varnothing, -, b\} = \{\text{unbounded}, \text{bounded above}, \text{bounded}\}$.
- $\mathsf{D}^*_{\mathrm{c}}$, $\mathsf{D}^*_{\mathrm{qc}}$, the derived categories of complexes having coherent and respectively quasi-coherent cohomologies, respectively.
- $\mathbb{D}$, the one-dimensional formal disk.
- $\mathbb{\Gamma}$, the Heisenberg group scheme $1 \to \mathbb{G}_m \to \mathbb{\Gamma} \to \mathbb{\Lambda}^\vee \to 0$ given by $\boldsymbol{B}$.
- $\Lambda \subset V$, a free abelian subgroup of rank $2g$.
- $\Lambda^\vee \subset \overline{V}$, the lattice of all $\xi \in \overline{V}^\vee$ satisfying $\mathrm{Im}(\langle \xi, \lambda \rangle) \in \mathbb{Z}$ for all $\lambda \in \Lambda$.
- $\mathbb{\Lambda}$, $\mathbb{\Lambda}^\vee$, the constant group schemes $\Lambda \times \mathbb{D}$ and $\Lambda^\vee \times \mathbb{D}$ over $\mathbb{D}$.
- $\mathcal{P}$, the normalized Poincaré line bundle on $X \times X^\vee$.
- $\mathcal{P}_{|X \times \{\alpha\}}$, the degree zero line bundle $\mathcal{P}_{|X \times \alpha}$ on X.
- $\boldsymbol{\Pi}$, a holomorphic Poisson structure on a complex manifold.
- $\phi_K^{[X \to Y]}$, the integral transform $\mathsf{D}^*(X) \to \mathsf{D}^*(Y)$ corresponding to a kernel object $K \in \mathsf{D}^*(X \times Y)$.
- $\mathcal{S}_X$, $\mathcal{S}_{X^\vee}$, the Fourier–Mukai transforms $\mathsf{D}^b(X) \to \mathsf{D}^b(X^\vee)$ and $\mathsf{D}^b(X^\vee) \to \mathsf{D}^b(X)$, respectively.
- V, a complex vector space of dimension g.
- $\overline{V}^\vee$, the complex space of conjugate linear homomorphisms from V to $\mathbb{C}$.
- $\mathbb{V}$, $\overline{\mathbb{V}}^\vee$, the formal spaces $V \times \mathbb{D}$ and $\overline{V}^\vee \times \mathbb{D}$.
- X, a complex torus of dimension g, typically $X = V/\Lambda$
- $X^\vee$, the dual complex torus of X, i.e. $X^\vee = \mathrm{Pic}^0(X) = \overline{V}^\vee/\Lambda^\vee$.
- $\mathbb{X}$, $\mathbb{X}^{\mathrm{op}}$, a formal non-commutative space $\mathbb{X} = (X, \mathscr{A}_X)$ and its opposite $\mathbb{X}^{\mathrm{op}} = (X, \mathscr{A}_X^{\mathrm{op}})$
- $\mathbb{X}_{\boldsymbol{\Pi}}$, the Moyal quantization of the Poisson torus $(X, \boldsymbol{\Pi})$.
- $\mathbb{X}^\vee$, the formal space $X^\vee \times \mathbb{D}$.

2. The classical situation

First let us recall the basic properties of complex tori that we will need. For a more detailed discussion of the properties of complex tori the reader may consult [Mum70, BL99, Pol03].

A complex torus is a compact complex manifold X which is isomorphic to a quotient V/Λ, where V is a g-dimensional complex vector space and $\Lambda \subset V$ is a free abelian subgroup of rank $2g$. Note that by construction X has a natural structure of an analytic group induced from the addition law on the vector space V.

To any complex torus X one can associate a dual complex torus $X^\vee$. If X is realized as V/Λ, the dual torus is defined to be $X^\vee = \overline{V}^\vee/\Lambda^\vee$. Here $\overline{V}^\vee$ denotes the space of conjugate linear homomorphisms from V to $\mathbb{C}$ and $\Lambda^\vee \subset \overline{V}^\vee$ is the lattice defined by

$$\Lambda^\vee = \{\xi \in \overline{V}^\vee \mid \mathrm{Im}(\xi(\lambda)) \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda\}.$$

In fact, the dual torus $X^\vee$ is intrinsically attached to X and does not depend on the realization of X as a quotient. Namely, one can define $X^\vee$ as the torus $\mathrm{Pic}^0(X)$ parameterizing all holomorphic line bundles $L \to X$ which are invariant for the translation action of X on itself. Equivalently, these are the holomorphic line bundles with the property $c_1(L) = 0 \in H^2(X, \mathbb{Z})$. It is known (see, e.g., [Mum70, Pol03]) that $X^\vee$ is a fine moduli space in the sense that we can find a line bundle

$$\mathcal{P} \to X \times X^\vee$$

with the property that for any $\alpha \in X^\vee$, the restriction $\mathcal{P}_{|X\times\{\alpha\}}$ is isomorphic to the degree zero line bundle corresponding to α. We write $\mathcal{P}_\alpha$ for the line bundle $\mathcal{P}_{|X\times\{\alpha\}}$. Such a $\mathcal{P}$ is called a Poincaré line bundle. If we further normalize $\mathcal{P}$ so that $\mathcal{P}_{|\{o\}\times X^\vee}$ is isomorphic to the trivial line bundle $\mathcal{O}_{X^\vee}$, then $\mathcal{P}$ is uniquely determined. Furthermore, the assignment $\alpha \mapsto \mathcal{P}_\alpha$ for the normalized Poincaré line bundle is compatible with group structures, e.g.

$$\mathcal{P}_{\alpha+\beta} \cong \mathcal{P}_\alpha \otimes \mathcal{P}_\beta.$$

Also, note that the normalized Poincaré sheaf gives rise to a canonical isomorphism $X^{\vee\vee} \xrightarrow{\sim} X$ (see [BL99]).

The main interest of this paper is a generalization of a powerful duality theorem of Mukai [Muk81, HV07]. First we need to set things up. For a complex manifold M, let $\mathsf{D}^b(M)$ be the bounded derived category of sheaves of $\mathcal{O}_M$-modules [Ver96]. An object in $\mathsf{D}^b(M)$ is a bounded complex

$$\cdots \to \mathscr{F}_{i-1} \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \cdots$$

of analytic sheaves of $\mathcal{O}_M$-modules. The morphisms are more complicated to define; see, e.g., [Ver96, GM96].

Given two compact complex manifolds M and N and an element $K \in \mathsf{D}^b(M \times N)$, define the integral transform

$$\phi_K^{[M\to N]} : \mathsf{D}^b(M) \to \mathsf{D}^b(N)$$

by

$$\phi_K^{[M\to N]}(G) = Rp_{N*}(p_M^* G \otimes^{\mathbb{L}} K).$$

The integral transform has the following convolution property (see [Muk81] or [Pol03, Proposition 11.1]). If M, N, and P are compact complex manifolds and $K \in \mathsf{D}^b(M\times N)$ and $L \in \mathsf{D}^b(N\times P)$, then one has a natural isomorphism of functors

$$\phi_L^{[N\to P]} \circ \phi_K^{[M\to N]} \cong \phi_{K*L}^{[M\to P]},$$

where

$$K * L = Rp_{M\times P*}(p_{M\times N}^* K \otimes^{\mathbb{L}} p_{N\times P}^* L) \in \mathsf{D}^b(M \times P),$$

and $p_{M\times N}$, $p_{M\times P}$, $p_{N\times P}$ are the natural projections $M \times N \times P \to M \times N$, etc.

For a complex torus X, the Poincaré sheaf on $X \times X^\vee$ provides natural integral transforms

$$\mathcal{S}_X := \phi_{\mathcal{P}}^{[X\to X^\vee]} : \mathsf{D}^b(X) \to \mathsf{D}^b(X^\vee)$$
$$\mathcal{S}_{X^\vee} := \phi_{\mathcal{P}}^{[X^\vee\to X]} : \mathsf{D}^b(X^\vee) \to \mathsf{D}^b(X).$$

Now we are ready to state Mukai's duality theorem. Since we were unable to find a reference treating this duality in the analytic context, we have sketched below the necessary modifications of Mukai's original proof [Muk81, Pol03].

THEOREM 2.1. *The integral transform* $\mathcal{S} : \mathsf{D}^b(X^\vee) \to \mathsf{D}^b(X)$ *is an equivalence of triangulated categories.*

Proof. The theorem follows from the existence of natural isomorphisms of functors:

$$\begin{aligned} \mathcal{S}_{X^\vee} \circ \mathcal{S}_X &\cong (-1)_X^*[-g] \\ \mathcal{S}_X \circ \mathcal{S}_{X^\vee} &\cong (-1)_{X^\vee}^*[-g], \end{aligned} \tag{2.1}$$

where $g = \dim_{\mathbb{C}} X$.

The existence of (2.1) follows from the identification $X \cong X^{\vee\vee}$ and the fact that the canonical adjunction morphism

$$\mathrm{Id} \to \phi^{[X\to X^\vee]}_{\mathcal{P}^{-1}[g]} \circ \phi^{[X^\vee \to X]}_{\mathcal{P}} \tag{2.2}$$

is an isomorphism. This is a formal consequence [Pol03, Theorem 11.4] of the cohomology and base change theorem for proper morphisms and of the fact that for a non-trivial holomorphic line bundle $L \in \mathrm{Pic}^0(X)$ one has $H^\bullet(X, L) = 0$. The cohomology and base change theorem for pushing forward a flat coherent sheaf under a proper analytic morphism can be found in [Sch72]. To show that $H^\bullet(X, L) = 0$ for a non-trivial $L \in \mathrm{Pic}^0(X)$ we first note that a degree zero holomorphic line bundle on X corresponds to a complex rank one local system $\mathbb{L}$ on X (see [Ati57]). Since every complex torus is Kähler, it follows that the natural map

$$H^k(X, \mathbb{L}) \to H^k(X, L)$$

is surjective for each k. If L is non-trivial, then so is $\mathbb{L}$ and so it corresponds to a non-trivial character $\mathbb{Z}^{2g} \to \mathbb{C}^\times$. Now by using the isomorphism $X \cong (S^1)^{2g}$, the Künneth formula, and the vanishing of the cohomology of $\mathbb{Z}$ with coefficients in a non-trivial character, we conclude that $H^k(X, \mathbb{L}) = 0$ for all k. □

3. Non-commutative complex tori and *B*-fields

In this section we introduce the relevant non-commutative and gerby deformations of the two sides of the Fourier–Mukai equivalence. The deformation of the equivalence itself is studied in the next section.

3.1 Non-commutative complex tori

Before we can extend the formalism of integral transforms to the realm of non-commutative geometry we need to make precise the notion of non-commutative space that we will be using. In the next section we introduce the first main player in our correspondence, the deformation quantization of a complex torus.

3.1.1 *Deformation quantization in the holomorphic setting.* Recall that a *deformation quantization* (see, e.g., [BFFLS77, NT01, Vai02]) of a complex analytic space X is a formal one-parameter deformation of the structure sheaf $\mathcal{O}_X$. Explicitly this means that we are given a sheaf $\mathscr{A}_X$ of associative unital $\mathbb{C}[[\hbar]]$-algebras, flat over $\mathbb{C}[[\hbar]]$, together with an algebra isomorphism

$$\mathscr{A}_X \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathcal{O}_X.$$

Note that usually one requires that $\mathscr{A}_X$ is not only flat but is moreover topologically free. We prefer the more relaxed flatness condition because it is better suited to the understanding of the category of quantizations, in particular for studying fiber products. This is a minor point which has only conceptual value and does not affect our considerations. In fact, all of the deformation quantizations we encounter in this paper will automatically be topologically free, because they are $\star$-quantizations (see Definition 3.1).

Geometrically one thinks of the data $(X, \mathscr{A}_X)$ as a non-commutative formal deformation of X over the one-dimensional formal disk $\mathbb{D} := (\{o\}, \mathbb{C}[[\hbar]]) = \mathrm{Spf}(\mathbb{C}[[\hbar]])$. In other words the data $(X, \mathscr{A}_X)$ should be viewed as defining a non-commutative formal space $\mathbb{X}$ which is equipped with a morphism $\mathbb{u} : \mathbb{X} \to \mathbb{D}$ and which specializes to X over the closed point $o \in \mathbb{D}$:

$$\begin{tikzcd} X \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{X} \arrow[d, "\mathbb{u}"] \\ o \arrow[r, phantom, "\in"] & \mathbb{D} \end{tikzcd}$$

Observe also that every formal non-commutative space $\mathbb{X} = (X, \mathscr{A}_X)$ has a natural companion $\mathbb{X}^{\mathrm{op}} := (X, \mathscr{A}_X^{\mathrm{op}})$ which has the same underlying analytic space but is equipped with the opposite sheaf of algebras.

A *morphism* $\mathbb{f} : \mathbb{X} \to \mathbb{Y}$ *of deformation quantizations* is defined to be a morphism between ringed spaces $(f, f^\sharp) : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$, so that $f^\sharp : f^{-1}\mathscr{A}_Y \to \mathscr{A}_X$ is continuous in the $\hbar$-adic topology and the induced morphism $(f, f^\sharp/\hbar) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of complex analytic spaces. Note that the category $\mathscr{F}\mathscr{S}/\mathbb{D}$ of formal analytic spaces over $\mathbb{D}$ is equivalent to the category of all commutative deformation quantizations.

A particularly tractable class of deformation quantizations $\mathbb{X}$ are the so-called $\star$-quantizations (see, e.g., [NT01], [BK04], and [Yek03, Definition 8.6]).

Definition 3.1. A *$\star$-quantization* $(X, \mathscr{A}_X)$ of a complex manifold X is a sheaf $\mathscr{A}_X$ of $\mathbb{C}[[\hbar]]$-algebras, which is flat over $\mathbb{C}[[\hbar]]$ and for which the following hold.

(a) There is an isomorphism $\varphi : \mathscr{A}_X/\hbar \widetilde{\to} \mathcal{O}_X$ of $\mathbb{C}[[\hbar]]$-algebras.

(b) Locally on X we have isomorphisms $\psi_U : \mathscr{A}_X|_U \widetilde{\to} \mathcal{O}_X|_U[[\hbar]]$ of sheaves of $\mathbb{C}[[\hbar]]$-modules, so that:

- the ψ_U are all compatible with φ;
- under this isomorphism ψ_U, $1_{\mathscr{A}_X}$ maps to $1_{\mathcal{O}_X}$ and the product on $\mathscr{A}_X|_U$ becomes a product $\star$ on $\mathcal{O}_X|_U[[\hbar]]$ so that for all $a, b \in \mathcal{O}_X|_U$ we have
$$a \star b = ab + \sum_{i=1}^{\infty} \beta_i(a,b)\hbar^i$$
with $\beta_i : \mathcal{O}_X|_U \otimes_{\mathbb{C}} \mathcal{O}_X|_U \to \mathcal{O}_X|_U$ being bidifferential operators;
- the composition maps $\psi_U \circ {\psi_V}^{-1}$ are given by a series in $\hbar$ of differential operators $\mathcal{O}_X|_{U\cap V} \to \mathcal{O}_X|_{U\cap V}$.

Remark 3.2. It is a consequence of the definition that for all $a, b \in \mathcal{O}_X|_U[[\hbar]]$ we have $a \star b \equiv ab \bmod \hbar$ and $\beta_i(1, a) = \beta_i(a, 1) = 0$. Also, the term of order zero in $\hbar$ of the series of differential operators which give the transition isomorphisms is the identity map on the sheaf $\mathcal{O}_X|_{U\cap V}$.

A *morphism between $\star$-quantizations* $\mathbb{f} : \mathbb{X} \to \mathbb{Y}$ is defined to be a morphism between ringed spaces $(f, f^\sharp) : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ for which $f^\sharp : f^{-1}\mathscr{A}_Y \to \mathscr{A}_X$ is continuous in the $\hbar$-adic topology and such that the induced morphism $(f, f^\sharp/\hbar) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of complex analytic spaces. Furthermore, we assume that $f^\sharp$ is *given by differential operators with respect to* $(f, f^\sharp/\hbar)$ in the following sense.

Suppose that $(g, g^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a complex analytic morphism. Then a differential operator of order 0 with respect to $(g, g^\sharp)$ is defined to be a map $g^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ given locally by $a \mapsto k \cdot (a \circ g)$ for some function $k \in \mathcal{O}_X$. A differential operator of order j with respect to $(g, g^\sharp)$ is defined inductively to be a $\mathbb{C}$-linear map $D : g^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ for which the assignment $a \mapsto D(g^{-1}(q) \cdot a) - q \cdot D(h)$ is a differential operator of order $j - 1$.

Consider, for any structure maps $\psi_U : \mathscr{A}_X|_U \widetilde{\to} \mathcal{O}_{X,U}[[\hbar]]$ and $\psi_W : \mathscr{A}_Y|_W \widetilde{\to} \mathcal{O}_{Y,W}[[\hbar]]$, the components of the composition
$$\psi_U \circ f^\sharp \circ f^{-1}({\psi_W}^{-1})|_{f^{-1}(W)\cap U} : f^{-1}\mathcal{O}_{Y,W}[[\hbar]]|_{f^{-1}(W)\cap U} \to \mathcal{O}_{X,U}[[\hbar]]|_{f^{-1}(W)\cap U}$$
as maps from $f^{-1}\mathcal{O}_{Y,W}|_{f^{-1}(W)\cap U}$ to $\mathcal{O}_{X,U}|_{f^{-1}(W)\cap U}$. The degree 0 part is just given by $f^\sharp/\hbar$, the pullback of functions, so it is a differential operator of order zero with respect to $(f, f^\sharp/\hbar)$. We say that $\mathbb{f}$ is given by differential operators when all of these components are differential operators with respect to $(f, f^\sharp/\hbar)$.

Remark 3.3. Every $\star$-quantization $(X, \mathscr{A}_X)$ induces a holomorphic Poisson structure $\mathbf{\Pi}$ on X defined by the formula

$$(df \wedge dg) \lrcorner \, \mathbf{\Pi} = \frac{1}{2\hbar}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f}) \bmod \hbar,$$

for local sections f and g in $\mathcal{O}_X$ and lifts $\tilde{f}, \tilde{g} \in \mathscr{A}_X$ of f and g. A morphism of $\star$-quantizations automatically induces a Poisson morphism. This remark remains true in the more general context of deformation quantizations.

Example 3.4. The basic example of a $\star$-quantization is the standard Moyal product on the holomorphic functions on a complex vector space V equipped with a constant Poisson structure $\mathbf{\Pi}$ (see [Moy49, BFFLS78]). By the constancy assumption, there are complex coordinates

$$(q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_l)$$

on V so that the Poisson structure is diagonal, that is

$$\mathbf{\Pi} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

Recall that a bidifferential operator on complex manifold X is a $\mathbb{C}$-linear map $\varphi : \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X$ which is a differential operator in each factor, i.e. for all $g \in \mathcal{O}_X$ we have $\varphi(\bullet \otimes g) \in \mathscr{D}_X$ and $\varphi(g \otimes \bullet) \in \mathscr{D}_X$. Given a differential operator $D \in \mathscr{D}_X$ we can promote it to a bidifferential operator in two ways with D acting on the first and the second factor, respectively. As usual we write $\overleftarrow{D}$ for the bidifferential operator $D \otimes \mathrm{id}$ and $\overrightarrow{D}$ for the bidifferential operator $\mathrm{id} \otimes D$. Note that the assignment $D \to \overrightarrow{D}$ is an algebra homomorphism, whereas the assignment $D \to \overleftarrow{D}$ is an algebra antihomomorphism.

With this notation we can now use $\mathbf{\Pi}$ to define the bidifferential operator P by

$$P = \sum_i \left(\overleftarrow{\frac{\partial}{\partial q_i}} \overrightarrow{\frac{\partial}{\partial p_i}} - \overleftarrow{\frac{\partial}{\partial p_i}} \overrightarrow{\frac{\partial}{\partial q_i}} \right). \tag{3.1}$$

Consider the sheaf $\mathcal{O}_V[[\hbar]]$ on V. For any open set $U \subset V$ and any $f, g \in \mathcal{O}_U[[\hbar]]$ we define their Moyal product

$$f \star g = \sum_k \frac{\hbar^k}{k!} f \cdot P^k \cdot g = f \cdot \exp(\hbar P) \cdot g = fg + \hbar\{f, g\} + \cdots .$$

Since the $\star$-product is defined by holomorphic bidifferential operators it maps holomorphic functions to holomorphic functions. Moreover, since bidifferential operators are local, the product sheafifies. We denote the resulting sheaf of $\mathbb{C}[[\hbar]]$-algebras on V by $\mathscr{A}_{V,\mathbf{\Pi}}$.

3.1.2 *Functional-analytic considerations.* For future reference we note that for a quantization, the sheaf $\mathscr{A}_X \to X$ is naturally a sheaf of multiplicatively convex nuclear Frechet algebras. Indeed, for a small enough open $U \subset X$ we can topologize $\mathscr{A}_X(U)$ by identifying it as a sheaf of vector spaces with $\mathcal{O}(U)[[\hbar]]$ and using the uniform topology on $\mathcal{O}(U)$ over compact subsets $K \subset U$ and the $\hbar$-adic topology on $\mathbb{C}[[\hbar]]$. It is well known [Trè67] that both the uniform topology on holomorphic functions and the $\hbar$-adic topology on $\mathbb{C}[[\hbar]]$ are nuclear Frechet and so their completed tensor product is also nuclear Frechet. To check that a Frechet algebra is multiplicatively convex, we also need to show the existence of a countable family of semi-norms p_n, satisfying $p_n(a \star b) \leqslant p_n(a)p_n(b)$. Choose an exhaustion $\{K_n\}_{n=0}^{\infty}$ of U by nested compact subsets. Given a local section $f = \sum_n f_n \hbar^n \in \mathscr{A}_U \cong \mathcal{O}(U)[[\hbar]]$

we define

$$p_n(f) := c_n \sum_{i=0}^{n} \sup_{x \in K_i} |f_i(x)|,$$

where $c_n \in \mathbb{R}_{>0}$ is an appropriately chosen normalization constant. With this definition one checks immediately that the p_n are multiplicatively convex seminorms.

Caution. The standard example of a nuclear Frechet algebra which is not multiplicatively convex is the Weyl algebra of a symplectic vector space. This is because the Weyl relation $[x, y] = 1$ can never hold in a Banach algebra. In the formal setting, however, we are saved by the fact that the relations are of the form $[x, y] = \hbar$ and $\hbar$ is a quasi-nilpotent element in our algebra.

The multiplicative convexity property of $\mathscr{A}_U$ is a key ingredient in the analogue of Grauert's direct image theorem in the context of formal deformation quantization. This theorem is instrumental in setting up integral transforms between the coherent derived categories.

3.1.3 *Formal Moyal products on the non-commutative torus.* Let $X = V/\Lambda$ be a complex torus, with a holomorphic Poisson structure $\mathbf{\Pi}$. Since the holomorphic tangent bundle of a complex torus is trivial, the bitensor $\mathbf{\Pi} \in H^0(X, \bigwedge^2 T_X)$ will necessarily be translation invariant and hence will be of constant rank on X. The formal $\star$-quantizations of a complex manifold equipped with a Poisson structure of constant rank are known to be parameterized [NT01, BK04, Yek03] by an affine space. This affine space is modeled on $F^2[[\hbar]]$, where F^2 is the second step of the Hodge filtration on the second de Rham cohomology of the symplectic Lie algebroid given by the sheaf of holomorphic vector fields tangent to the leaves of the Poisson foliation. In the case of a Poisson complex torus $(X, \mathbf{\Pi})$ the picture simplifies since one can use the Moyal product to construct a canonical point in the moduli space of quantizations of $(X, \mathbf{\Pi})$. We call this point *the Moyal quantization* of $(X, \mathbf{\Pi})$. The Moyal quantization is very concrete and easier to work with than the general constructions found, for example, in [Kon01, NT01, BK04, Yek03]. Since all of the essential features of the Fourier–Mukai duality are already present in the context of Moyal deformations, we chose to work mainly in this context.

To define the Moyal quantization $(X, \mathscr{A}_{X,\mathbf{\Pi}})$ of a holomorphic Poisson torus $(X, \mathbf{\Pi})$ we use the realization of X as a quotient $X = V/\Lambda$. Let $\pi : V \to X$ be the covering projection. Define the sheaf $\mathscr{A}_{X,\mathbf{\Pi}}$ of $\mathbb{C}[[\hbar]]$-algebras on X as follows. As a sheaf of $\mathbb{C}_X[[\hbar]]$-modules it will just be $\mathcal{O}_X[[\hbar]]$. To put a $\star$-product on this sheaf one only has to use the natural identification $\mathcal{O}_X[[\hbar]] := (\pi_*\mathcal{O}_V)^\Lambda$ and note that the $\mathbf{\Pi}$-Moyal product on V is translation invariant by construction. Explicitly, the sections of $\mathscr{A}_{X,\mathbf{\Pi}}$ over $U \subset X$ can be described as the invariant sections

$$\mathscr{A}_{X,\mathbf{\Pi}}(U) = \mathscr{A}_{V,\mathbf{\Pi}}(\pi^{-1}(U))^\Lambda \tag{3.2}$$

on the universal cover V. This is well defined since the Poisson structure $\mathbf{\Pi}$ is constant and thus the operator P is translation invariant.

3.1.4 *The group structure on non-commutative tori.* For understanding the convolution of sheaves on $\mathbb{X}_{\mathbf{\Pi}}$ it will be useful to have a lift of the group structure on X to a group law on $\mathbb{X}_{\mathbf{\Pi}}$. In contrast with the commutative case, we can not hope for the multiplication to live on a single non-commutative torus. This is because the multiplication on the commutative torus is not a Poisson map. However, this problem can be easily rectified if we replace the torus by X by $X \times \mathbb{Z}$ equipped with the Poisson structure which on the component $X \times \{k\}$ is $k\mathbf{\Pi}$.

In this approach we view $\mathbb{X}_{\mathbf{\Pi}}$ as a connected component of a non-commutative space

$$\aleph_{\mathbf{\Pi}} = \coprod_{k \in \mathbb{Z}} \mathbb{X}_{k\mathbf{\Pi}}.$$

This is a deformation quantization of the complex Poisson manifold

$$\Big(X \times \mathbb{Z}, \coprod_{k\in\mathbb{Z}} k\mathbf{\Pi}\Big).$$

The group structure on the space $\aleph_{\mathbf{\Pi}}$ is given by a map $\mathbb{m} : \aleph_{\mathbf{\Pi}} \times_{\mathbb{D}} \aleph_{\mathbf{\Pi}} \to \aleph_{\mathbf{\Pi}}$. Viewed as a map of ringed spaces $\mathbb{m}$ is a pair $\mathbb{m} = (m, m^\sharp)$ where m is the product group law on $X \times \mathbb{Z}$ and $m^\sharp = \{m^\sharp_{(a,b)}\}_{(a,b)\in\mathbb{Z}\times\mathbb{Z}}$ where

$$m^\sharp_{(a,b)} : m^{-1}_{(a,b)}\mathscr{A}_{(a+b)\mathbf{\Pi}} \to p_1^{-1}\mathscr{A}_{a\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{b\mathbf{\Pi}},$$

where $m_{(a,b)}$ is the natural addition map from $(X \times \{a\}) \times (X \times \{b\}) \to (X \times \{a + b\})$. For future reference we will write $\mathbb{m}_{(a,b)}$ for the component $(m_{(a,b)}, m^\sharp_{(a,b)})$. To define the group structure, we have to define the map $m^\sharp$ and verify that it gives a Hopf algebra structure on the structure sheaf of $\aleph_{\mathbf{\Pi}}$. Since the structure sheaf of $\aleph_{\mathbf{\Pi}}$ descends from the cover $V \times \mathbb{Z}$ it suffices to define the map $m^\sharp$ there.

Recall that on V the sheaf $\mathscr{A}_{k\mathbf{\Pi}}$ is simply the sheaf $\mathcal{O}_V[[\hbar]]$ equipped with the Moyal product $\star_{k\mathbf{\Pi}}$. Let $f \in \mathscr{A}_{(a+b)\mathbf{\Pi}}$, that is, f is a locally defined holomorphic function on V with values in $\mathbb{C}[[\hbar]]$. Now define

$$m^\sharp_{(a,b)}(m^{-1}f)(v_1, v_2) = f(v_1 + v_2).$$

Here $m^\sharp$ is thought of as a map $m^\sharp_{(a,b)} : m^{-1}_{(a,b)}\mathcal{O}_V[[\hbar]] \to p_1^{-1}\mathcal{O}_V[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_V[[\hbar]]$, and we use the fact that our completed tensor product $p_1^{-1}\mathcal{O}_V[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_V[[\hbar]]$ is naturally identified with $\mathcal{O}_{V\times V}[[\hbar]]$. Note that this product descends to the torus since the addition on V intertwines with the covering actions. Note that with this definition $m^\sharp$ is a coassociative and cocommutative coproduct. We now verify that $m^\sharp$ is a map of algebras.

Proposition 3.5. *The coproduct*

$$m^\sharp : m^{-1}\mathscr{A}_{\aleph_{\mathbf{\Pi}}} \to p_1^{-1}\mathscr{A}_{\aleph_{\mathbf{\Pi}}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\aleph_{\mathbf{\Pi}}}$$

is a morphism of sheaves of rings.

Proof. To check this, we need that for any $f = \{f_a\}$ and $g = \{g_b\}$ with $f_a, g_b \in \mathcal{O}_V[[\hbar]]$ we have

$$m^\sharp(m^{-1}f \star m^{-1}g)_{(a,b)}(v_1, v_2) = (f_{a+b} \star_{(a+b)\mathbf{\Pi}} g_{a+b})(v_1 + v_2). \tag{3.3}$$

To check the property (3.3) we use the fact that the $\star$-products on the different components of $\aleph_{\mathbf{\Pi}}$ are all Moyal products built out of Poisson structures that are proportional to $\mathbf{\Pi}$. Since $\mathbf{\Pi}$ is a constant Poisson structure we can choose a system $\{p_1, \ldots, p_n, q_1, \ldots q_n, c_1, \ldots, c_l\}$ of linear coordinates on the vector space V so that

$$\mathbf{\Pi} = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

Let

$$P = \sum_{i=1}^{n} \left(\overleftarrow{\frac{\partial}{\partial q_i}}\, \overrightarrow{\frac{\partial}{\partial p_i}} - \overleftarrow{\frac{\partial}{\partial p_i}}\, \overrightarrow{\frac{\partial}{\partial q_i}} \right).$$

The Moyal product $\star_{k\mathbf{\Pi}}$ is given by

$$f \star_{k\mathbf{\Pi}} g = f \exp(k\hbar P) g$$

for any two local sections f and g in $\mathcal{O}_V[[\hbar]]$. Now, viewing $m^\sharp(f)$ and $m^\sharp(g)$ as local sections in $\mathcal{O}_V[[\hbar]]$ we have $m^\sharp(f)(v_1, v_2) = f(v_1 + v_2)$ and $m^\sharp(g)(v_1, v_2) = g(v_1 + v_2)$. In these terms (3.3) becomes

$$m^\sharp(f\exp((a+b)\hbar P)g) = m^\sharp(f)\exp(\hbar(aP\otimes 1 + 1\otimes bP))m^\sharp(g).$$

This relation is a simple consequence of the chain rule as follows.

Consider a bi-differential operator L with constant coefficients on a vector space V. Denote the addition map on V by $a : V\times V\to V$. Applying the chain rule to differentiation of the functions $m^\sharp(f) = f\circ a$, and $m^\sharp(g) = g\circ a$ with respect to coordinates on the first or second copy of V we get

$$m^\sharp(f)\cdot(L\otimes 1)\cdot m^\sharp(g) = m^\sharp(f\cdot L\cdot g) = m^\sharp(f)\cdot(1\otimes L)\cdot m^\sharp(g).$$

Thus, we have established a group structure on $\aleph_{\mathbf{\Pi}}$, which completes the proof of the proposition. ☐

The usual definition of the antipode map of a (not necessarily commutative) Hopf algebra allows us to define the inversion on the group space $\aleph_{\mathbf{\Pi}}$. It is given by a morphism of ringed spaces $\aleph_{\mathbf{\Pi}}\to\aleph_{\mathbf{\Pi}}^{\mathrm{op}}$ represented by a pair $\mathbbm{inv} = (\mathrm{inv}, \mathrm{inv}^\sharp)$. Here $\mathrm{inv} : X\times\mathbb{Z}\to X\times\mathbb{Z}$ is the group inversion and the antipode $\mathrm{inv}^\sharp = \{\mathrm{inv}_a^\sharp\}_{a\in\mathbb{Z}}$ where inv_a is the algebra isomorphism

$$\mathrm{inv}_a : \mathrm{inv}^{-1}\mathscr{A}_{a\mathbf{\Pi}}\to\mathscr{A}_{-a\mathbf{\Pi}}^{\mathrm{op}}.$$

Similarly to the definition of $m^\sharp$, it suffices to define $\mathrm{inv}_a^\sharp$ on V. Again we identify the sheaves $\mathscr{A}_{a\mathbf{\Pi}}$ and $\mathscr{A}_{-a\mathbf{\Pi}}^{\mathrm{op}}$ with $\mathcal{O}[[\hbar]]$ and define

$$\mathrm{inv}_a^\sharp(\mathrm{inv}^{-1}f)(v) := f(-v)$$

for any local section $f\in\mathscr{A}_{a\mathbf{\Pi}}$ viewed as a $\mathbb{C}[[\hbar]]$-valued locally defined holomorphic function on V. The same reasoning as in the proof of Proposition 3.5 now implies that $\mathrm{inv}_a^\sharp$ is a ring homomorphism.

This concludes our general discussion on the properties of the formal non-commutative torus $\mathbb{X}_{\mathbf{\Pi}}$. The Moyal quantization $\mathbb{X}_{\mathbf{\Pi}} := (X, \mathscr{A}_{X,\mathbf{\Pi}})$ of the torus X is the first geometric input in the non-commutative Fourier–Mukai duality. As we mentioned above, it is instructive to view the ringed space $\mathbb{X}_{\mathbf{\Pi}}$ as a formal deformation of the complex manifold X:

$$\begin{tikzcd} X \arrow[d] \arrow[r, phantom, "\subset"] & \mathbb{X}_{\mathbf{\Pi}} \arrow[d, "\mathbbm{u}_{\mathbf{\Pi}}"] \\ o \arrow[r, phantom, "\in"] & \mathbb{D} \end{tikzcd}$$

which is parameterized by the one-dimensional formal disk $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[\hbar]])$. Now this suggests that the non-commutative duality we seek should be thought of as a formal deformation of the usual Mukai equivalence (see Theorem 2.1) between the derived categories of the dual complex tori X and $X^\vee$. Thus, we need to identify a dual object for $\mathbb{X}_{\mathbf{\Pi}}$ which is again defined over $\mathbb{D}$ and which specializes to $X^\vee$ at the closed point $o\in\mathbb{D}$.

3.2 Gerby complex tori

The first clue for what the dual object should be comes from the fact that to first order in the formal parameter $\hbar$ this dual object should again be determined by the Poisson structure $\mathbf{\Pi}$. It turns out that the correct dual object is an $\mathcal{O}^\times$-gerbe on the formal space $X^\vee\times\mathbb{D}$ which restricts to the trivial $\mathcal{O}^\times$-gerbe on the reduced space $X^\vee\times\{o\}$.

On the infinitesimal level this can be motivated as follows. As explained in § 3.1.1, the tangent space to the moduli of $\star$-deformations of X is $H^0(X, \bigwedge^2 T_X)$. However, since X is a complex torus, its holomorphic tangent bundle is trivial and so we have an identification

$$H^0\Big(X, \bigwedge^2 T_X\Big) = \bigwedge^2 T_{X,0} = \bigwedge^2 V. \tag{3.4}$$

On the other hand, recall (see [GH94]) that if Y is a complex torus with universal cover W, then the Dolbeault cohomology group $H_{\bar\partial}^{p,q}(Y) = H^q(Y, \Omega_Y^p)$ can be naturally identified with the vector space $\bigwedge^p W^\vee \otimes \bigwedge^q \overline{W}^\vee$. In particular, the cohomology space $H^{0,2}_{\bar\partial}(Y) = H^2(Y, \mathcal{O}_Y)$ is naturally identified with $\bigwedge^2 \overline{W}^\vee$. Applying this comment to the torus $X^\vee = \overline{V}^\vee/\Lambda^\vee$ gives an identification

$$H^2(X^\vee, \mathcal{O}) = \bigwedge^2 V. \tag{3.5}$$

Thus, we get an identification of the tangent space to the moduli of $\star$-deformations of X and the space $H^2(X^\vee, \mathcal{O})$, which in turn can be viewed as the tangent space to deformations of $X^\vee$ as an $\mathcal{O}^\times$-gerbe. One can check that the identification

$$H^0\Big(X, \bigwedge^2 T_X\Big) = H^2(X^\vee, \mathcal{O}) \tag{3.6}$$

coming from (3.4) and (3.5) is precisely the identification between the pieces of the Hochschild cohomology of X and $X^\vee$ given by the cohomological Fourier–Mukai transform $\widetilde{\mathcal{S}}_X$ that we discussed in the introduction. To see this one simply has to note that on the level of cohomology of polyvector fields, the map $\widetilde{\mathcal{S}}_X$ is given by the cohomological Fourier–Mukai transform $\alpha \mapsto p_{X^\vee *}(\exp(c_1(\mathcal{P})) \cup p_X^*\alpha)$.

Let $\boldsymbol{B} \in H^2(X^\vee, \mathcal{O})$ be the element corresponding to $\boldsymbol{\Pi} \in H^0(X, \bigwedge^2 T_X)$ via the isomorphism (3.6). The class $\boldsymbol{B}$ determines an $\mathcal{O}^\times$-gerbe ${}_{\boldsymbol{B}}\mathbb{X}^\vee$ over $\mathbb{X}^\vee := X^\vee \times \mathbb{D}$. In fact, this gerbe can be defined explicitly as a quotient gerbe. To streamline the discussion we introduce special notation for the formal analytic spaces we need. We write

$$\begin{aligned} \overline{\mathbb{V}}^\vee &:= \overline{V}^\vee \times \mathbb{D} \\ \mathbb{X}^\vee &:= X^\vee \times \mathbb{D} \end{aligned}$$

for the formal analytic spaces which are constant bundles over $\mathbb{D}$ with fibers $\overline{V}^\vee$ and $X^\vee$, respectively. We also write

$$\mathbb{\Lambda}^\vee := \Lambda^\vee \times \mathbb{D}$$

for the constant group space over $\mathbb{D}$ with fiber $\Lambda^\vee$.

We can think of the formal analytic space $\mathbb{X}^\vee$ as a quotient of the formal analytic space $\overline{\mathbb{V}}^\vee$ by the free action of the group space $\Lambda^\vee$, with $\Lambda^\vee$ acting trivially on $\mathbb{D}$. Similarly, viewing $\mathbb{X}^\vee \to \mathbb{D}$ as a relative space over $\mathbb{D}$ we can realize it as the quotient of the relative space $\overline{\mathbb{V}}^\vee \to \mathbb{D}$ by the relative action of the trivial bundle of commutative groups $\mathbb{\Lambda}^\vee \to \mathbb{D}$. For the construction of ${}_{\boldsymbol{B}}\mathbb{X}^\vee$ we first define a bundle $\mathbb{\Gamma} \to \mathbb{D}$ of non-commutative groups on $\mathbb{D}$, which is a Heisenberg extension:

$$1 \to \mathbb{G}_m \to \mathbb{\Gamma} \to \mathbb{\Lambda}^\vee \to 1, \tag{3.7}$$

where $\mathbb{G}_m$ is the multiplicative group scheme over $\mathbb{D}$. As a formal space $\mathbb{\Gamma} = \mathbb{G}_m \times_{\mathbb{D}} \mathbb{\Lambda}^\vee$. Given a formal space $S \to \mathbb{D}$ and sections $\xi, \xi' \in \mathbb{\Lambda}^\vee(S)$, $z, z' \in \mathbb{G}_m(S)$ the multiplication on $\mathbb{\Gamma}$ is given by the formula

$$(\xi, z) \cdot (\xi', z') = (\xi + \xi', zz'c(\xi, \xi')) \tag{3.8}$$

where we define $c(\xi, \xi')$ by

$$c(\xi, \xi') = \exp\left(\hbar\pi^2 \boldsymbol{B}(\xi', \xi)\right). \tag{3.9}$$

In this formula $\boldsymbol{B}$ is interpreted as a group cocycle of $\Lambda^\vee$ with values in $\mathbb{C}$. This involves two steps. First we use the canonical splitting

$$\begin{tikzcd} H^2(X^\vee, \mathbb{C}) \arrow[r] & H^2(X^\vee, \mathcal{O}) \arrow[l, bend left] \end{tikzcd}$$

of the Hodge filtration on $H^2(X^\vee, \mathbb{C})$ to interpret $\boldsymbol{B} \in H^2(X^\vee, \mathcal{O})$ as an element in $H^2(X^\vee, \mathbb{C})$, and then we us the fact that $X^\vee$ is a $K(\Lambda^\vee, 1)$ to identify $H^2(X^\vee, \mathbb{C})$ with the group cohomology $H^2(\Lambda^\vee, \mathbb{C})$. Explicitly, in these terms, $\boldsymbol{B}$ is viewed as a skew-symmetric biadditive map

$$\boldsymbol{B} : \Lambda^\vee \times \Lambda^\vee \to \mathbb{C} \tag{3.10}$$

which can be defined explicitly as follows. Recall that

$$\overline{V}^{\vee} = \left\{ l : V \to \mathbb{C} \;\middle|\; \begin{matrix} l(v_1 + v_2) = l(v_1) + l(v_2) \\ l(c \cdot v) = \bar{c} \cdot l(v) \end{matrix} \right\},$$

and

$$\Lambda^{\vee} = \{\xi \in \overline{V}^{\vee} \mid \operatorname{Im}(\xi(\lambda)) \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda\}.$$

Now, to every $l \in \overline{V}^{\vee}$ we can associate a natural complex linear map $\bar{l} : V^{\vee} \to \mathbb{C}$, given by $\bar{l}(v) := \overline{l(v)}$, and the alternating map (3.10) is given explicitly by

$$\boldsymbol{B}(\xi_1, \xi_2) = \boldsymbol{\Pi} \lrcorner (\bar{\xi}_1 \wedge \bar{\xi}_2).$$

The group space $\mathbb{\Gamma}$ still acts on $\overline{\mathbb{V}}^{\vee}$ by its image in $\mathbb{\Lambda}^{\vee}$. Every section $x : S \to \overline{\mathbb{V}}^{\vee}$ has a stabilizer equal to $\mathbb{G}_m(S)$. The quotient $[\overline{\mathbb{V}}^{\vee}/\mathbb{\Gamma}]$ is therefore a $\mathcal{O}^{\times}$-gerbe. We denote this gerbe by ${}_{\boldsymbol{B}}\mathbb{U}^{\vee} : {}_{\boldsymbol{B}}\mathbb{X}^{\vee} \to \mathbb{D}$. Since ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ is constructed as a quotient, we can compute the classifying element of ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ in $H^2(\mathbb{X}^{\vee}, \mathcal{O}^{\times})$ as the image of the $\mathbb{\Lambda}^{\vee}$-torsor $\overline{\mathbb{V}}^{\vee} \in H^1(\mathbb{X}^{\vee}, \mathbb{\Lambda}^{\vee})$ under the boundary map $H^1(\mathbb{X}^{\vee}, \mathbb{\Lambda}^{\vee}) \to H^2(\mathbb{X}^{\vee}, \mathcal{O}^{\times})$ associated with (3.7). From the definition of (3.7) it follows that ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ is classified by $c \in H^2(\mathbb{X}^{\vee}, \mathcal{O}^{\times})$. More simply, if we ignore the map to $\mathbb{D}$, we can think of the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ as the quotient of the formal space $\overline{\mathbb{V}}^{\vee}$ by the group $\Gamma = H^0(\mathbb{D}, \mathbb{\Gamma})$. Explicitly, Γ is given as the central extension

$$1 \to \mathbb{C}[[\hbar]]^{\times} \to \Gamma \to \Lambda^{\vee} \to 0$$

classified again by c which is now viewed as an element in the group cohomology $H^2(\Lambda^{\vee}, \mathbb{C}[[\hbar]]^{\times})$.

In § 6 we will show that the stack ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ can be identified with the relative Picard stack $\mathscr{P}ic^0(\mathbb{X}_{\boldsymbol{\Pi}}/\mathbb{D})$ of the formal non-commutative space $\mathbb{X}_{\boldsymbol{\Pi}} \to \mathbb{D}$.

4. Non-commutative line bundles and their moduli

In this section we investigate the family of translation invariant line bundles on the non-commutative torus $\mathbb{X}_{\boldsymbol{\Pi}}$. In particular, we exhibit a complete (Poincaré) family of such line bundles parameterized by the stack ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$.

4.1 Line bundles and factors of automorphy

Recall the classical picture for line bundles on complex tori in terms of factors of automorphy. We will describe this in a sufficiently general context so that it applies to all of the situations we need. Suppose that W is a locally compact space on which a discrete (not necessarily commutative) group Υ acts freely and properly discontinuously by homeomorphisms, (on the left). Denote by Y the quotient and let $\tau : W \to Y$ be the covering projection. Let $\mathscr{A}_W$ be a sheaf of unital not necessarily commutative algebras on W which is equivariant with respect to the action of Υ. This means that for every $\upsilon \in \Upsilon$ we are given an isomorphism $\mathfrak{a}_{\upsilon} : \mathscr{A}_W \to \upsilon^* \mathscr{A}_W$ of sheaf of algebras on W and that these isomorphisms satisfy $(\upsilon')^*(\mathfrak{a}_{\upsilon}) \circ \mathfrak{a}_{\upsilon'} = \mathfrak{a}_{\upsilon\upsilon'}$. Our convention is that Υ acts on sections on the right. For any open set $U \subset W$ and any $\upsilon \in \Upsilon$ we write

$$\begin{tikzcd} \mathscr{A}_W(\upsilon(U)) \arrow[r, "\cong"] & \mathscr{A}_W(U) \\ s \arrow[r, mapsto] & s \cdot \upsilon \end{tikzcd}$$

for the action of υ on $\mathscr{A}_W(\upsilon(U))$, i.e. $s \cdot \upsilon := \upsilon^*(\mathfrak{a}_{\upsilon^{-1}})(s)$.

We denote the product in $\mathscr{A}_W$ by $a \star b$. The sheaf $\mathscr{A}_W$ descends to a sheaf $\mathscr{A}_Y$ of algebras on Y defined by

$$\mathscr{A}_Y := (\tau_* \mathscr{A}_W)^{\Upsilon}.$$

Explicitly, given an open set $U \subset Y$ we have

$$\Gamma(U, \mathscr{A}_Y) = \{s \in \Gamma(\tau^{-1}(U), \mathscr{A}_W) \mid s \cdot \upsilon = s\}.$$

We are interested in describing sheaves of left (respectively right) $\mathscr{A}_Y$ modules that are locally free of rank one. In the usual way, the isomorphism classes of such modules are described by elements in the non-abelian cohomology set $H^1(Y, \mathscr{A}_Y^\times)$. Note that a Čech cocycle $[\gamma] \in Z^1(\mathfrak{U}, \mathscr{A}_Y^\times))$ for some open covering $\mathfrak{U}$ of Y gives a left (respectively right) rank one $\mathscr{A}_Y$-module if we let γ multiply elements in $\mathscr{A}_Y$ on the right (respectively left).

Alternatively we can describe sheaves of locally free rank one left (respectively right) $\mathscr{A}_Y$ modules in terms of factors of automorphy. The non-commutative factors of automorphy associated with Υ and W are just degree one group cocycles of Υ with values in the global sections of the sheaf $\mathscr{A}_W^\times$. Given such a cocycle $\boldsymbol{e} \in Z^1(\Upsilon, H^0(W, \mathscr{A}_W^\times))$ we can define a sheaf $L(\boldsymbol{e})$ of left $\mathscr{A}_W$ modules of rank one (respectively a sheaf $R(\boldsymbol{e})$ of right $\mathscr{A}_W$ modules of rank one) as follows. By definition, a non-commutative factor of automorphy $\boldsymbol{e}$ is a map $\boldsymbol{e} : \Upsilon \to H^0(W, \mathscr{A}_W^\times))$ satisfying the (left) cocycle condition

$$\boldsymbol{e}(\upsilon_1 \cdot \upsilon_2) = \boldsymbol{e}(\upsilon_2) \star (\boldsymbol{e}(\upsilon_1) \cdot \upsilon_2). \tag{4.1}$$

Now define the sheaves $L(\boldsymbol{e})$ and $R(\boldsymbol{e})$ by

$$\begin{aligned}\Gamma(U, L(\boldsymbol{e})) &= \{s \in \Gamma(\tau^{-1}(U), \mathscr{A}_W) \mid s \cdot \upsilon = s \star \boldsymbol{e}(\upsilon)\}\\ \Gamma(U, R(\boldsymbol{e})) &= \{s \in \Gamma(\tau^{-1}(U), \mathscr{A}_W) \mid s \cdot \upsilon = (\boldsymbol{e}(\upsilon^{-1}) \cdot \upsilon) \star s\}\end{aligned} \tag{4.2}$$

for an open $U \subset Y$. The fact that these formulas define sheaves easily follows from the cocycle condition (4.1). Clearly $L(\boldsymbol{e})$ and $R(\boldsymbol{e})$ have respectively left and right $\mathscr{A}_Y$-module structure since in the definition (4.2) the non-commutative factor of automorphy $\boldsymbol{e}$ multiplies on the right and left, respectively. Modulo the obvious equivalences the assignment $\boldsymbol{e} \mapsto L(\boldsymbol{e})$ (respectively $\boldsymbol{e} \mapsto R(\boldsymbol{e})$) gives the well-known map

$$H^1(\Upsilon, H^0(W, \mathscr{A}_W^\times)) \to H^1(Y, \mathscr{A}_Y^\times)$$

from cohomology classes of factors of automorphy to isomorphisms classes of rank one locally free left (respectively right) $\mathscr{A}_Y$-modules.

Remark 4.1. (i) It is instructive to point out that the sheaves $L(\boldsymbol{e})$ and $R(\boldsymbol{e})$ can be written as the invariants of appropriately defined actions of Υ on $\tau_*\mathscr{A}_W$. More precisely, given a non-commutative factor of automorphy $\boldsymbol{e} \in Z^1(\Upsilon, H^0(W, \mathscr{A}_W))$ we can define two new Υ-equivariant structures on the sheaf $\mathscr{A}_W$ by the formulas

$$\begin{aligned} s_\diamond \upsilon &:= (s \star \boldsymbol{e}(\upsilon^{-1})) \cdot \upsilon,\\ s_\vartriangle \upsilon &:= \boldsymbol{e}(\upsilon) \star (s \cdot \upsilon),\end{aligned}$$

for all $\upsilon \in \Upsilon$ and all sections s of $\mathscr{A}_W$ over Υ-invariant open sets on W. Now the automorphicity conditions in (4.2) become simply the condition of invariance with respect to these actions and so can describe $L(\boldsymbol{e})$ and $R(\boldsymbol{e})$ as the sheaves $(\tau_*\mathscr{A}_W)^{\Upsilon,\diamond}$ and $(\tau_*\mathscr{A}_W)^{\Upsilon,\vartriangle}$, respectively.

(ii) The somewhat mysterious formulas defining the automorphicity condition for $R(\boldsymbol{e})$ and the $\vartriangle$-action are forced on us by the non-commutative nature of $\mathscr{A}_W$. Indeed, the fact that the sheaf of groups $\mathscr{A}_W^\times$ is in general non-commutative implies that there are two natural notions of a factor of automorphy. First we have the *left* factors of automorphy $\boldsymbol{e} : \Upsilon \to H^0(W, \mathscr{A}_W^\times)$ satisfying the left cocycle condition (4.1). By the same token we have the *right* factors of automorphy $\boldsymbol{f} : \Upsilon \to H^0(W, \mathscr{A}_W^\times)$ satisfying the right cocycle condition

$$\boldsymbol{f}(\upsilon_1 \upsilon_2) = (\boldsymbol{f}(\upsilon_1) \cdot \upsilon_2) \star \boldsymbol{f}(\upsilon_2).$$

Clearly, given a right factor of automorphy we can write a right automorphicity condition on s, namely $s \cdot \upsilon = (\upsilon) \star \mathbf{s}$, and a new action of Υ on $\mathscr{A}_W$. However, the assignment $\boldsymbol{e}(\upsilon) \mapsto \boldsymbol{f}(\upsilon) := \boldsymbol{e}(\upsilon^{-1}) \cdot \upsilon$ transforms bijectively left factors of automorphy to right ones. Plugging this into the formulas defining $R(\boldsymbol{e})$ we obtain exactly the formulas in (i).

It is well known [Mum70] that in the case of a complex torus $X = V/\Lambda$ the map

$$H^1(\Lambda, H^0(V, \mathcal{O}_V^\times)) \to H^1(X, \mathcal{O}^\times) \tag{4.3}$$

is an isomorphism and thus gives a group cohomology description of the Picard group of X. Our goal is to obtain an analogous description for the non-commutative tori $\mathbb{X}_{\mathbf{\Pi}}$. To set this up, note that the relative spaces $\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{D}$ fit with the discussion of modules in the previous paragraph. Indeed, by definition the non-commutative torus $\mathbb{X}_{\mathbf{\Pi}}$ is the ringed space $(X, \mathscr{A}_X)$, which is the Moyal quantization of the Poisson torus $(X, \mathbf{\Pi})$. In § 3.1.3 this ringed space was constructed as the quotient of the Moyal ringed space $(V, \mathscr{A}_V)$ by the translation action of the lattice $\Lambda \subset V$. In particular, $\mathscr{A}_V^\times$-valued factors of automorphy for Λ will describe certain left (or right) locally free rank one modules on $\mathbb{X}_{\mathbf{\Pi}}$ and we will have a map of cohomology sets:

$$H^1(\Lambda, H^0(V, \mathscr{A}_V^\times)) \to H^1(X, \mathscr{A}_X^\times). \tag{4.4}$$

In fact, the map (4.4) is an isomorphism of pointed sets and so every left (or right) locally free rank one module admits a description via a factor of automorphy. The fact that (4.4) is an isomorphism will follow by the standard reasoning of [Mum70] from the fact that every $\mathscr{A}_V$ locally free left module of rank one is trivial. The latter statement can be proven by an order-by-order analysis of the non-commutative cocycles in $\check{Z}^1(V, \mathscr{A}_V^\times)$ or more generally in $\check{Z}^1(V, \mathrm{Aut}_{\mathscr{A}_V\text{-mod}}(\mathscr{A}_V^{\oplus n}))$. Since this argument is somewhat technical we have included it in Appendix B.

In contrast with the commutative case, in the non-commutative context the properties of being rank one and invertible no longer coincide. Therefore, it is important to differentiate between locally free rank one left modules and invertible bimodules, both of which can lay claim to be non-commutative line bundles. An invertible bimodule is often taken as the definition of a line bundle on a non-commutative space. However, we have found that on our non-commutative tori, bimodules do not behave flexibly enough when one looks at families. More precisely, it turns out that for a non-degenerate Poisson structure, non-constant holomorphic families of degree zero line bundles on X do not in general admit a consistent quantization.

PROPOSITION 4.2. *Let X be a complex torus and suppose that $\mathbf{\Pi} \in H^0(X, \bigwedge^2 T_X)$ is a non-degenerate holomorphic Poisson structure. Let S be a compact complex space and suppose that $L \to S \times X$ is a holomorphic line bundle whose restriction to each slice $\{s\} \times X$ has first Chern class zero. Suppose that we can find a holomorphic family $\mathscr{L} \to S \times \mathbb{X}_{\mathbf{\Pi}}$ of invertible bimodules with the property $\mathscr{L}/\hbar \cong L$. Then the classifying map $\kappa_L : S \to X^\vee$ corresponding to L is constant.*

Proof. An invertible bimodule on $\mathbb{X}_{\mathbf{\Pi}}$ is a sheaf of $\mathscr{A}_{X,\mathbf{\Pi}}$ bimodules on X which is locally free and of rank one when considered both as a left and a right module.

Note that for any bimodule $\mathscr{V}$ on $\mathbb{X}_{\mathbf{\Pi}}$ the sheaf $\mathscr{V}/\hbar\mathscr{V}$ is a Poisson module in the sense of [GK04, Appendix A.5]. Furthermore, since $\mathbf{\Pi}$ is non-degenerate the category of Poisson modules on $(X, \mathbf{\Pi})$ is equivalent to the category of D-modules on X (see [GK04, Appendix A.6] and [Kal04]) and so we get a well-defined functor from the category of finitely generated bimodules on the formal non-commutative space $\mathbb{X}_{\mathbf{\Pi}}$ to the category of finite rank complex local systems on X. Conversely, given a complex local system $\boldsymbol{V}$ on X we have an obvious $\mathscr{A}_{X,\mathbf{\Pi}}$ bimodule structure on the sheaf $\boldsymbol{V} \otimes_{\mathbb{C}} \mathscr{A}_{X,\mathbf{\Pi}}$. These functors are easily seen to be inverse equivalences of each other.

In particular, given a holomorphic family $\mathscr{L} \to S \times \mathbb{X}_{\mathbf{\Pi}}$ of invertible bimodules we get a holomorphic family $\boldsymbol{L} \to S \times X$ of rank one complex local systems on X which is parameterized by S.

By assumption, the family of holomorphic line bundles underlying these local systems is precisely $L \to S \times X$. In other words, the existence of $\mathscr{L}$ implies that the map $\kappa_L : S \to X^\vee$ lifts to a holomorphic map $\kappa_{\boldsymbol{L}} : S \to \mathrm{Loc}(X, 1)$ from S to the moduli of rank one local systems on X. In other words, we have a commutative diagram of complex spaces

$$\begin{tikzcd} & \mathrm{Loc}(X,1) \arrow[d, "p"] \\ S \arrow[ur, "\kappa_{\boldsymbol{L}}"] \arrow[r, "\kappa_L"'] & X^\vee \end{tikzcd}$$

with p being the natural projection and κ_L, $\kappa_{\boldsymbol{L}}$ being the classifying maps for L and $\boldsymbol{L}$. On the other hand the moduli space $\mathrm{Loc}(X, 1)$ is Stein (in fact, isomorphic to $(\mathbb{C}^\times)^{2g}$) and so the map $\kappa_{\boldsymbol{L}}$ must be constant. Hence, κ_L is constant and the proposition is proved. □

Our aim is to deform the Picard variety $X^\vee = \mathrm{Pic}^0(X)$ along with the non-commutative deformation $\mathbb{X}_{\boldsymbol{\Pi}}$ of X so that the Fourier–Mukai transform deforms as well. A natural choice will be to try and deform $X^\vee$ to the moduli of quantum line bundles on $\mathbb{X}_{\boldsymbol{\Pi}} \to \mathbb{D}$. If we attempt to do this with the interpretation of a quantum line bundle as an invertible bimodule, then we will end up with an obstructed moduli problem as explained in Lemma 4.2. As we will see below, this problem does not occur if we work with rank one locally free left $\mathscr{A}_{X,\boldsymbol{\Pi}}$-modules. This motivates the following.

Definition 4.3. A line bundle on a formal non-commutative space $(X, \mathscr{A}_X)$ is a sheaf $\mathscr{L} \to X$ of left-$\mathscr{A}_X$-modules which is locally isomorphic as a left module to $\mathscr{A}_X$.

The bimodule properties of a line bundle are not completely lost however.

Proposition 4.4. *Let $(X, \mathscr{A}_X)$ be a deformation quantization of $(X, \mathcal{O})$ and let L be a line bundle on $(X, \mathscr{A}_X)$. Then $(X, \mathscr{E}nd_{\mathscr{A}_X}(L))$ is again a deformation quantization of $(X, \mathcal{O})$ and thus L is a left-$(X, \mathscr{A}_X)$ and a right-$\mathscr{E}nd_{\mathscr{A}_X}(L)$ module. Furthermore these two actions commute with each other and L is a Morita equivalence bimodule.*

Proof. The algebra $\mathscr{E}nd_{\mathscr{A}_X}(L)$ is naturally a $\mathbb{C}[[\hbar]]$-algebra. Since flatness is a local condition and $\mathscr{E}nd_{\mathscr{A}|_U}(L|_U) \cong \mathscr{A}_X^{\mathrm{op}}|_U$ we see that $\mathscr{E}nd_{\mathscr{A}} L$ is a flat $\mathbb{C}[[\hbar]]$-module. Also

$$\mathscr{E}nd_{\mathscr{A}_X}(L)/\hbar \cong \mathscr{E}nd_{\mathscr{A}_X/\hbar}(L/\hbar) \cong \mathcal{O}$$

and hence $\mathscr{E}nd_{\mathscr{A}_X}(L)$ is a deformation quantization of X. The Morita equivalence statement is straightforward. □

Remark 4.5. The previous proposition shows that with our definition, a non-commutative line bundle L implements an equivalence between the category of $\mathscr{A}_X$ modules and the category of $\mathscr{E}nd_{\mathscr{A}_X}(L)^{\mathrm{op}}$ modules. In the commutative case this reduces to the standard interpretation of a line bundle as an autoequivalence of the category of sheaves. This idea of a line bundle is very natural physically and was exploited before in [JSW02]. Mathematically, it can be motivated by the natural expectation that a non-commutative space should not just be taken to be a ringed space $(X, \mathscr{A}_X)$ but should be a Morita equivalence class of such spaces (perhaps only locally defined). There is other evidence for this as well and we will hopefully pursue this in a future paper.

For future reference we record some simple properties of the deformation quantizations arising from non-commutative line bundles.

Proposition 4.6. *Let $(X, \mathscr{A}_X)$ be a deformation quantization of $(X, \mathcal{O})$. If $H^0(X, \mathcal{O}) = \mathbb{C}$, then for any line bundle L, the natural sheaf map $\mathcal{Z}(\mathscr{A}_X) \to \mathscr{E}nd_{\mathscr{A}_X}(L)$ given by the center of the algebra acting by left multiplication induces an isomorphism*

$$H^0(X; \mathcal{Z}(\mathscr{A}_X)) \to H^0(X, \mathscr{E}nd_{\mathscr{A}_X}(L)).$$

Proof. We will prove that the composition

$$\mathbb{C}[[\hbar]] \to H^0(X, \mathcal{Z}(\mathscr{A}_X)) \hookrightarrow H^0(X, \mathscr{E}nd_{\mathscr{A}_X}(L)) \tag{4.5}$$

is an isomorphism.

Suppose that $(X, \mathcal{O})$ is a complex manifold with $H^0(X; \mathcal{O}) = \mathbb{C}$ and let $(X, \mathscr{B})$ be a deformation quantization of $(X, \mathcal{O})$. Then the map $\mathbb{C}[[\hbar]] \to \mathscr{B}$ induces an isomorphism $H^0(X; \mathbb{C}[[\hbar]]) \to H^0(X, \mathscr{B})$.

Indeed, both $\mathbb{C}[[\hbar]]$ and $H^0(X; \mathscr{B})$ are complete filtered algebras where the filtrations are given by

$$H^0(X; \mathscr{B})_k = \hbar^k \cdot H^0(X; \mathscr{B})$$

and similarly for $\mathbb{C}[[\hbar]]$. The map $\mathbb{C}[[\hbar]] \to H^0(X; \mathscr{B})$ is a filtered map and induces a map

$$\operatorname{gr}(f) : \operatorname{gr}\mathbb{C}[[\hbar]] \cong \mathbb{C}[\hbar] \to \operatorname{gr} H^0(X; \mathscr{B}).$$

Now,

$$\begin{aligned} \operatorname{gr}_k H^0(X; \mathscr{B}) &= H^0(X, \mathscr{B})_k / H^0(X, \mathscr{B})_{k+1} \\ &= H^0(X, \hbar^k \mathscr{B}) / H^0(X, \hbar^{k+1} \mathscr{B}) \\ &\cong H^0(X, \mathscr{B}/\hbar\mathscr{B}) \cong H^0(X, \mathcal{O}) \cong \mathbb{C} \end{aligned} \tag{4.6}$$

because $H^0(\hbar^k \mathscr{B})$ surjects onto $H^0(\mathscr{B}/\hbar\mathscr{B})$, and because $\lambda \in \mathbb{C}$ is covered by $\hbar^k \lambda$. However, if $f : A \to B$ is a filtered map of complete filtered algebras and $\operatorname{gr}(f)$ is an isomorphism, then f is also an isomorphism. Indeed, recall that a complete filtered algebra A is a $\mathbb{C}$-algebra equipped with a decreasing filtration by two-sided ideals A_i, $A = A_0 \supseteq A_1 \supseteq \cdots$ which satisfy $A_i A_j \subseteq A_{i+j}$, and completeness: $A = \lim_{k\to\infty} A/A_k$.

Now by completeness, it is enough to show that the maps $f_k : A/A_k \to B/B_k$ induced by f are isomorphisms for all k. Since by assumption $\operatorname{gr}_k(f) : \operatorname{gr}_k(A) \to \operatorname{gr}_k(B)$ is an isomorphism for all k and since

$$A/A_1 = \operatorname{gr}_1(A) \to \operatorname{gr}_1(B) = B/B_1$$

is an isomorphism, the claim follows by induction using the following commutative diagram of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & A/A_{k-1} \arrow[r] \arrow[d] & A/A_k \arrow[r] \arrow[d] & \operatorname{gr}_k A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B/B_{k-1} \arrow[r] & B/B_k \arrow[r] & \operatorname{gr}_k B \arrow[r] & 0 \end{tikzcd}$$

This shows that (4.5) is an isomorphism and completes the proof of the lemma. ☐

We now begin to develop the necessary tools to analyze families of line bundles. We have the following statement, which parallels the classical see-saw lemma [Mum70].

Proposition 4.7. *Let $\mathbb{X} = (X, \mathscr{A}_X)$ be a deformation quantization of $(X, \mathcal{O}_X)$ and let $\mathbb{Y} = (Y, \mathcal{O}_{\mathbb{Y}})$ be a complex manifold over $\mathbb{D}$. Also, assume that $H^0(X, \mathcal{O}) \cong \mathbb{C}$. Consider the deformation quantization $\mathbb{X} \times_{\mathbb{D}} \mathbb{Y} = (X \times Y, \mathscr{A}_{X\times Y}) = (X \times Y, p_1^{-1}\mathscr{A}_X \mathbin{\widehat{\otimes}_{\mathbb{C}[[\hbar]]}} p_2^{-1}\mathcal{O}_{\mathbb{Y}})$. Let L_1 and L_2 be two line bundles on $\mathbb{X} \times_{\mathbb{D}} \mathbb{Y}$ such that for all $y \in Y$, there is an isomorphism*

$$\phi_y \in \mathcal{I}som(L_1|_{X\times\{y\}}, L_2|_{X\times\{y\}}).$$

Then:

(a) *there is a global isomorphism of sheaves of algebras*

$$\mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_1) \cong \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_2)$$

which on each fiber $X \times \{y\}$ *restricts to the isomorphism* $\phi_y \circ (-) \circ \phi_y^{-1}$;

(b) *there exists a line bundle* M *on* $\mathbb{Y}$ *and an isomorphism* $\mathbb{p}_{\mathbb{Y}}^* M \otimes_{\mathscr{A}_{X\times Y}} L_2 \widetilde{\rightarrow} L_1$.

Proof. Part (a) of this proposition is a new element of the see-saw principle, specific to the deformation quantization situation. To prove part (a), chose a cover $\mathfrak{U}$ of Y and elements $\phi_U \in \mathcal{I}som_{\mathscr{A}_{X\times Y}}(L_1, L_2)(X \times U)$ for all $U \in \mathfrak{U}$. Denote by $\psi_U : \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_1)|_{X\times U} \to \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_2)|_{X\times U}$ the induced local isomorphisms of algebras $\psi_U(f) = \phi_U \circ f \circ \phi_U^{-1}$. Using Proposition 4.6 we see that the elements

$$\phi_V^{-1} \circ \phi_U \in H^0(X \times (U \cap V), \mathcal{I}som_{\mathscr{A}_{X\times Y}}(L_1))$$

are in fact in $H^0(X \times (U \cap V), p_1^{-1}\mathcal{Z}(\mathscr{A}_X) \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_{U\cap V})$. Therefore, the ψ_U patch to a global isomorphism $\mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_1) \cong \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_2)$ of sheaves of algebras.

The proof of (b) is essentially the same as the see-saw proof found in [Mum70, § II.5]. We only discuss the modifications of the argument needed in the non-commutative setting. Set $\mathscr{E} := \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_1) \cong \mathscr{E}nd_{\mathscr{A}_{X\times Y}}(L_2)$. Then L_1 and L_2 are sheaves on $X \times Y$ which are locally free rank one right modules over the sheaf of algebras $\mathscr{E}$. Hence, the sheaf $L_2^\vee := \mathscr{H}om_{\mathscr{A}_{X\times Y}}(L_2, \mathscr{A}_{X\times Y})$ has a natural structure of a left $\mathscr{E}$-module and a right $\mathscr{A}_{X\times Y}$-module. Consider now the tensor product $L_1 \otimes_{\mathscr{E}} L_2^\vee$. This is a sheaf on $X \times Y$ which is a $\mathscr{A}_{X\times Y}$-bimodule and for every $y \in Y$ satisfies

$$(L_1 \otimes_{\mathscr{E}} L_2^\vee)_{|X\times\{y\}} \cong \mathscr{A}_X$$

as an $\mathscr{A}_X$-bimodule. By Proposition 4.6 applied to the trivial line bundle on X we have $H^0(X, \mathscr{A}_X) = \mathbb{C}[[\hbar]]$ and therefore $M := p_{Y*}(L_1 \otimes_{\mathscr{E}} L_2^\vee)$ is a rank one locally free left module over $\mathcal{O}_{\mathbb{Y}}$. By adjunction the identity endomorphism

$$\mathrm{id}_M \in \mathrm{Hom}_{\mathbb{Y}}(M, M) = \mathrm{Hom}_{\mathbb{Y}}(p_{Y*}(L_1 \otimes_{\mathscr{E}} L_2^\vee), p_{Y*}(L_1 \otimes_{\mathscr{E}} L_2^\vee))$$

corresponds to a map

$$\mathbb{p}_{\mathbb{Y}}^* M \otimes_{\mathscr{A}_{X\times Y}} L_2 \to L_1.$$

The check that this map is an isomorphism is exactly the same as in the commutative situation. □

4.2 The Poincaré sheaf

We now describe the factor of automorphy that defines the Poincaré sheaf in our context. Let $\mathbf{\Pi}$ be a holomorphic Poisson structure on X and let $\boldsymbol{B}$ denote the corresponding B-field on $X^\vee$. The Poisson structure $\mathbf{\Pi}$ lifts to a Poisson structure on V which will be denoted by the same letter. Consider the Poisson structure on $V \times \overline{V}^\vee$ which is $\mathbf{\Pi}$ on V and 0 on $\overline{V}^\vee$. We can then form the corresponding Moyal quantizations $(V \times \overline{V}^\vee, \mathscr{A}_{V\times\overline{V}^\vee,(\mathbf{\Pi},0)})$ and $(X \times X^\vee, \mathscr{A}_{X\times X^\vee,(\mathbf{\Pi},0)})$ (see § 3.1.1). In what follows we drop the labels $(\mathbf{\Pi}, 0)$ from our algebras since they will be clear from the context. Note that $(X \times X^\vee, \mathscr{A}_{X\times X^\vee,(\mathbf{\Pi},0)})$ is just the non-commutative space $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{X}^\vee$ which is the moduli space of the stack $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$.

Our goal is to construct a deformation of the Poincaré line bundle $\mathcal{P} \to X \times X^\vee$ to a line bundle on the non-commutative stack $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$. Why a stack? Even classically the moduli problem of topologically trivial line bundles on a space Z leads most naturally to an analytic stack $\mathscr{P}ic^0(Z)$ which is an $\mathcal{O}^\times$-gerbe over the usual Picard variety $\mathrm{Pic}^0(Z)$. This is rarely discussed since the gerbe $\mathscr{P}ic^0(Z) \to \mathrm{Pic}^0(Z)$ is trivial. Indeed, we can construct a trivialization of this gerbe by looking at the moduli problem of framed line bundles on Z with the framing specified at a fixed point $z \in Z$. This feature of the moduli of line bundles does not persist in families. If we look at the relative Picard problem for a smooth family $Z \to B$, then the moduli stack is not necessarily a trivial gerbe since now a trivialization depends on a framing along a section $B \to Z$, which may not exist. In our case, $\mathbb{X}^\vee \to \mathbb{D}$ should be thought of as the relative Picard variety $\operatorname{Pic}^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ and ${}_{\boldsymbol{B}}\mathbb{X}^\vee \to \mathbb{D}$ should be thought of as the stack of relative line bundles $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ (this will be justified in § 6). By construction the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^\vee \to \mathbb{X}^\vee$ is non-trivial, and indeed we do not expect $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D}) \to \operatorname{Pic}^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ to be trivial since $\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{D}$ has no sections.

To define the deformation $\mathscr{P} \to \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ of $\mathcal{P} \to X \times X^\vee$ we use the description of line bundles via factors of automorphy. Since the $\mathbb{X}_{\mathbf{\Pi}}$ is the Λ-quotient of the non-commutative space $(V, \mathscr{A}_{V,\mathbf{\Pi}})$ and ${}_{\boldsymbol{B}}\mathbb{X}^\vee$ is the Γ quotient of $(\overline{V}^\vee, \mathcal{O}[[\hbar]])$ we need a $\Lambda \times \Gamma$-factor of automorphy:

$$\phi : \Lambda \times \Gamma \to H^0(V \times \overline{V}^\vee, \mathscr{A}ut(\mathscr{A}_{V\times\overline{V}^\vee})) = \mathscr{A}^\times_{V\times\overline{V}^\vee}(V \times \overline{V}^\vee).$$

Here $\operatorname{Aut}(\mathscr{A}_{V\times\overline{V}^\vee})$ denotes the automorphisms of $\mathscr{A}_{V\times\overline{V}^\vee}$ considered as a left module over itself. We define ϕ by the formula

$$\begin{aligned}\phi(\lambda, (\xi, z))(v, l) &= z \exp(\pi\sqrt{-1}\operatorname{Im}(\langle \xi, \lambda\rangle)) \exp(\pi(\langle l, \lambda\rangle + \overline{\langle \xi, v\rangle})) \exp\left(\frac{\pi}{2}(\langle \xi, \lambda\rangle + \overline{\langle \xi, \lambda\rangle})\right) \\ &= z \exp(\pi(\langle l + \xi, \lambda\rangle + \overline{\langle \xi, v\rangle}))\end{aligned} \tag{4.7}$$

where $\lambda \in \Lambda$, $(\xi, z) \in \Gamma$ and $(v, l) \in V \times \overline{V}^\vee$, and $\langle l, v\rangle = l(v)$. In particular, $\langle l, v\rangle$ is complex anti-linear as a function of v and complex linear as a function of l. We note that the only term in this product that involves $\hbar$ is z. The exponentials and the products of terms in this formula can be viewed either as $\star$-exponentials and products, or as ordinary exponentials and products of functions on $V \times \overline{V}^\vee$. This is unambiguous as we will see in the proof of the next proposition.

PROPOSITION 4.8. *The factor ϕ is a factor of automorphy.*

Proof. To see that ϕ is a factor of automorphy, we have to check the condition (4.1), which says

$$\phi(\lambda_1 + \lambda_2, \xi_1 + \xi_2, z_1 z_2 c(\xi_1, \xi_2)) = \phi(\lambda_2, \xi_2, z_2) \star ([\phi(\lambda_1, \xi_1, z_1)] \cdot (\lambda_2, \xi_2, z_2)).$$

The computation which establishes the factor of automorphy condition for the classical Poincaré cochain [Pol03] immediately reduces us in this case to showing the equality

$$c(\xi_1, \xi_2) \exp(\pi\overline{\langle \xi_1 + \xi_2, v\rangle}) = \exp(\pi\overline{\langle \xi_2, v\rangle}) \star \exp(\pi\overline{\langle \xi_1, v\rangle}). \tag{4.8}$$

To that end consider two points f and g in $V^\vee$ viewed as linear functions on V. If P is the bidifferential operator associated with $\mathbf{\Pi}$, then using the formula (3.1) we compute (directly, or by the Campbell–Baker–Hausdorff formula)

$$(\exp f) \cdot P \cdot (\exp g) = \{f, g\} \exp(f + g).$$

Therefore,

$$(\exp f) \star (\exp g) = \exp(\hbar\{f, g\}) \exp(f + g). \tag{4.9}$$

Given $\xi \in \overline{V}^\vee$ let $f_\xi := \pi\overline{\langle \xi, \bullet\rangle}$ denote the corresponding linear function on V. Then (4.9) implies that the $\star$-inverse of $\exp(f_\xi)$ is $\exp(f_{-\xi})$.

Using (4.9) in the right-hand side of (4.8) and canceling $\exp(f_{\xi_1} + f_{\xi_2})$ from both sides, it remains only to show that

$$c(\xi_1, \xi_2) = \exp(\hbar\{f_{\xi_2}, f_{\xi_1}\}). \tag{4.10}$$

However,

$$\{f_{\xi_2}, f_{\xi_1}\} = \mathbf{\Pi} \lrcorner (df_{\xi_2} \wedge df_{\xi_1}) = \pi^2 \mathbf{\Pi} \lrcorner (\bar{\xi}_2 \wedge \bar{\xi}_1) = \pi^2 \boldsymbol{B}(\xi_2, \xi_1). \tag{4.11}$$

Now we simply recall that we defined c in (3.9) by $c(\xi, \xi') = \exp(\hbar\pi^2 \boldsymbol{B}(\xi', \xi))$ to see that (4.10) is satisfied so we are done. ☐

5. The equivalence of categories

We are now ready to compare the sheaf theories of the non-commutative torus $\mathbb{X}_{\boldsymbol{\Pi}}$ and the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$. Just as in the commutative case the equivalence question can be posed for different derived categories of sheaves.

5.1 Categories of sheaves

The basic categories we need for $\mathbb{X}_{\boldsymbol{\Pi}}$ will be:

- $\mathsf{Mod}(\mathbb{X}_{\boldsymbol{\Pi}})$, the category of all sheaves of left $\mathscr{A}_{\boldsymbol{\Pi}}$-modules on X;
- $\mathsf{Coh}(\mathbb{X}_{\boldsymbol{\Pi}})$, the category of all sheaves of coherent left $\mathscr{A}_{\boldsymbol{\Pi}}$-modules on X; by definition, these are sheaves in $\mathsf{Mod}(\mathbb{X}_{\boldsymbol{\Pi}})$ which locally in X admit a finite presentation by free modules.

We will write $\mathsf{D}^{*}(\mathbb{X}_{\boldsymbol{\Pi}})$ for the derived category of $\mathsf{Mod}(\mathbb{X}_{\boldsymbol{\Pi}})$ and $\mathsf{D}^{*}_{\mathsf{c}}(\mathbb{X}_{\boldsymbol{\Pi}})$ for the derived version of $\mathsf{Coh}(\mathbb{X}_{\boldsymbol{\Pi}})$. Here the decoration $*$ can be anything in the set $\{\varnothing, -, b\}$.

It is important to note that $\mathscr{A}_{X,\boldsymbol{\Pi}}$ is a coherent and Noetherian sheaf of rings, see [Bjö93, §§ A.II.6.27 and A.III.3.24] for the definitions. This implies that $\mathsf{Coh}(\mathbb{X}_{\boldsymbol{\Pi}})$ is a full abelian subcategory in $\mathsf{Mod}(\mathbb{X}_{\boldsymbol{\Pi}})$ and also that $\mathsf{D}^{*}_{\mathsf{c}}(\mathbb{X}_{\boldsymbol{\Pi}})$ is equivalent to the full subcategory of $\mathsf{D}^{*}(\mathbb{X}_{\boldsymbol{\Pi}})$ consisting of complexes with coherent cohomology. To check that $\mathscr{A}_{X,\boldsymbol{\Pi}}$ is coherent and Noetherian we use a very general criterion due to Björk [Bjö79, Lemma 8.2 and Theorem 9.6]. Even though the $\hbar$-filtration on $\mathscr{A}_{X,\boldsymbol{\Pi}}$ is decreasing, we can still apply the technology of [Bjö79] since his setup allows for filtrations infinite in both directions and we can simply relabel the filtration to make it increasing. To check the hypotheses of [Bjö79, Lemma 8.2] we need to make sure that the stalks of $\operatorname{gr}\mathscr{A}_{X,\boldsymbol{\Pi}}$ are left and right Noetherian. This is clear since (stalks of $\operatorname{gr}\mathscr{A}_{X,\boldsymbol{\Pi}}$) = (stalks of $\mathcal{O}_X[[\hbar]]$). To check the hypotheses of [Bjö79, Theorem 9.6] we need to show that given a left ideal L in a stalk of $\mathscr{A}_{X,\boldsymbol{\Pi}}$ and elements $a_1, \ldots, a_s \in L$, such that $\sigma(a_1), \ldots, \sigma(a_s)$ generate $\sigma(L)$, then

$$\Sigma_{\nu} \cap L = \Sigma_{\nu-\nu_1} a_1 + \cdots + \Sigma_{\nu-\nu_s} a_s \tag{5.1}$$

holds for all ν. Here Σ_{ν} is the νth step of the increasing filtration on the stalk of $\mathscr{A}_{X,\boldsymbol{\Pi}}$, i.e.

$$\Sigma_{\nu} = \begin{cases} \hbar^{-\nu}\mathscr{A} & \nu \leqslant 0, \\ \mathscr{A} & \nu > 0, \end{cases}$$

$\sigma : \mathscr{A}_{X,\boldsymbol{\Pi}} \to \operatorname{gr}\mathscr{A}_{X,\boldsymbol{\Pi}}$ is the symbol map, and ν_i is the order of a_i, i.e. $a_i \in \Sigma_{\nu_i} \setminus \Sigma_{\nu_{i-1}}$.

In fact the condition (5.1) holds for any deformation quantization. Indeed given $a_1, \ldots, a_s$ as above and $\ell \in \Sigma_{\nu} \cap L$, we can choose $\alpha_1, \ldots, \alpha_s$ in the stalk of $\mathscr{A}_{X,\boldsymbol{\Pi}}$ so that $\nu(\alpha_i) = \nu - \nu_i$ satisfying

$$\ell - (\alpha_1 \star a_1 + \cdots \alpha_s \star a_s) \in \Sigma_{\nu-1} \cap L.$$

Now induction and the $\hbar$-adic completeness of $\mathscr{A}_{X,\boldsymbol{\Pi}}$ finish the verification of (5.1).

For the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ the relevant categories are:

- $\mathsf{Mod}({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$, the category of sheaves of $\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$-modules;
- $\mathsf{Coh}({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$, the category of coherent $\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$-modules.

Since the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ admits a presentation ${}_{\boldsymbol{B}}\mathbb{X}^{\vee} = [\overline{\mathbb{V}}^{\vee}/\mathbb{\Gamma}]$ as a global quotient we can explicitly describe $\mathsf{Mod}({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$ and $\mathsf{Coh}({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$, the categories of $\mathbb{\Gamma}$-equivariant $\mathcal{O}_{\overline{\mathbb{V}}^{\vee}}$-modules and coherent $\mathbb{\Gamma}$-equivariant $\mathcal{O}_{\overline{\mathbb{V}}^{\vee}}$-modules, respectively. Since ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ is an $\mathcal{O}^{\times}$-gerbe, these categories decompose into orthogonal direct sums $\mathsf{Mod}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}) = \coprod_{k\in\mathbb{Z}} \mathsf{Mod}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}, k)$ and $\mathsf{Coh}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}) = \coprod_{k\in\mathbb{Z}} \mathsf{Coh}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}, k)$, labeled by the character k of the stabilizer $\mathbb{C}^{\times}$. Explicitly, $\mathsf{Mod}_k({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$ and $\mathsf{Coh}_k({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$ are, respectively, the categories of $\mathbb{\Gamma}$-equivariant sheaves and coherent, $\mathbb{\Gamma}$-equivariant sheaves for which the action of the center $\mathcal{O}_{\mathbb{D}}^{\times}$ is the kth power of the tautological action. These also admit an alternative interpretation as categories of $k\boldsymbol{B}$-twisted sheaves and coherent, $k\boldsymbol{B}$-twisted sheaves on $\mathbb{X}^{\vee}$ (see [Gir71, Căl02]). Finally, we write $\mathsf{D}^{*}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}, k)$ for the corresponding derived categories.

Digression on quasi-coherent sheaves. An unpleasant phenomenon of the sheaf theory in analytic geometry is the fact that for generic analytic spaces the categories of analytic coherent sheaves tend to be fairly small and boring [Ver04]. In particular, the category of analytic coherent sheaves is not a good invariant in the analytic world. This contrasts with the algebraic category where a Noetherian scheme can be reconstructed from its category of coherent sheaves [Gab62]. Therefore, the common wisdom is that for an analytic space X the geometry is captured better by more general categories of $\mathcal{O}_X$-modules, e.g. quasi-coherent ones or all $\mathcal{O}_X$-modules.

Unfortunately the notion of a quasi-coherent analytic sheaf is a bit murky and there are several competing definitions (see [RR74], [EP96], [Tay02, ch. 11.9], and [Orl99]) in the literature. The relationships among these definitions are not clear in general. We adopt the definition of [Tay02, ch. 11.9] which is best suited to our setup and we comment on how this definition compares to the others in our specific case.

Definition 5.1. Suppose that X is an analytic space and let F be an analytic sheaf of $\mathcal{O}_X$-modules. The sheaf F is called *quasi-coherent* if for every point $x \in X$ we can find a compact Stein neighborhood $x \in K \subset X$ having the Noether property, and a module M over $\Gamma(K, \mathcal{O}_K)$, so that

$$F_{|K} \cong \widetilde{M} := \mathcal{O}_K \otimes_{\Gamma(K,\mathcal{O}_K)} M.$$

As usual, in this formula, $\Gamma(K, \mathcal{O}_K)$ and M are viewed as constant sheaves on K.

Similarly, if $\mathbb{X}_{\boldsymbol{\Pi}} = (X, \mathscr{A}_{X,\boldsymbol{\Pi}})$ is a holomorphic deformation quantization of X, we will say that a sheaf $\mathscr{F} \in \mathsf{Mod}(\mathbb{X}_{\boldsymbol{\Pi}})$ of left $\mathscr{A}_{X,\boldsymbol{\Pi}}$-modules is *quasi-coherent* if for every $x \in X$ we can find a compact Stein neighborhood $x \in K \subset X$ having the Noether property, and a left module $\mathscr{M}$ over $\Gamma(K, \mathscr{A}_{X,\boldsymbol{\Pi}})$, so that

$$\mathscr{F}_{|K} \cong \widetilde{\mathscr{M}} := (\mathscr{A}_{X,\boldsymbol{\Pi}|K}) \otimes_{\Gamma(K,\mathscr{A}_{X,\boldsymbol{\Pi}})} \mathscr{M}.$$

Recall that a compact Stein set is a compact analytic subspace in some $\mathbb{C}^n$ which can be realized as the intersection of a nested sequence of Stein neighborhoods. The Noether property of a compact Stein space K is that $\Gamma(K, \mathcal{O}_K)$ is a Noetherian Stein algebra. It is known [Fri67, Lan77] that compact analytic polyhedra (i.e. subsets of a Stein space defined by finitely many inequalities of the form $|f| \leqslant 1$, for f holomorphic) are compact Stein sets that have the Noether property. In particular, polydisks are compact Stein and Noether. More generally every point in an analytic space has a basis of compact Stein Noether neighborhoods (see also [GR04]).

The above notion of quasi-coherence is somewhat unconventional from the point of view of Grothendieck sheaf theory on complex spaces. We chose to work with it, since it is compatible with the more standard points of view on quasi-coherence and, in addition, turns out to have a very good behavior with respect to pullbacks and pushforwards.

It is instructive to compare the quasi-coherence in the sense of Definition 5.1 to the quasi-coherence of [RR74], [EP96], and [Orl99].

The most general notion of quasi-coherence is that given by Orlov in [Orl99, Definition 2.6]. Orlov's definition works on an arbitrary ringed site and is conceptually the closest to the usual notion of quasi-coherence in algebraic geometry. In fact, Definition 5.1 is a special case of [Orl99, Definition 2.6]. For any complex space X we can consider the site cSt/X of compact Stein spaces, taken with the analytic topology and ringed by the sheaf of analytic functions. Any analytic sheaf F of $\mathcal{O}_X$-modules is a sheaf on St/X, gives rise to a sheaf $\mathsf{c}F$ on cSt/X, and the quasi-coherence of F in the sense of Definition 5.1 is simply the quasi-coherence of $\mathsf{c}F \to \mathsf{cSt}/X$ in the sense of [Orl99, Definition 2.6].

The other definition is the Ramis–Ruget definition of quasi-coherence [RR74, EP96] which is historically the first. In their definition, a sheaf F of $\mathcal{O}_X$ is called quasi-coherent if locally on X we can realize F as a transversal localization of a module. More precisely, F is *algebraically Ramis–Ruget quasi-coherent* if for every two Stein opens $V \subset U$ we have:

- $\Gamma(U, F)$ and $\Gamma(V, \mathcal{O})$ are Verdier transversal over $\Gamma(U, \mathcal{O})$, that is
$$\Gamma(V, \mathcal{O}) \overset{L}{\otimes}_{\Gamma(U,\mathcal{O})} \Gamma(U, F) = \Gamma(V, \mathcal{O}) \otimes_{\Gamma(U,\mathcal{O})} \Gamma(U, F);$$
- the natural map $\Gamma(V, \mathcal{O}) \overset{L}{\otimes}_{\Gamma(U,\mathcal{O})} \Gamma(U, F) \to \Gamma(V, F)$ is an isomorphism.

In fact, the Ramis–Ruget notion of quasi-coherence requires that the sheaf F is an analytic sheaf of nuclear Frechet $\mathcal{O}$-modules, and all of the tensor products are completed. This is necessary in their setup since they are concerned with the Grothendieck duality theory and, in particular, need the double dual of the space of sections of a sheaf to be isomorphic to itself. For our purposes, the nuclear Frechet condition is irrelevant, and so we talk only about the algebraic Ramis–Ruget quasi-coherence.

Note that the transversality condition in the Ramis–Ruget definition implies that any sheaf $F \in \mathsf{Mod}(X)$ which is algebraically Ramis–Ruget quasi-coherent is also quasi-coherent in the sense of Definition 5.1. So our notion of quasi-coherence is sandwiched between the Ramis–Ruget's function theoretic notion and Orlov's general categorical notion.

The main advantage of Definition 5.1 is that the localization functor on compact Stein spaces having the Noether property is an exact functor [Tay02, ch. 11.9]. In particular, given an analytic morphism $f : X \to Y$ the pushforward $f_* : \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$ preserves quasi-coherence. In the terminology of [Orl99] this means that the natural map from F to its coherator is an isomorphism. We write $\mathsf{Qcoh}(X)$ and $\mathsf{Qcoh}(Y)$ for the categories of quasi-coherent sheaves and f_* and f^* for the corresponding pullback and pushforward functors.

Finally, observe that the exactness of compact Stein localization of [Tay02, ch. 11.9] also holds for the sheaves in the category $\mathsf{Qcoh}(\mathbb{X}_{\boldsymbol{\Pi}})$. This is an immediate consequence of the exactness in [Tay02, ch. 11.9] and the fact that $\mathscr{A}_{X,\boldsymbol{\Pi}}$ is a coherent and Noetherian sheaf of complete $\mathbb{C}[[\hbar]]$-algebras.

For future reference we write $\mathsf{D}^*_{\mathrm{qc}}(\mathbb{X}_{\boldsymbol{\Pi}})$ and $\mathsf{D}^*_{\mathrm{qc}}({}_{\boldsymbol{B}}\mathbb{X}^\vee, k)$ for the derived categories of analytic sheaves with quasi-coherent cohomologies.

5.2 The main theorem

In § 4.2 we defined a Poincaré sheaf $\mathscr{P}$ on $\mathbb{X}_{\boldsymbol{\Pi}} \times_{\mathbb{D}} {}_{\boldsymbol{B}}\mathbb{X}^\vee$ deforming the classical Poincaré line bundle. By definition $\mathscr{P}$ is a sheaf on $X \times ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ which is a left $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}}$-module and a right $p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}$-module, i.e. $\mathscr{P} \in {}_{p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}}}\mathsf{Mod}_{p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}}$. We also need the (algebraic) dual sheaf $\mathscr{Q}$ of $\mathscr{P}$. For our purposes it will be convenient to define $\mathscr{Q}$ on the product of X and ${}_{\boldsymbol{B}}\mathbb{X}^\vee$ in which the order of the factors is transposed. In other words we define $\mathscr{Q}$ as the sheaf on $({}_{\boldsymbol{B}}\mathbb{X}^\vee) \times X$ given by

$$\mathscr{Q} = \mathscr{P}^\vee := \mathrm{Hom}_{{}_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}}\mathsf{Mod}_{p_1^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}}}(\boldsymbol{t}^*\mathscr{P}, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} p_1^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}).$$

Here $\boldsymbol{t} : ({}_{\boldsymbol{B}}\mathbb{X}^\vee) \times X \to X \times ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ is the isomorphism transposing the factors. The sheaf $\mathscr{Q}$ is in ${}_{p_1^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}}\mathsf{Mod}_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}}$ by definition. The left–right modules $\mathscr{P}$ and $\mathscr{Q}$ can be used as kernels of integral transforms between derived categories. More precisely we have functors

$$\begin{tikzcd} \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^\vee \to \mathbb{X}_{\boldsymbol{\Pi}}]} : \mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, -1) \arrow[r] & \mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}}) \\ \mathscr{F} \arrow[r] & Rp_{1*}(\mathscr{P} \otimes^{\mathbb{L}}_{p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}} p_2^{-1}\mathscr{F}) \end{tikzcd} \tag{5.2}$$

and

$$\begin{aligned}\phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]} : \mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}}) &\longrightarrow \mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)\\ \mathscr{F} &\longrightarrow Rp_{1*}(\mathscr{Q}\otimes^{\mathbb{L}}_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}} p_2^{-1}\mathscr{F}).\end{aligned} \tag{5.3}$$

Here p_1 and p_2 denote the projections on the first and second factors of the product $X \times ({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$. Alternatively, they can be thought of as the projections onto the two factors in the fiber product $\mathbb{X}_{\boldsymbol{\Pi}} \times_{\mathbb{D}} {}_{\boldsymbol{B}}\mathbb{X}^{\vee}$, but from that point of view we have to use the fact that $\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$ is commutative and regard $\mathscr{P}$ as a left $\mathcal{O}_{\mathbb{X}_{\boldsymbol{\Pi}}\times_{\mathbb{D}}{}_{\boldsymbol{B}}\mathbb{X}^{\vee}} = p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \mathbin{\widehat{\otimes}}_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$-module. The two points of view are equivalent but the second introduces a certain asymmetry in the treatment of $\mathscr{P}$ and $\mathscr{Q}$ and so we chose to consistently work with left–right modules rather than left modules over tensor product algebras.

The integral transform functors (5.2) and (5.3) are well-defined functors between all flavors of derived categories that we consider. Indeed these integral transforms are compositions of sheaf-theoretic pullbacks, tensor products over sheaves of rings, and derived direct images. The sheaf-theoretic pullbacks are always exact so they do not cause any trouble in the definition. The tensoring with the Poincaré sheaves $\mathscr{P}$ (or with $\mathscr{Q}$) is also exact since by definition $\mathscr{P}$ is a flat $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \mathbin{\widehat{\otimes}}_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$ module and both $\mathscr{A}_{\boldsymbol{\Pi}}$ and $\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^{\vee}}$ are flat over $\mathbb{C}[[\hbar]]$. Finally, since the derived pushforward is defined by means of injective resolutions, it always makes sense as a functor between bounded derived categories. In fact, the pushforward makes sense as a functor between unbounded or bounded above derived categories since our maps are proper maps of finite homological dimension and thus satisfy the sufficient condition [Har66, § II.2]. Alternatively, one can use the technique of Spaltenstein which allows us to define all functors on the unbounded derived category by means of K-injective and K-flat resolutions [Spa88].

Note also that, by the discussion in the quasi-coherent digression above, all of these functors preserve quasi-coherence. To see that the integral transforms restrict to functors between the corresponding coherent categories $\mathsf{D}^*_{\mathrm{c}}$ we have to argue that the analogue of Grauert's direct image theorem holds in our case. By now there is plenty of technology in the literature to handle this; see, for example, [Sch94]. The details are routine so to keep down the size of the paper we do not include them. The essential hypotheses to check in order to apply [Sch94] is that $\mathscr{A}_{\boldsymbol{\Pi}}$ is a sheaf of nuclear Frechet, multiplicatively convex algebras. We have noted all of these conditions above.

Now we are ready to state our main equivalence result.

THEOREM 5.2. *Suppose that X is a g-dimensional complex torus equipped with a holomorphic Poisson structure $\boldsymbol{\Pi}$. Let $\mathbb{X}_{\boldsymbol{\Pi}} \to \mathbb{D}$ be the corresponding Moyal quantization, and let ${}_{\boldsymbol{B}}\mathbb{X}^{\vee} \to \mathbb{D}$ be the dual $\mathcal{O}^{\times}$-gerbe on $\mathbb{X}^{\vee}$. Then we have isomorphisms of functors*

$$\begin{aligned}\phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]} \circ \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to \mathbb{X}_{\boldsymbol{\Pi}}]} &\cong \mathrm{id}_{\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)}[-g]\\ \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to \mathbb{X}_{\boldsymbol{\Pi}}]} \circ \phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]} &\cong \mathrm{id}_{\mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}})}[-g].\end{aligned}$$

In particular, $\mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}})$ and $\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$ are triangulated equivalent. Similarly, $\mathsf{D}^*_{\mathrm{c}}(\mathbb{X}_{\boldsymbol{\Pi}})$ (respectively $\mathsf{D}^*_{\mathrm{qc}}(\mathbb{X}_{\boldsymbol{\Pi}})$) and $\mathsf{D}^*_{\mathrm{c}}({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)$ (respectively $\mathsf{D}^*_{\mathrm{qc}}({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)$) are triangulated equivalent, and so $\mathbb{X}_{\boldsymbol{\Pi}}$ and ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ are Fourier–Mukai partners.

Proof. Similarly to the classical case discussed in § 2, the theorem will follow from the convolution property of the integral transform functors. Specifically, we have natural isomorphisms of functors:

$$\begin{aligned}\phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]} \circ \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to \mathbb{X}_{\boldsymbol{\Pi}}]} &\cong \phi_{\mathscr{Q}*\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]}\\ \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to \mathbb{X}_{\boldsymbol{\Pi}}]} \circ \phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]} &\cong \phi_{\mathscr{P}*\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to \mathbb{X}_{\boldsymbol{\Pi}}]}.\end{aligned} \tag{5.4}$$

Our strategy will be to first compute the convolution $\mathscr{P} * \mathscr{Q}$ of the kernel objects $\mathscr{P}$ and $\mathscr{Q}$ and use the result to show that

$$\phi_{\mathscr{P}}^{[\boldsymbol{B}\mathbb{X}^\vee \to \mathbb{X}_{\mathbf{\Pi}}]} \circ \phi_{\mathscr{Q}}^{[\mathbb{X}_{\mathbf{\Pi}} \to \boldsymbol{B}\mathbb{X}^\vee]} \cong \mathrm{id}_{\mathsf{D}^*(\mathbb{X}_{\mathbf{\Pi}})}[-g].$$

After that we finish the proof of the theorem by using the Bondal–Orlov criterion to check that $\phi_{\mathscr{P}}^{[\boldsymbol{B}\mathbb{X}^\vee \to \mathbb{X}_{\mathbf{\Pi}}]}$ is fully faithful.

To compute $\mathscr{P} * \mathscr{Q}$ consider the triple product $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}}$, which for the purposes of handling left–right modules we view as the product $X \times \boldsymbol{B}\mathbb{X}^\vee \times X$ equipped with the structure sheaf

$$\mathcal{O}_{\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}} = p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_2^{-1}\mathcal{O}_{\boldsymbol{B}\mathbb{X}^\vee} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}}$$

where we write $p_1 : X \times \boldsymbol{B}\mathbb{X}^\vee \times X \to X$, $p_{12} : X \times \boldsymbol{B}\mathbb{X}^\vee \times X \to X \times \boldsymbol{B}\mathbb{X}^\vee$, etc., for the projections onto the corresponding spaces or stacks. We also write $\mathbb{p}_1 : \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}} \to \mathbb{X}_{\mathbf{\Pi}}$, $\mathbb{p}_{12} : \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}} \to \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee$ for the corresponding maps of ringed objects.

Now, according to § 3.1.4 the difference map $d : X \times X \to X$, $d(x,y) = x - y$ for the additive structure on the complex torus X, induces a natural map of ringed spaces

$$\mathbb{d}_{\mathbf{\Pi}} : \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}} \to \mathbb{X}.$$

After inserting $\boldsymbol{B}\mathbb{X}^\vee$ as a middle factor in the triple product $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}}$ we write $\mathbb{p}_{1-3,2}$ for the composition map

$$\begin{tikzcd}
\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}} \arrow[r, "(\mathbb{d}_{\mathbf{\Pi}}(\mathbb{p}_1(\bullet)\text{,}\mathbb{p}_3(\bullet))\text{,}\mathbb{p}_2(\bullet))"] \arrow[dr, "\mathbb{p}_{1-3,2}"'] & \mathbb{X} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \arrow[d, "\mathbb{p}_1 \times \mathbb{r}"] \\
& \mathbb{X} \times_{\mathbb{D}} \mathbb{X}^\vee
\end{tikzcd}$$

where $\mathbb{r} : \boldsymbol{B}\mathbb{X}^\vee \to \mathbb{X}$ is the natural structure map. Being a morphism of ringed objects, the map $\mathbb{p}_{1-3,2} : \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \boldsymbol{B}\mathbb{X}^\vee \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}^{\mathrm{op}} \to \mathbb{X} \times_{\mathbb{D}} \mathbb{X}^\vee$ is given as a pair $\mathbb{p}_{1-3,2} = (p_{1-3,2}, p_{1-3,2}^\sharp)$ where $p_{1-3,2} : X \times \boldsymbol{B}\mathbb{X}^\vee \times X \to X \times X^\vee$ and a morphism of sheaves of algebras

$$p_{1-3,2}^\sharp : \qquad p_{1-3,2}^{-1}\mathcal{O}_{X \times X^\vee} \longrightarrow p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_2^{-1}\mathcal{O}_{\boldsymbol{B}\mathbb{X}^\vee} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}}.$$

As usual, given a coherent sheaf $\mathcal{M}$ on $X \times X^\vee$ we define its pullback via $\mathbb{p}_{1-3,2}$ to be the sheaf on $X \times \boldsymbol{B}\mathbb{X}^\vee \times X$ given by

$$\mathbb{p}_{1-3,2}^*\mathcal{M} := (p_{1-3,2}^{-1}\mathcal{M}) \otimes_{p_{1-3,2}^{-1}\mathcal{O}_{X \times X^\vee}} (p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_2^{-1}\mathcal{O}_{\boldsymbol{B}\mathbb{X}^\vee} \ \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \ p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}}).$$

Since $\mathcal{O}_{X \times X^\vee}$ is commutative, the sheaf $\mathbb{p}_{1-3,2}^*\mathcal{M}$ will have a natural structure of a left $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}$-module and a right $p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}$ module.

With this notation we have the following.

Lemma 5.3. *Let $\mathcal{P} \to X \times X^\vee$ be the classical Poincaré bundle for the pair of dual complex tori $(X, X^\vee)$. On the triple product $X \times (\boldsymbol{B}\mathbb{X}^\vee) \times X$, there is a natural isomorphism of sheaves*

$$p_{12}^{-1}\mathscr{P} \otimes_{p_2^{-1}\mathscr{O}_{\boldsymbol{B}\mathbb{X}^\vee}} p_{23}^{-1}\mathscr{Q} \cong \mathbb{p}_{1-3,2}^*\mathcal{P}[[\hbar]], \tag{5.5}$$

which is also an isomorphism in ${}_{p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}}\mathsf{Mod}_{p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}}$.

Proof. First observe that the sheaves appearing in the two sides of the identity (5.5) can all be specified via factors of automorphy. Thus, the question of proving the lemma reduces to computing and comparing the factors of automorphy of the left-hand side and the right-hand side of (5.5).

As explained in § 4.2 the Poincaré sheaf $\mathscr{P}$ is defined by a $\Lambda \times \mathbf{\Gamma}$ factor of automorphy $\phi_{\mathscr{P}}$ on the vector space $V \times \overline{V}^{\vee}$ taking values in the invertible global sections of the sheaf of algebras $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}$. Explicitly,

$$\begin{aligned} \phi_{\mathscr{P}}:\quad \Lambda \times \mathbf{\Gamma} &\longrightarrow \Gamma(V \times \overline{V}^{\vee},\ (p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}})^{\times}) \\ (\lambda; (\xi, z)) &\longrightarrow ((v, l) \mapsto z e^{\pi\langle \xi, \lambda\rangle} e^{\pi\overline{\langle \xi, v\rangle}} e^{\pi\langle l, \lambda\rangle}). \end{aligned} \tag{5.6}$$

Similarly,

$$\begin{aligned} \phi_{\mathscr{Q}}:\quad \mathbf{\Gamma} \times \Lambda &\longrightarrow \Gamma(\overline{V}^{\vee} \times V,\ (p_1^{-1}\mathcal{O}_{\overline{V}^{\vee}} \,\widehat{\otimes}_{\mathbb{C}}\, p_2^{-1}\mathscr{A}_{\mathbf{\Pi}})^{\times}) \\ (\lambda; (\xi, z)) &\longrightarrow ((l, v) \mapsto z^{-1} e^{-\pi\langle \xi, \lambda\rangle} e^{-\pi\langle l, \lambda\rangle} e^{\pi\overline{\langle \xi, v\rangle}}). \end{aligned} \tag{5.7}$$

Let

$$\phi : \Lambda \times \mathbf{\Gamma} \times \Lambda \to \Gamma(V \times \overline{V}^{\vee} \times V,\ \mathrm{Aut}(p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}}))$$

denote the factor of automorphy for the left–right module $p_{12}^{-1}\mathscr{P} \otimes_{p_2^{-1}\mathscr{O}_{\boldsymbol{B}\mathbb{X}^{\vee}}} p_{23}^{-1}\mathscr{Q}$. The formulas (5.6) and (5.7) now combine in an explicit expression for ϕ. If f is a local section of $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \otimes_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}$, then

$$[\phi(\lambda, (\xi, z), \mu)(f)](v, x, w) = [p_2^{-1}(e^{\pi\langle \bullet+\xi, \lambda-\mu\rangle}) p_3^{-1}(e^{-\pi\overline{\langle \xi, \bullet\rangle}}) \star f \star p_1^{-1}(e^{\pi\overline{\langle \xi, \bullet\rangle}})](v, x, w).$$

We now compute the factor of automorphy for $\mathbb{p}_{1-3,2}^{*}\mathcal{P}[[\hbar]]$. Let $\phi_{\mathcal{P}}$ denote the factor of automorphy for the classical Poincaré sheaf $\mathcal{P} \to X \times X^{\vee}$

$$\begin{aligned} \phi_{\mathcal{P}}:\quad \Lambda \times \Lambda^{\vee} &\longrightarrow \Gamma(V \times \overline{V}^{\vee},\ (p_1^{-1}\mathcal{O}_{V} \,\widehat{\otimes}_{\mathbb{C}}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}})^{\times}) \\ (\lambda, \xi) &\longrightarrow ((v, l) \mapsto e^{\pi\langle l+\xi, \lambda\rangle} e^{\pi\overline{\langle \xi, v\rangle}}). \end{aligned} \tag{5.8}$$

Consider the map

$$\begin{aligned} \mathsf{diff}:\ V \times \overline{V}^{\vee} \times V &\to V \times \overline{V}^{\vee} \\ (v, x, w) &\mapsto (v - w, x). \end{aligned} \tag{5.9}$$

The factor of automorphy ψ for $\mathbb{p}_{1-3,2}^{*}\mathcal{P}[[\hbar]]$ is a map

$$\psi : \Lambda \times \mathbf{\Gamma} \times \Lambda \to \Gamma(V \times \overline{V}^{\vee} \times V,\ \mathscr{A}ut(p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}})),$$

where the automorphism $\psi(\lambda; (\xi, z); \mu)$ is given by right–left multiplication by the invertible section

$$p_{1-3,2}^{\sharp}(p_{1-3,2}^{-1}\phi_{\mathcal{P}} \circ \mathsf{diff})(\lambda; (\xi, z); \mu)$$

of the sheaf of algebras $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}$. Since the map $p_{1-3,2}^{\sharp}$ is defined in terms of the coproduct structure of § 4.2 we compute explicitly:

$$\begin{aligned} &[p_{1-3,2}^{\sharp}(p_{1-3,2}^{-1}\phi_{\mathcal{P}} \circ \mathsf{diff})(\lambda; (\xi, z); \mu)](v, l, w) \\ &\quad = (1 \otimes e^{\pi\langle l+\xi, \lambda-\mu\rangle} \otimes 1) \star (e^{\pi\overline{\langle \xi, v\rangle}} \otimes 1 \otimes 1) \star (1 \otimes 1 \otimes e^{-\pi\overline{\langle \xi, w\rangle}}) \\ &\quad = e^{\pi\overline{\langle \xi, v\rangle}} \otimes e^{\pi\langle l+\xi, \lambda-\mu\rangle} \otimes e^{-\pi\overline{\langle \xi, w\rangle}}. \end{aligned} \tag{5.10}$$

Thus, $\psi(\lambda; (\xi, z); \mu)$ is the automorphism of $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathcal{O}_{\overline{V}^{\vee}}[[\hbar]] \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_3^{-1}\mathscr{A}_{\mathbf{\Pi}}^{\mathrm{op}}$ given by $p_1^{-1}e^{\pi\overline{\langle \xi, \bullet\rangle}}$ acting by a right multiplication, $p_3^{-1}e^{-\pi\overline{\langle \xi, \bullet\rangle}}$ acting by a left multiplication, and the function $p_2^{-1}e^{\pi\langle \bullet+\xi, \lambda-\mu\rangle}$ acting in the unambiguous, obvious way. Since this agrees precisely with the description of ϕ, we are done. $\square$

To finish the computation of the convolution $\mathscr{P} * \mathscr{Q}$ we use the fiber-product diagram:

$$\begin{tikzcd}
\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} {}_{\boldsymbol{B}}\mathbb{X}^{\vee} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}} \arrow[r, "\mathbb{p}_{1-3,2}"] \arrow[d, "p_{13}"'] & \mathbb{X}_0 \times_{\mathbb{D}} {}_0\mathbb{X}^{\vee} \arrow[d, "p_1"] \\
\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}} \arrow[r, "\mathbb{p}_{1-2}"'] & \mathbb{X}_0
\end{tikzcd}$$

where $\mathbb{X}_0 = X \times \mathbb{D}$ denotes the trivial formal deformation of X and ${}_0\mathbb{X}^{\vee}$ denotes the trivial $\mathcal{O}^{\times}$-gerbe on the space $\mathbb{X}^{\vee} = X^{\vee} \times \mathbb{D}$.

Alternatively, viewing the maps $\mathbb{p}_{1-3,2} = (p_{1-3,2}, p^{\sharp}_{1-3,2})$, etc., as morphisms of ringed spaces we can apply the base change property and Lemma 5.3 to conclude that

$$\begin{aligned}
\mathscr{P} * \mathscr{Q} &:= Rp_{13*}(p_{12}^{-1}\mathscr{P} \otimes_{p_2^{-1}\mathcal{O}_{\boldsymbol{B}\mathbb{X}^{\vee}}} p_{23}^{-1}\mathscr{Q}) \\
&= Rp_{13*}\mathbb{p}^{*}_{1-3,2}\mathcal{P}[[\hbar]] \\
&= \mathbb{p}^{*}_{1-2}(Rp_{1*}\mathcal{P}[[\hbar]]) \\
&= \mathbb{p}^{*}_{1-2}(\mathcal{O}_0[[\hbar]])[-g],
\end{aligned}$$

where $\mathcal{O}_0$ denotes the skyscraper sheaf on X supported at the origin $0 \in X$. In particular, the identity

$$\phi_{\mathscr{P}}^{[\boldsymbol{B}\mathbb{X}^{\vee} \to \mathbb{X}_{\mathbf{\Pi}}]} \circ \phi_{\mathscr{Q}}^{[\mathbb{X}_{\mathbf{\Pi}} \to \boldsymbol{B}\mathbb{X}^{\vee}]} \cong \mathrm{id}_{\mathsf{D}^{*}(\mathbb{X}_{\mathbf{\Pi}})}[-g]$$

will follow immediately from the convolution property (5.4) and the following.

LEMMA 5.4. *We have*

$$\phi^{[\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{X}_{\mathbf{\Pi}}]}_{\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]]} = \mathrm{id}_{\mathsf{D}^{*}(\mathbb{X}_{\mathbf{\Pi}})}.$$

Proof. Consider the sheaf $\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]]$ on the topological space $X \times X$. By construction it is naturally a left $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}$-module and a right $p_2^{-1}\mathscr{A}_{\mathbf{\Pi}}$-module. The element $\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]] \in {}_{p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}}\mathsf{Mod}_{p_2^{-1}\mathscr{A}_{\mathbf{\Pi}}}$ is easy to compute. Recall that the sheaf $\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]]$ is defined as the tensor product

$$\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]] = (p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \mathbin{\widehat{\otimes}}_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathscr{A}^{\mathrm{op}}_{\mathbf{\Pi}}) \otimes_{p^{-1}_{1-2}\mathcal{O}_X[[\hbar]]} (p^{-1}_{1-2}\mathcal{O}_0[[\hbar]])$$

of $p^{-1}_{1-2}\mathcal{O}_0[[\hbar]]$ with the sheaf of algebras $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \otimes_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathscr{A}^{\mathrm{op}}_{\mathbf{\Pi}}$, where the tensor product is taken over the algebra $p^{-1}_{1-2}\mathcal{O}[[\hbar]]$ via the coproduct homomorphism

$$p^{\sharp}_{1-2} : p^{-1}_{1-2}\mathcal{O}_X[[\hbar]] \to p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \mathbin{\widehat{\otimes}}_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathscr{A}^{\mathrm{op}}_{\mathbf{\Pi}}$$

defined in § 3.1.4. In these terms the left–right module structure on $\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]]$ arises from the left multiplication of $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}} \mathbin{\widehat{\otimes}}_{\mathbb{C}[[\hbar]]} p_2^{-1}\mathscr{A}^{\mathrm{op}}_{\mathbf{\Pi}}$ on itself.

Consider the following commutative diagram of topological spaces.

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d] & X \times X \arrow[d, "p_{1-2}"] \\
0 \arrow[r] & X
\end{tikzcd}$$

From this diagram, we deduce $p^{-1}_{1-2}\mathcal{O}_0[[\hbar]] \cong \Delta_{*}\mathbb{C}[[\hbar]]$. Below we use this identification to argue that the element $\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]] \in {}_{p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}}\mathsf{Mod}_{p_2^{-1}\mathscr{A}_{\mathbf{\Pi}}}$ is given as

$$\mathbb{p}^{*}_{1-2}\mathcal{O}_0[[\hbar]] \cong \Delta_{*}\mathscr{A}_{\mathbf{\Pi}}. \tag{5.11}$$

Here we view $\Delta_{*}\mathscr{A}_{\mathbf{\Pi}}$ as a sheaf on $X \times X$ supported on the diagonal and equipped with natural left–right module structure. Namely the left $p_1^{-1}\mathscr{A}_{\mathbf{\Pi}}$-module structure corresponding to left multiplication by elements in $\Delta^{-1}p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} = \mathscr{A}_{\boldsymbol{\Pi}}$ and the right $p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}$-module structure corresponding to right multiplication by elements in $\Delta^{-1}p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}} = \mathscr{A}_{\boldsymbol{\Pi}}$.

The isomorphism claimed in (5.11) is given by the following mutually inverse maps in ${}_{p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}}}\mathsf{Mod}_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}}$

$$\begin{aligned} (p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}}) \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]]) &\longrightarrow \Delta_*\mathscr{A}_{\boldsymbol{\Pi}} \\ a\otimes b\otimes(\Delta_* c) &\longmapsto \Delta_*((\Delta^{-1}(b))\star(\Delta^{-1}(a))\star c) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \Delta_*\mathscr{A}_{\boldsymbol{\Pi}} &\longrightarrow (p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}}) \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]]) \\ \Delta_* f &\longmapsto 1\otimes p_1^{-1}(f)\otimes 1. \end{aligned} \tag{5.13}$$

The composition of (5.13) followed by (5.12) tautologically gives the identity. Composing in the reverse order also gives the identity. Indeed, the composed map is

$$a\otimes b\otimes(\Delta_* c) \mapsto p_1^{-1}(\Delta^{-1}(b)\star\Delta^{-1}(a)\star c)\otimes 1\otimes 1.$$

Consider now the element

$$a\otimes 1\otimes \Delta_* c \in p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}} \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]]).$$

Taking into account that $p_1^{-1}(\Delta^{-1}(a)) = a$, $p_2^{-1}(\Delta^{-1}(b)) = b$, and using the definition of the left–right module structure on $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}} \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]])$, we get

$$\begin{aligned} a\otimes b\otimes\Delta_* c &= p_1^{-1}((\Delta^{-1}(a))\star c)\otimes b\otimes 1 \\ &= (p_1^{-1}((\Delta^{-1}(a))\star c)\otimes 1\otimes 1)\cdot b. \end{aligned}$$

Moreover, for any $x\in p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}}$, we have the identity

$$(x\otimes 1\otimes 1)\cdot b = p_1^{-1}(\Delta^{-1}(b))\cdot(x\otimes 1\otimes 1), \tag{5.14}$$

valid in the left–right module $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\, p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}} \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]])$. This identity follows immediately from the definition of the coproduct map $p_{1-2}^{\sharp}$ and from the fact that this module is supported on the diagonal in $X\times X$.

Combining the identity (5.14) with the definition of the left $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}}$ action on $p_1^{-1}\mathscr{A}_{\boldsymbol{\Pi}} \,\widehat{\otimes}_{\mathbb{C}[[\hbar]]}$ $p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}^{\mathrm{op}} \otimes_{p_{1-2}^{-1}\mathcal{O}_X[[\hbar]]} (\Delta_*\mathbb{C}[[\hbar]])$, yields

$$(x\otimes 1\otimes 1)\cdot b = p_1^{-1}(\Delta^{-1}(b))\cdot(x\otimes 1\otimes 1) = (p_1^{-1}(\Delta^{-1}(b))\star x)\otimes 1\otimes 1,$$

and so $a\otimes b\otimes\Delta_* c = p_1^{-1}(\Delta^{-1}(b)\star\Delta^{-1}(a)\star c)\otimes 1\otimes 1$.

Now that we have the isomorphism $\mathbb{p}_{1-2}^*\mathcal{O}_0[[\hbar]] \cong \Delta_*\mathscr{A}_{\boldsymbol{\Pi}}$ we can easily compute that for any element $\mathscr{M}\in\mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}})$

$$\begin{aligned} Rp_{1*}(\mathbb{p}_{1-2}^*\mathcal{O}_0[[\hbar]] \otimes_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}} p_2^{-1}\mathscr{M}) &\cong Rp_{1*}(R\Delta_*\mathscr{A}_{\boldsymbol{\Pi}} \otimes_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}} p_2^{-1}\mathscr{M}) \\ &\cong Rp_{1*}R\Delta_*(\mathscr{A}_{\boldsymbol{\Pi}} \otimes_{\Delta^{-1}p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}} \Delta^{-1}p_2^{-1}\mathscr{M}) \end{aligned}$$

in other words we have

$$Rp_{1*}(\mathbb{p}_{1-2}^*\mathcal{O}_0[[\hbar]] \otimes_{p_2^{-1}\mathscr{A}_{\boldsymbol{\Pi}}} p_2^{-1}\mathscr{M}) \cong Rp_{1*}R\Delta_*(\mathscr{A}_{\boldsymbol{\Pi}} \otimes_{\mathscr{A}_{\boldsymbol{\Pi}}} \mathscr{M}) \cong Rp_{1*}R\Delta_*(\mathscr{M}) \cong \mathscr{M}.$$

This completes the proof of the lemma. □

In the opposite direction we must verify the identity

$$\phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}}\to {}_{\boldsymbol{B}}\mathbb{X}^\vee]} \circ \phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^\vee\to\mathbb{X}_{\boldsymbol{\Pi}}]} \cong \mathrm{id}_{\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee,-1)}[-g].$$

By Lemma 5.4 we have

$$\phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to\mathbb{X}_{\mathbf{\Pi}}]}\circ\phi_{\mathscr{Q}}^{[\mathbb{X}_{\mathbf{\Pi}}\to{}_{\boldsymbol{B}}\mathbb{X}^{\vee}]}\cong\mathrm{id}_{\mathsf{D}^{*}(\mathbb{X}_{\mathbf{\Pi}})}[-g],$$

and so $\phi_{\mathscr{P}}:=\phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to\mathbb{X}_{\mathbf{\Pi}}]}$ is essentially surjective. Therefore, if we can check that $\phi_{\mathscr{P}}$ is fully faithful we can conclude that $\phi_{\mathscr{P}}$ is an equivalence and that $\phi_{\mathscr{Q}}[g]$ is the inverse.

LEMMA 5.5. *The functor* $\phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee}\to\mathbb{X}_{\mathbf{\Pi}}]}$ *is fully faithful.*

To argue that $\phi_{\mathscr{P}}$ is fully faithful we first identify an orthogonal spanning class of objects for the category $\mathsf{D}^{*}({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)$, and then check that $\phi_{\mathscr{P}}$ satisfies the Bondal–Orlov faithfulness criterion [BO95, Bri99] on the spanning class. To apply the criterion we need to know that the functor $\phi_{\mathscr{P}}$ has left and right adjoints. For this we only need to note that $\phi_{\mathscr{P}}$ is given as the composition of a pullback, a tensoring with $\mathscr{P}$, and a pushforward. Since the pullback and the pushforward have left and right adjoints and the tensoring with $\mathscr{P}$ has a left and right adjoint given by the tensoring with $\mathscr{Q}$, we get that $\phi_{\mathscr{P}}$ has both left and right adjoints. This puts us in a position to apply the Bondal–Orlov criterion. We proceed in several steps.

Step 1. Consider the category $\mathsf{D}^{*}({}_{\boldsymbol{B}}\mathbb{X}^{\vee},-1)$ of weight (-1) sheaves on the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$. We have a natural collection of objects in this category labeled by the points of the torus $X^{\vee}$.

Indeed, suppose that $s\in X^{\vee}$ is a point in the torus $X^{\vee}$ and let $\mathfrak{s}:\mathbb{D}\to\mathbb{X}^{\vee}$ be the constant section of $\mathbb{X}^{\vee}=X^{\vee}\times\mathbb{D}\to\mathbb{D}$ passing through s. The first observation is that the pullback of the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ by $\mathfrak{s}$ is a trivial $\mathcal{O}^{\times}$-gerbe on $\mathbb{D}$. Indeed, the gerbe ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ was defined as a global quotient $[\overline{\mathbb{V}}^{\vee}/\mathbb{\Gamma}]$ with $\mathbb{\Gamma}$ acting through the projection $\mathbb{\Gamma}\to\mathbb{\Lambda}^{\vee}$ and the natural free action of $\mathbb{\Lambda}^{\vee}$ on $\overline{\mathbb{V}}^{\vee}$. In particular, for any space S and any map $f:S\to X^{\vee}$ for which $S\times_{X^{\vee}}\overline{V}^{\vee}$ is a trivial $\Lambda^{\vee}$ cover of S, the f-pullback of ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ will be a trivial $\mathcal{O}^{\times}$-gerbe. This implies that for any weight $k\in\mathbb{Z}$ we can view the sheaf $\mathfrak{s}_{*}\mathcal{O}_{\mathbb{D}}=\mathcal{O}_{s\times\mathbb{D}}$ as a sheaf on ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ of pure weight k.

In fact, the presentation of ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ as a global quotient provides a canonical way of endowing $\mathfrak{s}_{*}\mathcal{O}_{\mathbb{D}}$ with the structure of a weight k sheaf on the gerbe. To that end, consider the universal covering map $\pi:\overline{V}^{\vee}\to X^{\vee}$. The preimage $F_s:=\pi^{-1}(s)\subset\overline{V}^{\vee}$ is a $\Lambda^{\vee}$-orbit in $\overline{V}^{\vee}$, and the pullback gerbe

$$\mathfrak{s}^{*}({}_{\boldsymbol{B}}\mathbb{X}^{\vee}):={}_{\boldsymbol{B}}\mathbb{X}^{\vee}\times_{\mathbb{X}^{\vee},\mathfrak{s}}\mathbb{D}$$

is naturally realized as the global quotient $[(F_s\times\mathbb{D})/\mathbb{\Gamma}]$. Looking at $\mathbb{D}$-points it is clear that in order to equip $\mathfrak{s}_{*}\mathcal{O}_{\mathbb{D}}$ with the structure of a weight (-1)-sheaf on ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$, it suffices to describe a Γ equivariant structure on $\mathcal{O}_{F_s}[[\hbar]]$ in which every central element $z\in\mathbb{C}[[\hbar]]^{\times}\subset\Gamma$ acts as multiplication by z^{-1} viewed as a section in $\mathcal{O}_{F_s}[[\hbar]]$.

To achieve this as before we let $c:\Lambda^{\vee}\times\Lambda^{\vee}\to\mathbb{C}[[\hbar]]^{\times}$ denote the cocycle defining the group Γ. Since c was given as the exponential of a multiple of the $\mathbb{R}$-linear map $B:\bigwedge^2\overline{V}^{\vee}\to\mathbb{C}$, we can use $\mathbb{R}$-linearity to extend c to a multiplicative map $\tilde{c}:\Lambda^{\vee}\otimes\overline{V}^{\vee}\to\mathbb{C}[[\hbar]]^{\times}$. Now, suppose $(\xi,z)\in\Gamma$ and let $f\in\mathcal{O}_{F_s}[[\hbar]]$. Since F_s is a discrete set of points, we can write f as a collection $\{f_w\}_{w\in F_s}$ with $f_w\in\mathcal{O}_{F_s,w}[[\hbar]]=\mathbb{C}[[\hbar]]$. With this notation (ξ,z) gives rise to an automorphism

$$\rho_{(\xi,z)}:H^0(F_s,\mathcal{O}_{F_s}[[\hbar]])\to H^0(F_s,\mathcal{O}_{F_s}[[\hbar]]),$$

where for each $w\in F_s$ we define

$$\rho_{(\xi,z)}=\{(\rho_{(\xi,z)}f)_w\}_{w\in F_s},\quad\text{with }(\rho_{(\xi,z)}f)_w:=z^{-1}\tilde{c}(w,\xi)f_{w-\xi}.$$

To check that ρ is a Γ-action we compute

$$\begin{aligned}(\rho_{(\xi',z')}\rho_{(\xi,z)}f)_w&=z'^{-1}\tilde{c}(w,\xi')(\rho_{(\xi,z)}f)_{w-\xi'}\\&=z'^{-1}z^{-1}\tilde{c}(w,\xi')\tilde{c}(w-\xi',\xi)f_{w-\xi'-\xi}\\&=(z'z)^{-1}\tilde{c}(-\xi',\xi)\tilde{c}(w,\xi+\xi')f_{w-\xi'-\xi}.\end{aligned}$$

On the other hand, $(\xi', z') \cdot (\xi, z) = (\xi + \xi', zz'c(\xi', \xi))$ and so

$$(\rho_{(\xi',z')\cdot(\xi,z)} f)_w = (z'z)^{-1} \tilde{c}(\xi', \xi)^{-1} \tilde{c}(w, \xi + \xi') f_{w-\xi'-\xi}.$$

taking into account the fact that c is bilinear on $\Lambda^\vee$ we conclude that $\rho_{(\xi',z')\cdot(\xi,z)} = \rho_{(\xi',z')}\rho_{(\xi,z)}$.

For future reference we denote the weight (-1)-sheaf given by ρ by $(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}$. Clearly this generalizes to all weights $k \in \mathbb{Z}$ yielding weight k sheaves $(\mathfrak{s}, k)_*\mathcal{O}_{\mathbb{D}}$ on ${}_{\boldsymbol{B}}\mathbb{X}^\vee$. In fact, we have a pair of adjoint functors $(L(\mathfrak{s}, k)^*, (\mathfrak{s}, k)_*)$ between the categories $D^*(\mathbb{D})$ and $\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, k)$, defined for all integers k.

The existence of such functors is a basic fact about any morphism between a space and an $\mathcal{O}^\times$ gerbe. Indeed, suppose that S is a space and $\mathscr{T}$ is an $\mathcal{O}^\times$ gerbe on a space T. Suppose that we are given a morphism of spaces $f : S \to T$ with the property that $f^*\mathscr{T}$ is trivializable. Choosing a trivialization $f^*\mathscr{T} \cong {}_0S$ we get a well-defined pair of adjoint functors (Lf^*, Rf_*) between $D^*(S)$ and $\mathsf{D}^*(\mathscr{T})$. By construction these functors are compatible with the weight decompositions so we get induced adjoint pairs between the corresponding weight pieces. These can, in turn, be combined with the canonical equivalences $D^*(S) = \mathsf{D}^*({}_0S, k)$ which are also defined for all k. This results in adjoint pairs of functors

$$\begin{tikzcd} D^*(S) \arrow[r, shift left, "R(f{,}k)_*"] & \mathsf{D}^*(\mathscr{T}, k) \arrow[l, shift left, "L(f{,}k)^*"] \end{tikzcd} \tag{5.15}$$

which we frequently use below. Note that the construction of (5.15) depends on the choice of trivialization of the gerbe $f^*\mathscr{T}$. In the particular case of the map $\mathfrak{s} : \mathbb{D} \to \mathbb{X}^\vee$, we used a special trivialization constructed out of the quotient presentation of ${}_{\boldsymbol{B}}\mathbb{X}^\vee$. This trivialization is precisely encoded in the map $\tilde{c}$ used above.

Consider now the collection of objects $\{(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}\}_{s \in X^\vee} \subset \mathrm{ob}(\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, -1))$. We argue that this is an orthogonal spanning class of $\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, -1)$.

The orthogonality is obvious since for any two points $s \neq t \in X^\vee$ the supports of the sheaves $(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}$ and $(\mathfrak{t}, -1)_*\mathcal{O}_{\mathbb{D}}$ are disjoint substacks in ${}_B\mathbb{X}^\vee$.

To show that these sheaves span the category, we need to check that if A is a complex of sheaves on ${}_B\mathbb{X}^\vee$ of pure weight (-1), with the property that A is left (respectively right) orthogonal to all $(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}$, then $A = 0$ in $\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, -1)$. Suppose first $R\operatorname{Hom}((\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}, A) = 0$ for all $s \in X^\vee$. Let $j : X^\vee \to {}_B\mathbb{X}^\vee$ be the natural closed immersion. We have a distinguished triangle

$$\begin{tikzcd} A \arrow[r, "\hbar"] & A \arrow[r] & (j, -1)_*L(j, -1)^*A \arrow[r] & A[1] \end{tikzcd} \tag{5.16}$$

and since the complex of vector spaces $R\operatorname{Hom}((\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}, A)$ is exact, it follows that the complex $R\operatorname{Hom}((\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}, (j, -1)_*L(j, -1)^*A)$ is exact.

By adjunction we get that

$$R\operatorname{Hom}_X(L(j, -1)^*(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}, L(j, -1)^*A) = 0. \tag{5.17}$$

However, $L(j, -1)^*(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}}$ can be computed explicitly:

$$L(j, -1)^*(\mathfrak{s}, -1)_*\mathcal{O}_{\mathbb{D}} = Lj^*\mathfrak{s}_*\mathcal{O}_{\mathbb{D}} = \mathcal{O}_s. \tag{5.18}$$

The first equality in (5.18) follows tautologically from the definition of the functors $L(j, -1)^*$ and $(\mathfrak{s}, -1)_*$ and the second follows immediately from the base change identity $Lj^*\mathfrak{s}_*\mathcal{O}_{\mathbb{D}} = Rs_*Li^*\mathcal{O}_{\mathbb{D}} = Rs_*\mathcal{O}_0 = \mathcal{O}_s$. Here $i : 0 \to \mathbb{D}$ denotes the inclusion of the closed point and $s : 0 \to X^\vee$ is the map given by the point s.

Now (5.17) and (5.18) imply that $R\operatorname{Hom}_X(\mathcal{O}_s, L(j, -1)^*A) = 0$ for all $s \in X^\vee$. Since the structure sheaves of points form a spanning class in the derived category of $X^\vee$ it follows that $L(j, -1)^*A = 0$. Thus, $(j, -1)_*L(j, -1)^*A = 0$ in $\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee, -1)$. Now the exact triangle (5.16) implies that multiplication by $\hbar$ is an isomorphism on all cohomology sheaves of the complex A. By Nakayama's lemma this implies that the cohomology sheaves of A are all zero and so A is quasi-isomorphic to the zero complex.

Step 2. Given a section $\mathfrak{s} : \mathbb{D} \to \mathbb{X}^\vee$ of $\mathbb{X}^\vee \to \mathbb{D}$, we check that

$$\phi_{\mathscr{P}}((\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}}) \cong \mathscr{P}_{\mathfrak{s}}, \tag{5.19}$$

where $\mathscr{P}_{\mathfrak{s}}$ is the line bundle

$$\mathscr{P}_{\mathfrak{s}} := L(\mathrm{id} \times \mathfrak{s}, 1)^* \mathscr{P} = (\mathrm{id} \times \mathfrak{s}, 1)^* \mathscr{P}$$

on $\mathbb{X}_{\boldsymbol{\Pi}}$.

Indeed, if we write $\mathbb{u}_{\boldsymbol{\Pi}} : \mathbb{X}_{\boldsymbol{\Pi}} \to \mathbb{D}$ for the structure morphism of $\mathbb{X}_{\boldsymbol{\Pi}}$ we get a natural cartesian diagram

$$\begin{tikzcd}
\mathbb{X}_{\boldsymbol{\Pi}} \arrow[r, "\mathrm{id} \times \mathbb{s}"] \arrow[d, "\mathbb{u}_{\boldsymbol{\Pi}}"'] & \mathbb{X}_{\boldsymbol{\Pi}} \times_{\mathbb{D}} {}_{\boldsymbol{B}}\mathbb{X}^\vee \arrow[d, "\mathbb{p}_2"] \\
\mathbb{D} \arrow[r, "\mathbb{s}"'] & {}_{\boldsymbol{B}}\mathbb{X}^\vee
\end{tikzcd}$$

where $\mathbb{s} : \mathbb{D} \to {}_{\boldsymbol{B}}\mathbb{X}^\vee$ denotes the map corresponding to our preferred trivialization of the $\mathcal{O}^\times$-gerbe $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ on $\mathbb{D}$.

Base changing along this diagram now gives $p_2^{-1}(\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}} = R(\mathrm{id} \times \mathbb{s}, -1)_* \mathbb{u}_{\boldsymbol{\Pi}}^{-1} \mathcal{O}_{\mathbb{D}}$. Thus,

$$\begin{aligned}
\mathscr{P} \otimes_{p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}} p_2^{-1}(\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}} &= \mathscr{P} \otimes_{p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}} R(\mathrm{id} \times \mathfrak{s}, -1)_* \mathbb{u}_{\boldsymbol{\Pi}}^{-1} \mathcal{O}_{\mathbb{D}} \\
&= R(\mathrm{id} \times \mathfrak{s}, 0)_* (L(\mathrm{id} \times \mathfrak{s}, 1)^* \mathscr{P} \otimes_{\mathbb{u}_{\boldsymbol{\Pi}}^{-1}\mathcal{O}_{\mathbb{D}}} \mathbb{u}_{\boldsymbol{\Pi}}^{-1} \mathcal{O}_{\mathbb{D}}) \\
&= R(\mathrm{id} \times \mathfrak{s}, 0)_* \mathscr{P}_{\mathfrak{s}},
\end{aligned} \tag{5.20}$$

where for the second equality we used the projection formula applied to the map id $\times \mathbb{s}$.

We are now in a position to check (5.19). By definition, we have

$$\phi_{\mathscr{P}}((\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}}) = Rp_{1*}(\mathscr{P} \otimes_{p_2^{-1}\mathcal{O}_{{}_{\boldsymbol{B}}\mathbb{X}^\vee}} p_2^{-1}((\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}}))$$

and therefore by (5.20)

$$\begin{aligned}
\phi_{\mathscr{P}}((\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}}) &= Rp_{1*} R(\mathrm{id} \times \mathfrak{s}, 0)_* \mathscr{P}_{\mathfrak{s}} \\
&= R(p_1 \circ (\mathrm{id} \times \mathfrak{s}, 0))_* \mathscr{P}_{\mathfrak{s}} \\
&= R(\mathrm{id})_* \mathscr{P}_{\mathfrak{s}} \\
&= \mathscr{P}_{\mathfrak{s}}.
\end{aligned}$$

Remark 5.6. This calculation only uses the fact that $\mathfrak{s} : \mathbb{D} \to \mathbb{X}^\vee$ is a section and is insensitive to whether this section is constant or not. In particular, our proof shows that the identity (5.19) is valid for all (not necessarily constant) sections $\mathfrak{s} : \mathbb{D} \to \mathbb{X}^\vee$.

Step 3. Finally we check that

$$R\,\mathrm{Hom}_{\mathsf{D}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)}((\mathfrak{s}, -1)_* \mathcal{O}_{\mathbb{D}}, (\mathfrak{t}, -1)_* \mathcal{O}_{\mathbb{D}}) = R\,\mathrm{Hom}_{\mathsf{D}^*(\mathbb{X}_{\boldsymbol{\Pi}})}(\mathscr{P}_{\mathfrak{s}}, \mathscr{P}_{\mathfrak{t}})$$

for all constant sections $\mathfrak{s}, \mathfrak{t} : \mathbb{D} \to {}_{\boldsymbol{B}}\mathbb{X}^\vee$.

Since $\mathscr{P}_{\mathfrak{s}}$ and $\mathscr{P}_{\mathfrak{t}}$ are translation invariant line bundles on $\mathbb{X}_{\boldsymbol{\Pi}}$, they are naturally $\mathscr{A}_{\boldsymbol{\Pi}}$ bimodules. In particular, there is a well-defined inner hom

$$\mathscr{H}om_{\mathscr{A}_{\boldsymbol{\Pi}}-\mathrm{mod}}(\mathscr{P}_{\mathfrak{s}}, \mathscr{P}_{\mathfrak{t}}) = \mathscr{P}_{\mathfrak{s}}^\vee \otimes_{\mathscr{A}_{\boldsymbol{\Pi}}} \mathscr{P}_{\mathfrak{t}}$$

which is also a translation invariant line bundle on $\mathbb{X}_{\boldsymbol{\Pi}}$. In fact, by writing the factors of automorphy for $\mathscr{P}_{\mathfrak{s}}^\vee$ and $\mathscr{P}_{\mathfrak{t}}$ it is clear that $\mathscr{P}_{\mathfrak{s}}^\vee \otimes_{\mathscr{A}_{\boldsymbol{\Pi}}} \mathscr{P}_{\mathfrak{t}} = \mathscr{P}_{\mathfrak{t}-\mathfrak{s}}$.

Thus, for the global homomorphisms in the derived category we get

$$R\operatorname{Hom}^\bullet(\mathscr{P}_{\mathfrak{s}}, \mathscr{P}_{\mathfrak{t}}) \cong H^\bullet(X, \mathscr{P}_{\mathfrak{t}-\mathfrak{s}}).$$

Now by the computation in Appendix D we conclude that $H^\bullet(X, \mathscr{P}_{\mathfrak{t}-\mathfrak{s}}) = 0$ unless $\mathscr{P}_{\mathfrak{t}-\mathfrak{s}}/\hbar \cong \mathcal{O}_X$ or, equivalently, $\mathfrak{s}(0) = \mathfrak{t}(0)$. Since our sections $\mathfrak{s}$ and $\mathfrak{t}$ are constant, this can happen only when $\mathfrak{s} = \mathfrak{t}$.

Finally, when $\mathfrak{s} = \mathfrak{t}$, we have $\mathscr{P}_{\mathfrak{t}-\mathfrak{s}} = \mathscr{P}_{\mathfrak{o}} = \mathscr{A}_{\mathbf{\Pi}}$, and so as a sheaf on X we have $\mathscr{P}_{\mathfrak{o}} = \mathcal{O}_X[[\hbar]]$. In other words $H^\bullet(X, \mathscr{P}_{\mathfrak{o}}) = H^\bullet(X, \mathcal{O})[[\hbar]]$. This completes the proof of Step 3.

Lemma 5.4 together with Lemma 5.5 now yield a proof of the theorem. ☐

Remark 5.7. It was pointed out by the anonymous referee that in the case of abelian varieties, the proof of the previous theorem can be simplified considerably. The first observation is that the two composition functors in the statement of Theorem 5.2 can be shown to be equivalences by deformation theory. Up to a shift, the composition functor is an integral transform whose kernel is a deformation of the structure sheaf of the diagonal. One can verify order by order that this kernel is a bimodule and then check that after pushing forward with either of the two projections it becomes an invertible bimodule. This will imply that the compositions $\phi_{\mathscr{Q}} \circ \phi_{\mathscr{P}}$ and $\phi_{\mathscr{P}} \circ \phi_{\mathscr{Q}}$ are both equivalences and, in particular, will show that $\phi_{\mathscr{P}}$ and $\phi_{\mathscr{Q}}$ are equivalences. This argument can be used as a replacement for the explicit verification of the Bondal–Orlov conditions in Lemma 5.5 and shows that the first identity in the statement of Theorem 5.2 implies the second.

Finally, to prove the first identity one can use a method employed by Gaitsgory in the context of Fourier transforms for perverse sheaves. The key observation here is that once we know that a functor is an equivalence, its inverse will coincide with its adjoints (both left and right). Therefore, it suffices to check that $\phi_{\mathscr{Q}}$ is the adjoint of $\phi_{\mathscr{P}}$. In the algebraic situation this follows from relative duality theory (and some special care needed to deal with the quasi-coherent sheaves). In the analytic situation this argument has only a chance of working for the Ramis–Ruget notion of quasi-coherence and leads to technical difficulties which we have not analyzed.

5.3 Remarks on classical supports

The reader may have noticed by now that there are some suggestive similarities between the module theory on a deformation quantization and the theory of D-modules. Here we point out a particular aspect of this similarity that has to do with the supports of quantum modules.

Definition 5.8. (i) Suppose that $\mathbb{M} = (M, \mathscr{A}_M)$ is a deformation quantization of a complex manifold M. Let

$$\begin{tikzcd} M \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d] & \mathbb{M} \arrow[d] \\ o \arrow[r, phantom, "\in"] & \mathbb{D} \end{tikzcd}$$

be the inclusion of the closed fiber and let F be a coherent sheaf on $\mathbb{M}$. We define the *classical support of* F as the support of the complex $Li^*F \in \mathsf{D}^b_c(M)$.

(ii) Suppose that N is a complex manifold and suppose $\mathfrak{N} \to \mathbb{N} = N \times \mathbb{D}$ is an $\mathcal{O}^\times$-gerbe which is trivialized on the closed fiber $N \times \{o\} \subset \mathbb{N}$. Let

$$\begin{tikzcd} N \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d] & \mathfrak{N} \arrow[d] \\ o \arrow[r, phantom, "\in"] & \mathbb{D} \end{tikzcd}$$

be the inclusion corresponding to this trivialization and let G be a coherent sheaf of pure weight on $\mathfrak{N}$. We define the *classical support of* G as the support of the complex $Li^*G \in \mathsf{D}^b_{\mathrm{c}}(N)$.

Observe that if the sheaf F on $\mathbb{M}$ is flat over $\mathbb{D}$, then the classical supports of F is just the support of the sheaf i^*F. Similarly, if G is flat over $\mathfrak{N}$, then the classical support of G is the support of the sheaf i^*G.

Claim 5.9. (a) Suppose that $\mathbb{M} = (M, \mathscr{A}_M)$ is a deformation quantization of a complex manifold M and suppose that F is a coherent sheaf on $\mathbb{M}$ which is flat over $\mathbb{D}$. Let $\mathbf{\Pi} \in H^0(M, \bigwedge^2 T_M)$ be the holomorphic Poisson structure associated with $\mathbb{M}$, and let $\{\bullet, \bullet\}_{\mathbf{\Pi}} : \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M \to \mathcal{O}_M$ be the corresponding bracket on functions. Then the classical support $S \subset M$ of F is coisotropic with respect to $\mathbf{\Pi}$, i.e. the ideal sheaf $\mathcal{I}_S \subset \mathcal{O}_M$ satisfies $\{\mathcal{I}_S, \mathcal{I}_S\}_{\mathbf{\Pi}} \subset \mathcal{I}_S$.

(b) Suppose that N is a complex manifold and suppose that $\mathfrak{N} \to \mathbb{N} = N \times \mathbb{D}$ is an $\mathcal{O}^\times$-gerbe which is trivialized on the closed fiber. Let $\boldsymbol{B} \in H^2(N, \mathcal{O})$ be the infinitesimal class of the gerbe $\mathfrak{N}$ and let G be a pure weight one coherent sheaf on $\mathfrak{N}$ which is flat over $\mathbb{D}$. Then the classical support $T \subset N$ of G is $\boldsymbol{B}$-isotropic, i.e. $\boldsymbol{B}_{|T} = 0 \in H^2(T, \mathcal{O}_T)$.

Remark 5.10. There is a puzzling asymmetry in the notions of coisotropic and isotropic defined above. Whereas the property of being coisotropic is geometric, the property of being isotropic appears to be only homological. To justify why our notion of isotropic is meaningful note that if $\boldsymbol{B} \in H^2(N, \mathcal{O}_N)$, and if we choose a Hermitian metric on N, then $\boldsymbol{B}$ can be represented by a $\bar{\partial}$-harmonic $(0,2)$-form $\boldsymbol{\beta}$. When the metric is Kähler and N is compact, the form $\boldsymbol{\beta}$ is d-harmonic and hence closed. In other words, $\boldsymbol{B}$ is represented by a presymplectic form $\boldsymbol{\beta}$ on the C^∞-manifold N. If, now, $T \subset N$ is a complex submanifold for which $\boldsymbol{B}_{|T} = 0 \in H^2(N, \mathcal{O}_N)$, it follows that the restriction $\boldsymbol{\beta}_{|T} \in \Gamma(T, A^{0,2}_T)$ is the zero form, i.e. T is isotropic in the sense of presymplectic geometry.

Note also, that when $N = X^\vee$ was a complex torus and $\mathfrak{N} = {}_{\boldsymbol{B}}\mathscr{X}^\vee$, then the class $\boldsymbol{B}$ had a canonical harmonic representative, since $\boldsymbol{B}$ was given by an element in $\bigwedge^2 V$.

Proof of the Claim. Part (a) is a deformation quantization analogue of the corresponding result for D-modules. It is, in fact, a special case of a general version of Gabber's theorem [Bjö93, Appendix III, Theorem 3.7]. The result [Bjö93, Appendix III, Theorem 3.7] implies the usual Gabber's theorem on D-modules when applied to $\mathcal{R}/t^2$, where $\mathcal{R}$ is the Rees algebra of the sheaf of differential operators. It also implies part (a) of our claim when applied to $\mathscr{A}_M/\hbar^2$. The only thing we need to check is the hypothesis $\ker[F/\hbar^2 \xrightarrow{\hbar\cdot} F/\hbar^2] = \operatorname{im}[F/\hbar^2 \xrightarrow{\hbar\cdot} F/\hbar^2]$ which is immediate from the fact that F is flat over $\mathbb{D}$.

Part (b) follows from [Tod05, Proposition 6.1] applied to $\mathfrak{N}/\hbar^2$. Indeed, according to [Tod05, Proposition 6.1], for any sheaf Φ on N, the existence of a flat extension of Φ to $\mathfrak{N}/\hbar^2$ is equivalent to the vanishing of the product of the exponential $\exp(a(\Phi)) \in \bigoplus_k \operatorname{Ext}^k(G, G \otimes \Omega^k_N)$ of the Atiyah class $a(\Phi)$ of Φ, and the $\boldsymbol{B} \in H^2(N, \mathcal{O}_N) \subset HT^2(N) = \bigoplus_{p+q=2} H^p(N, \bigwedge^q T_N)$. Here the product is defined as the image under the natural map

$$\left(\bigoplus_k \operatorname{Ext}^k(G, G \otimes \Omega^k_N)\right) \otimes \left(\bigoplus_{p,q} H^p(N, \wedge^q T_N)\right) \xrightarrow{\cup} \bigoplus_{a,b} \operatorname{Ext}^a(G, G \otimes \Omega^b_N) \to \bigoplus_a \operatorname{Ext}^a(G, G).$$

In particular, $\exp(a(\Phi)) \cdot \boldsymbol{B}$ can be zero only if $\boldsymbol{B}$ restricts to zero on the support of Φ. Applying this to $\Phi := i^*G$ and taking into account that G is flat, we get the statement (b). The claim is proved. ☐

The claim together with the classical Fourier–Mukai duality impose non-trivial conditions on the support of a sheaf on M, that are necessary for quantizing it. For instance, part (b) of the claim immediately implies that an ample line bundle on an abelian variety can not be quantized since its

Fourier–Mukai transform is a vector bundle on the dual abelian variety. The supports of modules over deformation quantizations was recently investigated in [NT04].

6. The gerbe $\mathscr{P}\boldsymbol{ic}^{\mathbf{0}}(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$

With the equivalence of categories in place we are now ready to identify ${}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ geometrically as the relative Picard stack $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ of degree zero line bundles on $\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{D}$.

Before we define $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ let us recall the classical notion of a Picard variety. In the classical situation, the Picard variety $\mathrm{Pic}^0(Z)$ of a smooth space Z is defined as the moduli space of isomorphism classes of degree zero line bundles. More precisely, consider the site $\mathscr{S}_{\mathrm{an}}$ of analytic spaces with the analytic topology and let $\mathscr{P}ic^0(Z)$ denote the moduli stack on $\mathscr{S}_{\mathrm{an}}$ associated to the prestack:

$$Y \to \left\{\begin{matrix}\text{the groupoid whose objects are holomorphic line bundles on } Z \times Y \text{ of degree} \\ \text{zero relative to } Y\text{, and morphisms are isomorphisms}\end{matrix}\right\}. \tag{6.1}$$

The associated sheaf of sets $\pi_0(\mathscr{P}ic^0(Z))$ associated to $\mathscr{P}ic^0(Z)$ is representable by a space $\mathrm{Pic}^0(Z)$. The stack $\mathscr{P}ic^0(Z)$ is an $\mathcal{O}^{\times}$-gerbe over $\mathrm{Pic}^0(Z)$ which encodes the fact that the automorphisms of a line bundle are given by multiplication by an invertible holomorphic function. Explicitly, for any analytic space Y one introduces an equivalence relation $\sim_Y$ on the collection of all line bundles on $Z \times Y$. Two line bundles $L, M \to Z \times Y$ are considered to be equivalent if L is isomorphic to $M \otimes p_Y^* A$ for some line bundle A on Y. The variety $\mathrm{Pic}^0(Z)$ represents the functor

$$\pi_0(\mathscr{P}ic^0(Z)) : \mathscr{S}_{\mathrm{an}} \to \mathsf{Sets}$$

given by

$$Y \to \left\{\begin{matrix}\text{the set of all } \sim_Y\text{-equivalence classes of holomorphic line bundles on } Z \times Y \text{ of} \\ \text{degree zero relative to } Y\end{matrix}\right\}.$$

Moreover the moduli problem (6.1) can be rigidified in a simple way (by considering line bundles on $Z \times Y$ equipped with a trivialization on $\{o\} \times Y$ for some fixed point $o \in Z$) which trivializes this gerbe $\mathscr{P}ic^0(Z)$. Interestingly, in the non-commutative case we can not resort to a rigidification trick since the non-commutative space $\mathbb{X}_{\mathbf{\Pi}}$ need not have any points. In fact, as we will see below, $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ is a gerbe on $\mathbb{X}^{\vee}$ which is no longer trivializable.

To define line bundles of degree zero on $\mathbb{X}_{\mathbf{\Pi}}$ we look at the translation action of a torus X on itself. Since the holomorphic Poisson structures are constant, this action lifts to the sheaf $\mathscr{A}_{X,\mathbf{\Pi}}$. In particular, X acts on the non-commutative space $\mathbb{X}_{\mathbf{\Pi}}$. In the classical situation, the degree zero line bundles can be characterized as those that are translation invariant. We use this as our definition of degree 0 in the non-commutative case.

Definition 6.1. A line bundle (a locally free rank one left $\mathscr{A}_{X,\mathbf{\Pi}}$-module) on $\mathbb{X}_{\mathbf{\Pi}}$ is said to be *of degree zero* when it is translation invariant.

It turns out that a line bundle $\mathscr{L}$ on $\mathbb{X}_{\mathbf{\Pi}}$ is of degree zero if and only if its classical part $\mathscr{L}/\hbar\mathscr{L}$ has zero first Chern class (see Lemma C.2). It is a coincidence that such an $\mathscr{L}$ is also a bimodule. We will not use these bimodule structures. As discussed in § 4.2, if one wants line bundles to vary in families, the bimodule structures on the individual line bundles can not be chosen in a consistent way.

By definition $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ is the moduli stack of line bundles of relative degree zero on $\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{D}$. To spell this out we need the analytic site $(\mathscr{F}\mathscr{S}/\mathbb{D})_{an}$ of formal analytic spaces over $\mathbb{D}$. Formal analytic spaces are the analytic counterpart of Knutson's formal algebraic spaces. The theory of formal analytic spaces is parallel to [Knu71, ch. V] but with commutative rings replaced with

Stein algebras. A convenient way to look at the formal analytic spaces over $\mathbb{D}$ is as commutative deformation quantizations of analytic spaces. A morphism $\mathbb{f} = (f, f^\sharp) : \mathbb{X} \to \mathbb{Y}$ in the category $\mathscr{F}\mathscr{S}/\mathbb{D}$ is an *analytic open immersion* if the map $f = \mathbb{f}/\hbar : X \to Y$ is an open analytic map, and the map $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism. The analytic topology on the category of formal analytic spaces is the Grothendieck topology associated (in the sense of say [Knu71]) with the subcategory of *analytic open immersions*.

Given a formal analytic space $\mathbb{Y}$ over $\mathbb{D}$, with closed fiber Y we form the ringed space

$$\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{Y} = (X \times Y, p_X^{-1}\mathscr{A}_{X,\mathbf{\Pi}} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} p_Y^{-1}\mathscr{A}_Y)$$

where $\widehat{\otimes}$ denotes the completed tensor product of sheaves of nuclear Frechet algebras.

Given a formal analytic space $\mathbb{Y}$ over $\mathbb{D}$, we say that a line bundle $\mathscr{L}$ on $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{Y}$ is of degree zero relative to $\mathbb{Y}$ if for any section $\sigma : \mathbb{D} \to \mathbb{Y}$, the pullback $(1 \times \sigma)^*(\mathscr{L})$ is of degree zero as a line bundle on $\mathbb{X}_{\mathbf{\Pi}}$.

Definition 6.2. The moduli stack $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ is the stack on $(\mathscr{F}\mathscr{S}/\mathbb{D})_{an}$ associated with the prestack:

$$(\mathbb{Y} \to \mathbb{D}) \to \left\{ \begin{array}{l} \text{the groupoid whose objects are line bundles over } \mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{Y} \text{ of degree zero relative} \\ \text{to } \mathbb{Y}, \text{ and morphisms are isomorphisms} \end{array} \right\}.$$

By definition the Poincaré sheaf $\mathscr{P}$ on $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ is a line bundle of relative degree zero along $\mathbb{X}_{\mathbf{\Pi}}$. Indeed if we pullback $\mathscr{P}$ by a translation by a point in X, then we get a line bundle isomorphic to $\mathscr{P}$. This is easily seen in terms of factors of automorphy. If $w \in V$ is any point, then translation of the factor of automorphy by w results into a cohomologous factor of automorphy. The two factors are related by the coboundary of the element $g \in C^0(\Lambda \times \Gamma, \mathscr{A}^\times_{V\times\overline{V}^\vee}(V \times \overline{V}^\vee)) = \mathscr{A}^\times_{V\times\overline{V}^\vee}(V \times \overline{V}^\vee)$, where $g(l, v) := \exp(\pi\overline{\langle l, w\rangle})$.

In particular, the Poincaré sheaf $\mathscr{P}$ gives rise to a natural morphism of stacks

$$\mathfrak{c} : {}_{\boldsymbol{B}}\mathbb{X}^\vee \to \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D}), \tag{6.2}$$

which sends $\mathbb{f} : \mathbb{Y} \to {}_{\boldsymbol{B}}\mathbb{X}^\vee$ to the line bundle $\mathbb{f}^*\mathscr{P}$ on $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{Y}$.

Our goal is to show that $\mathfrak{c}$ is an isomorphism. The first step is to analyze the relationship between ${}_{\boldsymbol{B}}\mathbb{X}^\vee$ and $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ on the level of $\mathbb{D}$-points.

Proposition 6.3. *The map $\mathfrak{c}$ induces an equivalence between the groupoid ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ of all sections of ${}_{\boldsymbol{B}}\mathbb{X}^\vee \to \mathbb{D}$ and the groupoid $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})$ of global translation invariant line bundles on $\mathbb{X}_{\mathbf{\Pi}}$.*

Proof. The set of isomorphism classes of the groupoid ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ is the set $\mathbb{X}^\vee(\mathbb{D})$ of $\mathbb{D}$-points of the formal space $\mathbb{X}^\vee \to \mathbb{D}$. The natural map from ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ to the discrete groupoid $\mathbb{X}^\vee(\mathbb{D})$ is actually split. This is a special feature of the groupoid of $\mathbb{D}$-points and no such splitting exists for general test spaces $\mathbb{Y} \in \mathscr{F}\mathscr{S}/\mathbb{D}$. For our purposes it will be important to exhibit a distinguished splitting of ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D}) \to \mathbb{X}^\vee(\mathbb{D})$. In fact, we have already done this in the proof of Step 1 of Lemma 5.5. Specifically, to lift a section $\mathfrak{s} : \mathbb{D} \to \mathbb{X}^\vee$ to a section $\mathfrak{s} : \mathbb{D} \to {}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$, we have to produce a natural trivialization of the pullback gerbe $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$. For this we can use the global quotient presentation of ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$. Write $s \in X^\vee$ for the image $\mathfrak{s}(0)$ of the closed point $0 \in \mathbb{D}$. Then in the notation of the proof of Step 1 of Lemma 5.5 we have a quotient presentation $[(F_s \times \mathbb{D})/\mathbb{\Gamma}]$ of $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D}))$. Now a trivialization of the gerbe $[(F_s \times \mathbb{D})/\mathbb{\Gamma}]$ is simply a Γ-equivariant structure on $\mathcal{O}_{F_s}[[\hbar]]$ in which $z \in \mathbb{C}[[\hbar]] \subset \Gamma$ acts as multiplication by z (viewed as a section of $\mathcal{O}_{F_s}[[\hbar]]$). The cocycle $c : \bigwedge^2 \Lambda^\vee \to \mathbb{C}[[\hbar]]^\times$ gives rise to such an equivariant structure: a group element $(\xi, z) \in \Gamma$ acts on $f = \{f_w\}_{w\in F_s}$ via the formula $f \mapsto \{z\tilde{c}(w, \xi)f_{w+\xi}\}_{w\in F_s}$. Thus, we get a natural section $\mathfrak{a} : \mathbb{X}^\vee(\mathbb{D}) \to {}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ and hence an isomorphism between the groupoid ${}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ and the groupoid $\mathbb{X}^\vee(\mathbb{D}) \times B(\mathbb{C}[[\hbar]]^\times)$.

On the non-commutative side, the set $\pi_0(\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D}))$ of isomorphism classes of global degree zero quantum line bundles on $\mathbb{X}_{\mathbf{\Pi}} \to \mathbb{D}$, coincides with the set $\mathrm{Pic}^0(\mathbb{X}/\mathbb{D})(\mathbb{D})$ of isomorphism classes of global degree zero line bundles on $\mathbb{X} \to \mathbb{D}$. This fact is not obvious but can be established as follows. First of all, as explained in Lemma C.2, every element in the set $X^\vee \times (\overline{V}^\vee)^{\mathbb{Z}_{>0}}$ gives rise to a quantum line bundle of degree zero, and this procedure induces a bijection between $X^\vee \times (\overline{V}^\vee)^{\mathbb{Z}_{>0}}$ and the set of isomorphism classes of quantum line bundles of degree zero. On the other hand, the set $\mathrm{Pic}^0(\mathbb{X}/\mathbb{D})(\mathbb{D})$ is simply the set of formal arcs in $X^\vee = \mathrm{Pic}^0(X)$ and so can be described [Voj04] explicitly in terms of Hasse–Schmidt higher derivations [HS37, Mat89].

Recall that a Hasse–Schmidt higher $\mathbb{C}$-derivation from $\mathcal{O}_{X^\vee}$ to $\mathbb{C}$ is a pair (s, D), where $s \in X^\vee$ is a point and D is (bounded or unbounded) sequence $D = (D_1, D_2, D_3, \dots)$ of $\mathbb{C}$-linear maps $D_i : \mathcal{O}_{X^\vee,s} \to \mathbb{C}$ satisfying

$$D_k(fg) = \sum_{i=0}^{k} D_i(f) D_{k-i}(g)$$

for any two germs f and g of $\mathcal{O}_{X^\vee,s}$. Here $\mathcal{O}_{X^\vee,s}$ denotes the local ring at $s \in X^\vee$ and $D_0 : \mathcal{O}_{X^\vee,s} \to \mathbb{C}$ is the evaluation map.

We denote the infinite-order higher-order derivations by $\mathrm{Der}^\infty_{\mathbb{C}}(\mathcal{O}_{X^\vee}, \mathbb{C})$. It is not hard to identify these derivations with the set of formal arcs in $X^\vee$. Indeed, to specify a formal arc $\mathfrak{s} : \mathbb{D} \to X^\vee$, we need to specify a point $s \in X^\vee$ and a $\mathbb{C}$-algebra homomorphism $\mathfrak{s}^\sharp : \mathcal{O}_{X^\vee,s} \to \mathbb{C}[[\hbar]]$. In these terms, the identification

$$\mathrm{Der}^\infty_{\mathbb{C}}(\mathcal{O}_{X^\vee}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}, \mathbb{X}^\vee) = \mathrm{Hom}(\mathbb{D}, X^\vee) \tag{6.3}$$

is given by

$$(s, D) \mapsto \left(s, \mathfrak{s}^\sharp := \sum_{k=0}^{\infty} \hbar^k D_k\right).$$

On the other hand, we have a bijection

$$X^\vee \times (\overline{V}^\vee)^{\mathbb{Z}_{>0}} \to \mathrm{Der}^\infty_{\mathbb{C}}(\mathcal{O}_{X^\vee}, \mathbb{C}) \tag{6.4}$$

given by

$$(s, l_1, l_2, l_3, \dots) \mapsto (D_0 = \mathrm{ev}_s, D_1, D_2, D_3, \dots)$$

where

$$D_0 \circ \exp(\pi(\hbar l_1 + \hbar^2 l_2 + \hbar^3 l_3 + \cdots)) = D_0 + \hbar D_1 + \hbar^2 D_2 + \cdots$$

and the $l_i \in \overline{V}^\vee$ are thought of as translation invariant vector fields on $X^\vee$. The exponential in the left-hand side of this formula is defined by the usual power series using the composition of differential operators. This is an extension of the standard fact that any tangent germ l, defined at a smooth point m in some complex analytic space M, can be exponentiated to a formal arc $\mathfrak{e} : \mathbb{D} \to M$ in M:

$$\mathfrak{e} := \left(D_0, D_0 \circ l, \frac{1}{2} D_0 \circ l^2, \frac{1}{3!} D_0 \circ l^3, \frac{1}{4!} D_0 \circ l^4, \dots\right) \in \mathrm{Der}^\infty_{\mathbb{C}}(\mathcal{O}_M, \mathbb{C}),$$

where D_0 denotes the evaluation map at the point m and l^k denotes the kth iterated Lie derivative.

Combining the bijections (6.3) and (6.4) with the fact that $X^\vee \times (\overline{V}^\vee)^{\mathbb{Z}_{>0}}$ parameterizes isomorphism classes of degree zero quantum line bundles (see Lemma A.1), we obtain a natural identification

$$\pi_0(\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})) \cong \mathbb{X}^\vee(\mathbb{D}).$$

Thus, we get a map $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D}) \to \mathbb{X}^\vee(\mathbb{D})$ which similarly to the map $_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D}) \to \mathbb{X}^\vee(\mathbb{D})$ admits a preferred splitting $\mathfrak{b} : \mathbb{X}^\vee(\mathbb{D}) \to \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})$, defined to be the composition of the correspondence of Lemma C.2 with the identification (6.4). In particular, we get a natural equivalence of groupoids between $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})$ and the groupoid $\mathbb{X}^\vee(\mathbb{D}) \times B(\mathbb{C}[[\hbar]]^\times)$.

Therefore, we get an explicit equivalence

$$ {}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D}) \cong \mathbb{X}^\vee(\mathbb{D}) \times B(\mathbb{C}[[\hbar]]^\times) \cong \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D}). \tag{6.5} $$

Finally, to show that the equivalence (6.5) is given by the map $\mathfrak{c}$ we need to chase through the sequence of bijections defining (6.5). This is tedious but straightforward. The key observation here is that $\mathfrak{c} : {}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D}) \to \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})$ maps the discrete subgroupoid $\mathbb{X}^\vee(\mathbb{D}) \subset {}_{\boldsymbol{B}}\mathbb{X}^\vee(\mathbb{D})$ identically to the discrete subgroupoid $\mathbb{X}^\vee(\mathbb{D}) \subset \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D})$. This is sufficient to conclude that (6.5) is given by $\mathfrak{c}$ since $\mathfrak{c}$ is a morphism of groupoids. The statement that $\mathfrak{c}$ induces the identity on $\mathbb{X}^\vee(\mathbb{D})$ is easy to check directly. Specifically we need to check that the two sections $\mathfrak{c} \circ \mathfrak{a}$ and $\mathfrak{b}$ of $\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D}) \to \mathbb{X}^\vee(\mathbb{D})$ are isomorphic, i.e. we need to exhibit an isomorphism of functors $\boldsymbol{\iota}$:

$$\begin{tikzcd}
\mathbb{X}^\vee(\mathbb{D}) \arrow[rr, bend left, "\mathfrak{c}\circ\mathfrak{a}"] \arrow[rr, bend right, "\mathfrak{b}"'] & \Downarrow \boldsymbol{\iota} & \mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})(\mathbb{D}).
\end{tikzcd}$$

For this we need the explicit form of the functors $\mathfrak{c} \circ \mathfrak{a}$ and $\mathfrak{b}$. By definition, the splitting $\mathfrak{b}$ is the composition of the correspondence of Lemma C.2 with the identification (6.4). Given an arc $\mathfrak{s}$ in $X^\vee$, we can describe the degree zero line bundle $\mathfrak{b}(\mathfrak{s})$ on $\mathbb{X}_{\mathbf{\Pi}}$ explicitly by a factor of automorphy. Use (6.4) to write $\mathfrak{s} = (s, \mathfrak{s}^\sharp)$, $\mathfrak{s}^\sharp = \mathrm{ev}_s \circ \exp(\sum_{i=1}^\infty \hbar^i l_i)$ for some collection of $l_i \in \overline{V}^\vee$. Now according to Lemma C.2 the factor of automorphy for $\mathfrak{b}(\mathfrak{s})$ is the map

$$\begin{aligned} \Lambda &\longrightarrow \Gamma(V, \mathscr{A}_{V,\mathbf{\Pi}}^\times) \\ \lambda &\longmapsto \chi_s(\lambda) \exp\left(\pi \textstyle\sum_{i=1}^\infty \hbar^i \langle l_i, \lambda\rangle\right), \end{aligned} \tag{6.6}$$

where $\chi_s : \Lambda \to U(1) \subset \mathbb{C}^\times$ is the unitary character corresponding to $s \in X^\vee$. To compare this factor of automorphy with that corresponding to $\mathfrak{c} \circ \mathfrak{a}(\mathfrak{s})$, it is convenient to realize $\mathbb{X}_{\mathbf{\Pi}}$ not as the quotient $\mathbb{V}_{\mathbf{\Pi}}/\mathbb{\Lambda}$ but rather as the quotient $(\mathbb{V}_{\mathbf{\Pi}} \times F_s)/(\mathbb{\Lambda} \times \Lambda^\vee)$. Using the the identification $\mathbb{X}_{\mathbf{\Pi}} = (\mathbb{V}_{\mathbf{\Pi}} \times F_s)/(\mathbb{\Lambda} \times \Lambda^\vee)$ we can now rewrite the factor of automorphy (6.6) as a factor of automorphy for the group $\Lambda \times \Lambda^\vee$ with values in $\Gamma(V, \prod_{w \in F_s} \mathscr{A}_{V,\mathbf{\Pi}}^\times)$. In these terms (6.6) becomes

$$\begin{aligned} \Lambda \times \Lambda^\vee &\longrightarrow \textstyle\prod_{w \in F_s} \Gamma(V, \mathscr{A}_{V,\mathbf{\Pi}}^\times) \\ (\lambda, \xi) &\longmapsto \{e^{(2\pi\sqrt{-1}\,\mathrm{Im}\langle w, \lambda\rangle)} e^{(\pi \sum_{i=1}^\infty \hbar^i \langle l_i, \lambda\rangle)}\}_{w \in F_s}. \end{aligned} \tag{6.7}$$

The factor of automorphy defining $\mathfrak{c} \circ \mathfrak{a}(\mathfrak{s})$ is also easy to describe. Since the map $\mathfrak{c}$ is given by the Poincaré sheaf $\mathscr{P}$, the degree zero line bundle $\mathfrak{c} \circ \mathfrak{a}(\mathfrak{s})$ on $\mathbb{X}_{\mathbf{\Pi}}$ can be described as first restricting the sheaf $\mathscr{P}$ to the product $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} (\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee))$ and then pulling back this restriction by our preferred trivialization of the gerbe $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$. Equivalently, we can tensor the restriction of $\mathscr{P}$ to $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} (\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee))$ by the structure sheaf on $\mathbb{D}$ viewed as a weight (-1) line bundle on $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ (via the preferred trivialization). Writing $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ as the quotient $[(\mathbb{V} \times F_s)/(\mathbb{\Lambda} \times_{\mathbb{D}} \mathbb{\Gamma})]$ we can now describe the restriction of $\mathscr{P}$ by the $\Lambda \times \Gamma$ factor of automorphy

$$ \mathfrak{s}^\sharp \phi : \Lambda \times \Gamma \to \prod_{w \in F_s} \Gamma(V, \mathscr{A}_{V,\mathbf{\Pi}}^\times). $$

Here as usual ϕ is the factor of automorphy for $\mathscr{P}$ (see (4.7)) and the terms in $\mathfrak{s}^\sharp$ act on the functions $\phi(\lambda, (\xi, z))$ as iterated Lie derivatives. Similarly, we can use this quotient presentation, to write the factor of automorphy for the structure sheaf on $\mathbb{D}$ viewed as a weight (-1) line bundle on $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$. As explained in the proof of Step 1 of Lemma 5.5, this twisted line bundle corresponds to the Γ equivariant structure on $\prod_{w\in F_s}\mathcal{O}_{\mathbb{D}}$ given by the representation ρ. In particular, the pullback of the twisted line bundle to the product $\mathbb{X}_{\mathbf{\Pi}}\times_{\mathbb{D}}\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$ is given by the factor of automorphy obtained by applying ρ to the constant section 1, i.e. by

$$\begin{tikzcd}
\Lambda\times\Gamma \arrow[r] & \prod_{w\in F_s}\Gamma(V,\mathscr{A}^{\times}_{V,\mathbf{\Pi}}), \\
(\lambda,(\xi,z)) \arrow[r] & \rho_{(\xi,z)}(1)=\{z^{-1}\tilde{c}(w,\xi)\}_{w\in F_s}.
\end{tikzcd} \tag{6.8}$$

Multiplying $\mathfrak{s}^\sharp\phi$ and (6.8) gives rise to a factor of automorphy on $\Lambda\times\Gamma$, which is a pullback of a factor of automorphy on $\Lambda\times\Lambda^\vee$ which describes the non-commutative line bundle $\mathfrak{c}\circ\mathfrak{a}(\mathfrak{s})$. The latter factor is easily computed. The formula (4.7) describing ϕ, together with the definition of $\mathfrak{s}^\sharp$, give

$$(\mathfrak{s}^\sharp\phi)(\lambda,(\xi,z))_w(v)=\exp(\pi\langle\xi+w,\lambda\rangle)\exp(\pi\overline{\langle\xi,v\rangle})\exp\left(\pi\sum_{i=1}^{\infty}\hbar^i\langle l_i,\lambda\rangle\right).$$

Hence, the factor of automorphy of $\mathfrak{c}\circ\mathfrak{a}(\mathfrak{s})$ is given by the formula

$$\begin{tikzcd}
\Lambda\times\Lambda^\vee \arrow[r] & \prod_{w\in F_s}\Gamma(V,\mathscr{A}^{\times}_{V,\mathbf{\Pi}}), \\
(\lambda,\xi) \arrow[r] & \{e^{(\pi\langle\xi+w,\lambda\rangle)}\tilde{c}(w,\xi)e^{(\pi\overline{\langle\xi,\bullet\rangle})}e^{(\pi\sum_{i=1}^{\infty}\hbar^i\langle l_i,\lambda\rangle)}\}_{w\in F_s}.
\end{tikzcd} \tag{6.9}$$

We are now ready to describe the isomorphism of functors $\boldsymbol{\iota}$. Specifying $\boldsymbol{\iota}$ is equivalent to specifying isomorphisms $\boldsymbol{\iota}_{\mathfrak{s}}:\mathfrak{c}\circ\mathfrak{a}(\mathfrak{s})\to\mathfrak{b}(\mathfrak{s})$ of non-commutative line bundles. In terms of the factors of automorphy (6.7) and (6.9) the isomorphism $\boldsymbol{\iota}_{\mathfrak{s}}$ can be viewed as a group cochain $\boldsymbol{\iota}_{\mathfrak{s}}\in C^0(\Lambda\times\Lambda^\vee,\prod_{w\in F_s}\Gamma(V,\mathscr{A}^{\times}_{V,\mathbf{\Pi}}))=\prod_{w\in F_s}\Gamma(V,\mathscr{A}^{\times}_{V,\mathbf{\Pi}})$ which makes the factor of automorphy (6.7) cohomologous to (6.9). A straightforward computations shows that

$$\boldsymbol{\iota}_{\mathfrak{s}}:=\{e^{-\pi\overline{\langle w,\bullet\rangle}}\}_{w\in F_s}$$

does the job. This completes the proof of the proposition. □

Proposition 6.3 together with Theorem 5.2 readily imply the following.

THEOREM 6.4. *The morphism* $\mathfrak{c}:{}_{\boldsymbol{B}}\mathbb{X}^\vee\to\mathscr{P}ic^0(\mathbb{X}_{\mathbf{\Pi}}/\mathbb{D})$ *is an isomorphism of analytic stacks on* $(\mathscr{F}\mathscr{S}/\mathbb{D})_{\mathrm{an}}$.

Proof. The proof follows the reasoning of [Pol03, Theorem 11.2] with some necessary modifications since we work in the context of formal deformation quantizations. The essential difficulties are already dealt with in Proposition 6.3, Theorem 5.2, and [Pol03, Theorem 11.2], but some additional work is required to package the argument properly. Given a formal analytic space $\mathbb{Y}\to\mathbb{D}$ and a line bundle $L\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}}$ we need to construct a morphism $\mathbb{f}:\mathbb{Y}\to{}_{\boldsymbol{B}}\mathbb{X}^\vee$ and an isomorphism $(\mathbb{f}\times\mathrm{id})^*\mathscr{P}\cong L$. Consider the integral transform

$$\phi_{\mathscr{Q}}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}}\to\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee)]}:\mathsf{D}^b_{\mathrm{c}}(\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}})\to\mathsf{D}^b_{\mathrm{c}}(\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee),-1).$$

and the object $\phi_{\mathscr{Q}}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}}\to\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee)]}(L)\in\mathsf{D}^*_{\mathrm{c}}(\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee),-1)$. By Proposition 6.3 it follows that for every $\mathbb{D}$ point $\mathfrak{y}:\mathbb{D}\to\mathbb{Y}$ of $\mathbb{Y}$, the pullback $\mathfrak{y}^*L$ is isomorphic to $(\mathsf{s}\times\mathrm{id})^*\mathscr{P}$ for some $\mathbb{D}$-point $\mathsf{s}:\mathbb{D}\to{}_{\boldsymbol{B}}\mathbb{X}^\vee$. Specifying the $\mathbb{D}$-point s amounts to specifying a $\mathbb{D}$-point $\mathfrak{s}:\mathbb{D}\to\mathbb{X}^\vee$ together with a trivialization of the gerbe $\mathfrak{s}^*({}_{\boldsymbol{B}}\mathbb{X}^\vee)$. Now $(\mathfrak{y}\times\mathrm{id})^*\phi_{\mathscr{Q}}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}}\to\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee)]}(L)=\phi_{\mathscr{Q}}^{[\mathbb{X}_{\mathbf{\Pi}}\to{}_{\boldsymbol{B}}\mathbb{X}^\vee]}((\mathsf{s}\times\mathrm{id})^*\mathscr{P})$, which in turn is isomorphic to $(\mathfrak{s},-1)_*\mathcal{O}_{\mathbb{D}}[-g]$, as explained in the proof of Theorem 5.2. Taking homs into the various elements of our orthogonal spanning class for $\mathsf{D}^*_{\mathrm{c}}({}_{\boldsymbol{B}}\mathbb{X}^\vee,-1)$, we conclude that the object $\phi_{\mathscr{Q}}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}_{\mathbf{\Pi}}\to\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee)]}(L)$ is of the form $\mathscr{F}[-g]$ for some (-1)-twisted sheaf $\mathscr{F}$ on $\mathbb{Y}\times_{\mathbb{D}}({}_{\boldsymbol{B}}\mathbb{X}^\vee)$.

The key point of the argument is to show that the stack-theoretic support of $\mathscr{F}$ is the graph of a map $\mathbb{f} : \mathbb{Y} \to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}$ and that $\mathscr{F}$ is a line bundle on its support. As in the proof of [Pol03, Theorem 11.2], this will follow from the property

$$(\mathfrak{y} \times \mathrm{id})^{*}\mathscr{F} \cong (\mathfrak{s}, -1)_{*}\mathcal{O}_{\mathbb{D}}, \quad \text{for all } \mathbb{D}\text{-points } \mathfrak{y} : \mathbb{D} \to \mathbb{Y} \tag{6.10}$$

provided that we can show that $\mathscr{F}$ is finite and flat over $\mathbb{Y}$.

The property (6.10) implies that $\mathscr{F}$ is a (-1)-twisted line bundle on its support. In other words, we can find a closed analytic subspace $\mathbb{i} : \mathbb{S} \hookrightarrow \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}^{\vee}$, so that the stack-theoretic support of $\mathscr{F}$ is the gerbe $\mathbb{S} \times_{\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}} [\mathbb{Y} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^{\vee})]$ and $\mathscr{F}$ trivializes this gerbe. In particular, $\mathscr{F}$ is (non-canonically) isomorphic to a sheaf of the form $(\mathbb{i}, -1)_{*}\mathscr{G}$ for some line bundle $\mathscr{G}$ on $\mathbb{S}$. Thus, $\mathscr{F}$ will be finite and flat over $\mathbb{Y}$ if and only if the sheaf $\mathbb{i}_{*}\mathscr{G}$ on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}^{\vee}$ is finite and flat over $\mathbb{Y}$. To check this we use the following lemma.

Lemma 6.5. *Suppose that L is a line bundle on $\mathbb{X}_{\mathbf{\Pi}} \times_{\mathbb{D}} \mathbb{Y}$ which is of degree zero relative to $\mathbb{Y}$. Let $\mathbb{S} \subset \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}^{\vee}$ and $\mathscr{G}$ be as above. Then there exists a line bundle M on $\mathbb{X} \times_{\mathbb{D}} \mathbb{Y}$ which is of degree zero relative to Y and for which*

$$\phi_{\mathcal{P}^{\vee}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}]}(M) \cong \mathbb{i}_{*}\mathscr{G}[-g].$$

Proof. Consider the natural projection and addition maps

$$\begin{tikzcd} \mathbb{Y} \times_{\mathbb{D}} \mathbb{X} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}} \arrow[r, shift left, "\mathbb{p}_{1,2+3}"] \arrow[r, shift right, "\mathbb{p}_{1,3}"'] & \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}. \end{tikzcd}$$

Here $\mathbb{p}_{1,3}$ is the projection onto the first and third factors and $\mathbb{p}_{1,2+3} = \mathbb{p}_{1} \times \mathbb{m}_{(0.1)}$ is the product of the projection onto $\mathbb{Y}$ and the ringed space map $\mathbb{m}_{(0,1)} : \mathbb{X} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}} \to \mathbb{X}_{\mathbf{\Pi}}$ described in § 3.1.4.

Since by hypothesis $L \to \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\mathbf{\Pi}}$ is of degree zero relative to $\mathbb{Y}$, i.e. L is translation invariant in the $\mathbb{X}_{\mathbf{\Pi}}$ direction, it follows that the line bundles $\mathbb{p}_{1,3}^{*}L$ and $\mathbb{p}_{1,2+3}^{*}L$ satisfy the assumptions of the see-saw principle Proposition 4.7. Therefore, we can find a line bundle M on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}$ so that

$$\mathbb{p}_{1,2}^{*}M \otimes \mathbb{p}_{1,3}^{*}L \cong \mathbb{p}_{1,2+3}^{*}L.$$

Now a straightforward diagram chase shows that the Fourier–Mukai transform of M with respect to the Poincaré sheaf $\mathbb{p}_{2,3}^{*}\mathcal{P}^{\vee}[[\hbar]]$ on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X} \times_{\mathbb{D}} \mathbb{X}^{\vee}$ is the object $\mathbb{i}_{*}\mathscr{G}[-g]$. ☐

With M in hand we can now proceed to reason as in the proof of [Pol03, Theorem 11.2]. Since $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X} \to \mathbb{Y}$ and $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}^{\vee} \to \mathbb{Y}$ are dual family of complex tori, the usual argument [Pol03, § 11.2] shows that $\phi_{\mathcal{P}^{\vee}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}]}$ is left-adjoint to $\phi_{\mathcal{P}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}]}$, and that the adjunction map

$$\mathrm{Id} \to \phi_{\mathcal{P}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}]} \circ \phi_{\mathcal{P}^{\vee}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}]}$$

is an isomorphism.

Now, to check that $\mathbb{i}_{*}\mathscr{G}$ is finite and flat over $\mathbb{Y}$ we can assume that the closed fiber $Y \subset \mathbb{Y}$ of $\mathbb{Y} \to \mathbb{D}$ is a Stein space and check that the global sections of $\mathbb{i}_{*}\mathscr{G}$ on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}$ are a flat module of finite rank over the Stein algebra $\Gamma(\mathbb{Y}, \mathcal{O}_{\mathbb{Y}})$.

For the global sections $\Gamma(\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}, \mathbb{i}_{*}\mathscr{G})$ we use the above adjunction to compute

$$\begin{aligned} \Gamma(\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}, \mathbb{i}_{*}\mathscr{G}) &= \mathrm{Hom}(\mathcal{O}_{\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}}, \mathbb{i}_{*}\mathscr{G}) \\ &= \mathrm{Hom}(\phi_{\mathcal{P}^{\vee}[[\hbar]]}^{[\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}\to\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}^{\vee}]}(\mathcal{O}_{\mathbb{Y}\times_{\mathbb{D}}\mathbb{X}}), M) \\ &= \mathrm{Hom}((\mathrm{id} \times \mathfrak{o})_{*}\mathcal{O}_{\mathbb{Y}}[-g], M) \\ &= \mathrm{Ext}^{g}((\mathrm{id} \times \mathfrak{o})_{*}\mathcal{O}_{\mathbb{Y}}, M), \end{aligned}$$

where $\mathfrak{o} : \mathbb{D} \to \mathbb{X}$ is the constant section corresponding to the origin $o \in X$. Now, on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}$ we have $(\mathrm{id} \times \mathfrak{o})_* \mathcal{O}_{\mathbb{Y}} = \mathbb{p}_2^* \mathfrak{o}_* \mathcal{O}_{\mathbb{D}}$ and since $\mathfrak{o}_* \mathcal{O}_{\mathbb{D}}$ is supported on a section of $\mathbb{X} \to \mathbb{D}$ we get

$$\begin{aligned} \mathrm{Ext}^g((\mathrm{id} \times \mathfrak{o})_* \mathcal{O}_{\mathbb{Y}}, M) &= \mathrm{Ext}^g(\mathbb{p}_2^* \mathfrak{o}_* \mathcal{O}_{\mathbb{D}}, M) \\ &= \mathrm{Ext}^g(\mathfrak{o}_* \mathcal{O}_{\mathbb{D}}, \mathbb{p}_{2*} M) \\ &= H^0(\mathbb{X}, \mathscr{E}xt^g(\mathfrak{o}_* \mathcal{O}_{\mathbb{D}}, \mathbb{p}_{2*} M)). \end{aligned}$$

Now, it only remains to observe that since M is a line bundle, $\mathbb{p}_{2*} M$ is a flat $\mathcal{O}_{\mathbb{X}}$-module, and so $\mathscr{E}xt^g(\mathfrak{o}_* \mathcal{O}_{\mathbb{D}}, \mathbb{p}_{2*} M) = \mathscr{E}xt^g(\mathfrak{o}_* \mathcal{O}_{\mathbb{D}}, \mathcal{O}_{\mathbb{X}}) \otimes_{\mathcal{O}_{\mathbb{X}}} \mathbb{p}_{2*} M = \mathfrak{o}_* \mathcal{O}_{\mathbb{D}} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathbb{p}_{2*} M$. In other words

$$\Gamma(\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}, \mathfrak{i}_* \mathscr{G}) = H^0(\mathbb{X}, \mathfrak{o}_* \mathcal{O}_{\mathbb{D}} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathbb{p}_{2*} M)$$

which is clearly finite and flat as an $\Gamma(\mathbb{Y}, \mathcal{O}_{\mathbb{Y}})$-module.

This implies that the sheaf $\mathscr{F}$ on $\mathbb{Y} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^{\vee})$ is finite and flat over $\mathbb{Y}$, which combined with the property (6.10) implies that the support $\mathbb{S}$ is the graph of a morphism $\mathbb{f} : \mathbb{Y} \to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}$. Hence, $\phi_{\mathscr{P}}^{[\mathbb{Y} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^{\vee}) \to \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\boldsymbol{\Pi}}]}(\mathscr{F})$ is a line bundle on $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\boldsymbol{\Pi}}$ which is isomorphic to $(\mathrm{id} \times \mathbb{f})^* \mathscr{P}$ along the fibers of $\mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\boldsymbol{\Pi}} \to \mathbb{Y}$. However, in the proof of Theorem 5.2 we checked that the composition functor $\phi_{\mathscr{P}}^{[{}_{\boldsymbol{B}}\mathbb{X}^{\vee} \to \mathbb{X}_{\boldsymbol{\Pi}}]} \circ \phi_{\mathscr{Q}}^{[\mathbb{X}_{\boldsymbol{\Pi}} \to {}_{\boldsymbol{B}}\mathbb{X}^{\vee}]}$ is isomorphic to $\mathrm{id}[-g]$. This implies that the canonical adjunction morphism

$$\phi_{\mathscr{P}}^{[\mathbb{Y} \times_{\mathbb{D}} ({}_{\boldsymbol{B}}\mathbb{X}^{\vee}) \to \mathbb{Y} \times_{\mathbb{D}} \mathbb{X}_{\boldsymbol{\Pi}}]}(\mathscr{F}) \to L$$

is an isomorphism since it is an isomorphism over every $\mathbb{D}$ point of $\mathbb{Y}$. Applying again the see-saw principle Proposition 4.7, we conclude that L is isomorphic to $\mathbb{p}_1^* A \otimes (\mathrm{id} \times \mathbb{f})^* \mathscr{P}$ for a line bundle A on $\mathbb{Y}$ which is unique up to a unique isomorphism. This shows that $\mathfrak{c}$ is an isomorphism of stacks and concludes the proof of the theorem. □

In the appendices, we prove some general facts concerning the module theory for deformation quantizations of Hausdorff analytic spaces. In some cases we specialize to complex tori, or the Moyal quantization in particular. It should be noted that many of the proofs in these appendices carry over immediately to other contexts. In particular we expect that our results will be applicable to deformation quantizations in characteristic p (see [BK04]) and the case of C^∞ real manifolds. In particular, our deformation theory analysis recovers some results from [BW00, BW04, BW05].

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Appendix A. Quantum invertible sheaves

Let S be a Hausdorff analytic space. Let $\mathbb{S} := (S, \mathscr{A})$ be a deformation quantization of $(S, \mathcal{O})$. We assume that $\mathscr{A}$ and $\mathcal{O}[[\hbar]]$ are isomorphic as sheaves of $\mathbb{C}[[\hbar]]$-modules and that the product on $\mathscr{A}$ is given by a $\star$-product on $\mathcal{O}[[\hbar]]$. Suppose further that for some subsheaf of $\mathbb{C}$-algebras $\mathcal{C} \subset \mathcal{O}$, the left or right $\star$-multiplication action of $\mathcal{C}[[\hbar]] \subset \mathscr{A}$ on $\mathscr{A}$ agree with the commutative action of multiplication of $\mathcal{C}[[\hbar]] \subset \mathcal{O}[[\hbar]]$ on $\mathcal{O}[[\hbar]]$. In particular, $\mathcal{C}[[\hbar]] \subset \mathscr{A}$ is a subsheaf of central $\mathbb{C}[[\hbar]]$-subalgebras in A.

To any locally free left rank one $\mathcal{C}[[\hbar]]$-module M we can associate a natural line bundle $L := \mathscr{A} \otimes_{\mathcal{C}[[\hbar]]} M$ on $\mathbb{S}$. In this way we get a functor from the groupoid of locally free left rank one $\mathcal{C}[[\hbar]]$-modules to the groupoid of line bundles on $\mathbb{S}$. We would like to understand the map that this functor induces on isomorphism classes.

The isomorphism classes of locally free left rank one $\mathcal{C}[[\hbar]]$-modules are in a natural bijection with the cohomology group $H^1(S, \mathcal{C}[[\hbar]]^\times)$, whereas the isomorphism classes of line bundles on $\mathbb{S}$ are in bijection with the cohomology set $H^1(S, \mathscr{A}^\times)$. Denote by G the image of $H^1(S, \mathcal{C}[[\hbar]]^\times)$ in $H^1(S, \mathscr{A}^\times)$, by I_0 the image of $H^1(S, \mathcal{C}^\times)$ in $H^1(S, \mathcal{O}^\times)$, and by I the image of $H^1(S, \mathcal{C})$ in $H^1(S, \mathcal{O})$. Then we seek to understand the structure of G, in terms of I_0 and I.

Lemma A.1. *There is a natural bijection $I_0 \times \prod_{k=1}^\infty I \to G$. In particular, G is a commutative group.*

Proof. First consider the isomorphism of sheaves of groups

$$\begin{tikzcd}
\mathrm{Exp} : & \mathcal{C}^\times \times \prod_{k=1}^\infty \mathcal{C} \arrow[r] & \mathcal{C}[[\hbar]]^\times \\
& (a_0, a_1, a_2, a_3, \ldots) \arrow[r, mapsto] & a_0 \exp(a_1\hbar + a_2\hbar^2 + a_3\hbar^3 + \cdots).
\end{tikzcd}$$

For any open covering $\mathfrak{U}$ of S we have an induced isomorphism of groups

$$\begin{tikzcd}
\mathrm{Exp} : & \check{Z}^1(S, \mathfrak{U}, \mathcal{C}^\times) \times \prod_{i=1}^\infty \check{Z}^1(S, \mathfrak{U}, \mathcal{C}) \arrow[r] \arrow[d, equal] & \check{Z}^1(S, \mathfrak{U}, \mathcal{C}[[\hbar]]^\times) \\
& \check{Z}^1(S, \mathfrak{U}, \mathcal{C}^\times \times \prod_{k=1}^\infty \mathcal{C}). &
\end{tikzcd}$$

We also have a bijection of sets

$$\begin{tikzcd}
\mathrm{Exp} : & \check{C}^0(S, \mathfrak{U}, \mathcal{O}^\times) \times \prod_{k=1}^\infty \check{C}^0(S, \mathfrak{U}, \mathcal{O}) \arrow[r] \arrow[d, equal] & \check{C}^0(S, \mathfrak{U}, \mathcal{O}[[\hbar]]^\times) \arrow[d, "\cong"] \\
& \check{C}^0(S, \mathfrak{U}, \mathcal{O}^\times \times \prod_{k=1}^\infty \mathcal{O}) & \check{C}^0(S, \mathfrak{U}, \mathscr{A}^\times).
\end{tikzcd}$$

given by the same formula (for the regular exponential, not the $\star$-exponential). Suppose now that we have two cocycles $a, a' \in \check{Z}^1(S, \mathfrak{U}, \mathcal{C}^\times) \times \prod_{k=1}^\infty \check{Z}^1(S, \mathfrak{U}, \mathcal{C})$ which satisfy

$$a(a')^{-1} = \delta(b)$$

in $\check{Z}^1(S, \mathfrak{U}, \mathcal{O}^\times) \times \prod_{k=1}^\infty \check{Z}^1(S, \mathfrak{U}, \mathcal{O})$, with some $b \in \check{C}^0(S, \mathfrak{U}, \mathcal{O}^\times) \times \prod_{k=1}^\infty \check{C}^0(S, \mathfrak{U}, \mathcal{O})$. Then we have $a_{ij} b_j = b_i a'_{ij}$ and so

$$\mathrm{Exp}(a_{ij}) \cdot \mathrm{Exp}(b_j) = \mathrm{Exp}\,(b_i) \cdot \mathrm{Exp}\,(a'_{ij}).$$

Since $\mathrm{Exp}(a_{ij})$ and $\mathrm{Exp}(a'_{ij})$ are in $\mathcal{C}[[\hbar]]$, this means that

$$(\mathrm{Exp}(b_i))^{-1} \star \mathrm{Exp}(a_{ij}) \star \mathrm{Exp}(b_j) = \mathrm{Exp}(a'_{ij})$$

in $\check{Z}^1(S, \mathfrak{U}, \mathscr{A}^\times)$, where the inverse here is taken with respect to the $\star$-product. Therefore, we have produced a well-defined and surjective map

$$\mathrm{Exp} : I_0 \times \prod_{k=1}^\infty I \to G.$$

Now suppose that $a \in \check{Z}^1(S, \mathfrak{U}, \mathcal{C}^\times) \times \prod_{k=1}^\infty \check{Z}^1(S, \mathfrak{U}, \mathcal{C})$ is a cocycle for which $\mathrm{Exp}(a) \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}^\times)$ is a coboundary. Then $\mathrm{Exp}(a_{ij}) = (\mathrm{Exp}(a))_{ij} = c_i \star c_j^{-1}$ for some $c \in \check{C}^0(S, \mathfrak{U}, \mathscr{A}^\times)$. Consequently $(\mathrm{Exp}(a_{ij}))c_j = c_i$ and so if we choose $b \in \check{C}^0(S, \mathfrak{U}, \mathcal{O}^\times) \times \prod_{k=1}^\infty \check{C}^0(S, \mathfrak{U}, \mathcal{O})$ satisfying $\mathrm{Exp}(b) = c$, then we have $a = \delta(b)$. Therefore, Exp is in fact an isomorphism. By taking the limit over all coverings $\mathfrak{U}$, we arrive at the desired statement. □

Remark A.2. One important example of the above is the case when S is a complex manifold, $\mathscr{A}$ is a deformation quantization of the structure sheaf $\mathcal{O}_S$ which is globally isomorphic to $\mathcal{O}_S[[\hbar]]$, and $\mathbf{\Pi}$ is the corresponding global holomorphic Poisson structure on S. In this case we can take $\mathcal{C} \subset \mathcal{O}$ to be the subsheaf of functions constant along the leaves of the foliation of S by symplectic leaves. Explicitly, $\mathcal{C}$ is the kernel of the map $\mathcal{O} \to T_S$ of $\mathbb{C}$-sheaves given by the composition of the de Rham differential $d : \mathcal{O} \to \Omega^1_{S,\mathrm{cl}} \subset \Omega^1_S$ with the contraction $\mathbf{\Pi} \lrcorner : \Omega^1_S \to T_S$. If we further define $\Omega^1_{S,\mathbf{\Pi},\mathrm{cl}} \subset \Omega^1_{S,\mathrm{cl}}$ as the kernel of the map $\mathbf{\Pi} \lrcorner : \Omega^1_{S,\mathrm{cl}} \to T_S$, then we have a short exact sequence of sheaves of groups

$$1 \to \mathcal{C}[[\hbar]]^\times \to \mathcal{O}[[\hbar]]^\times \to (\Omega^1_{S,\mathrm{cl}}/\Omega^1_{S,\mathbf{\Pi},\mathrm{cl}})[[\hbar]] \to 0.$$

Combining the relevant part of the associated long exact sequence of cohomology groups with Lemma A.1 we obtain a five term exact sequence of groups

$$\begin{tikzcd} 1 \arrow[r] & H^0(S,\mathcal{C}[[\hbar]]^\times) \arrow[r] & H^0(S,\mathscr{A}^\times) \arrow[r] & H^0(S,(\Omega^1_{S,\mathrm{cl}}/\Omega^1_{S,\mathbf{\Pi},\mathrm{cl}})[[\hbar]] \arrow[dll, rounded corners] \\ & H^1(S,\mathcal{C}[[\hbar]]^\times) \arrow[r] & I_0 \times \prod_{k=1}^{\infty} I \arrow[r] & 1 \end{tikzcd} \tag{A.1}$$

where the identification $I_0 \times \prod_{k=1}^{\infty} I \cong G \subset H^1(S,\mathscr{A}^\times)$ is described in Lemma A.1, and we have used the natural identification

$$\mathrm{image}[H^1(S,\mathcal{C}[[\hbar]]^\times) \to H^1(S,\mathscr{A}^\times)] \cong \mathrm{image}[H^1(S,\mathcal{C}[[\hbar]]^\times) \to H^1(S,\mathcal{O}[[\hbar]]^\times)].$$

Note that the first two groups in (A.1) are often isomorphic, e.g. for a compact S. In such cases, the sequence (A.1) reduces to a short exact sequence and we get a concrete description of $H^1(S,\mathcal{C}[[\hbar]]^\times)$.

For line bundles on a quantization of a complex torus we obtain the more specific results in Lemma C.2 using factors of automorphy on the universal cover, instead of Čech cohomology.

Appendix B. Quantized vector bundles

In this appendix, we provide the promised proof of the triviality of locally free left $\mathscr{A}_V$-modules where $\mathscr{A}_V$ is a Moyal quantization of a vector space V. In fact, we prove a more general statement, which is applicable in many contexts that arise in deformation quantization.

Let S be a Hausdorff analytic space and let $\mathscr{A}$ be a sheaf of $\mathbb{C}[[\hbar]]$-algebras such that $\mathscr{A} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathcal{O}_S$. Consider the multiplicative sheaf of groups $\mathscr{A}ut_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}^{\oplus m})$ of left $\mathscr{A}$-module automorphisms of $\mathscr{A}^{\oplus m}$. Isomorphism classes of left $\mathscr{A}$-modules which are locally free of rank m are in a natural bijective correspondence with $H^1(S, \mathscr{A}ut_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}^{\oplus m}))$. It turns out that the vanishing of the first cohomology of the multiplicative sheaf of groups $\mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})$ and the first cohomology of the additive sheaf of groups $\mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m})$ guarantee that all such left $\mathscr{A}$-modules are isomorphic to the trivial one. Below we treat only left modules, and so we simply write $\mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m})$ in place of $\mathscr{A}ut_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}^{\oplus m})$ and also $\mathscr{E}nd_{\mathscr{A}}(\mathscr{A}^{\oplus m})$ in place of $\mathscr{E}nd_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}^{\oplus m})$.

Lemma B.1. *If $H^1(S, \mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = \{0\}$ and $H^1(S, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = \{1\}$, then the set $H^1(S, \mathscr{A}ut_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}^{\oplus m}))$ has only one element.*

Proof. Fix a cofinal system of coverings $\mathfrak{S}$, such that if $\mathfrak{U}$ is any covering in $\mathfrak{S}$ we have $\check{H}^1(S, \mathfrak{U}, \mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = \{0\}$, $\check{H}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = \{1\}$, and also such that for any covering $\mathfrak{U}$ in $\mathfrak{S}$ and for any open set U in the covering $\mathfrak{U}$, there is a splitting (a morphism of sheaves of $\mathbb{C}$-vector spaces) σ_U over U of the projection $\rho : \mathscr{A} \to \mathcal{O}$. Then if we have a covering $\mathfrak{U} = \{U_i \mid i \in I\}$ in the system $\mathfrak{S}$ it is enough to show that any Čech 1-cochain $G \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m}))$ is a coboundary. We do this by constructing a sequence $\{G =: G^0, G^1, G^2, \dots\}$ of cocycles in $\check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m}))$ with the following properties:

- G^{j+1} is cohomologous to G^j for $j \geqslant 0$;
- $\rho(G^j) = \mathrm{id}$ for $j \geqslant 1$;
- $\rho((G^j - \mathrm{id})/\hbar^k) = 0$ for every $j \geqslant 1$ and every $k = 1, \ldots, j-1$.

To carry out the construction chose splittings $\sigma_i := \sigma_{U_i} : \mathcal{O}_{|U_i} \to \mathscr{A}_{|U_i}$ over each U_i as above. The cocycle relationship says that $G_{ij} \in \mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m})(U_{ij})$ satisfy

$$G_{ik} = G_{ij} \circ G_{jk}.$$

Reducing modulo $\hbar$ we see that

$$\rho(G) \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus n})).$$

Since $\check{H}^1(S, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = 0$ we can choose

$$\phi^0 = \{\phi_i^0\}_{i \in I} \in \check{C}^0(S, \mathfrak{U}, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m}))$$

satisfying

$$(\phi_i^0)^{-1} \circ \rho(G_{ij}) \circ \phi_j^0 = 1_{\mathcal{O}_{|U_{ij}}}.$$

Consider the cochain $G^1 \in \check{C}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m}))$ defined by

$$G_{ij}^1 = (\sigma_i(\phi_i^0))^{-1} \circ G_{ij} \circ (\sigma_j(\phi_j^0)).$$

Clearly G^1 is a cocycle which is cohomologous to G. By construction, G^1 satisfies $\rho(G^1) = \mathrm{id}$ and therefore $\rho(G^1 - \mathrm{id}) = 0$.

Assume that by induction we have produced $G^1, \ldots, G^n$ which are cohomologous to each other and such that for each $1 \leqslant j \leqslant n$ we have $\rho(G^j) = \mathrm{id}$ and for $k < j$ we have $\rho((G^j - \mathrm{id})/\hbar^k) = 0$. Now we have

$$(G_{ij}^n - \mathrm{id}) \circ (G_{jk}^n - \mathrm{id}) = G_{ik}^n - G_{ij}^n - G_{jk}^n + \mathrm{id} = (G_{ik}^n - \mathrm{id}) - (G_{ij}^n - \mathrm{id}) - (G_{jk}^n - \mathrm{id}). \qquad \text{(B.1)}$$

Note that

$$\rho(((G_{ij}^n - \mathrm{id}) \circ (G_{jk}^n - \mathrm{id}))/\hbar^n) = \rho((G_{ij}^n - \mathrm{id})/\hbar^{n-1}) \circ \rho((G_{jk}^n - \mathrm{id})/\hbar) = 0.$$

Therefore, dividing by $\hbar^n$ and applying ρ to (B.1) we conclude that

$$\rho((G_{ik}^n - \mathrm{id})/\hbar^n) = \rho((G_{ij}^n - \mathrm{id})/\hbar^n) + \rho((G_{jk}^n - \mathrm{id})/\hbar^n).$$

In other words $\rho((G^n - \mathrm{id})/\hbar^n) \in \check{Z}^1(S, \mathfrak{U}, \mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m}))$. Therefore, taking into account the fact that $\check{H}^1(S, \mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m})) = 0$, we can choose a cochain

$$\phi^n = \{\phi_i^n\}_{i \in I} \in \check{C}^0(S, \mathfrak{U}, \mathscr{E}nd_{\mathcal{O}}(\mathcal{O}^{\oplus m}))$$

satisfying

$$\phi_i^n - \phi_j^n = \rho((G_{ij}^n - \mathrm{id})/\hbar^n).$$

Consider now the cocycle $G^{n+1} \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}}(\mathscr{A}^{\oplus m}))$ defined by

$$G_{ij}^{n+1} = (\mathrm{id} + \hbar^n \sigma_i(\phi_i^n))^{-1} \circ G_{ij}^n \circ (\mathrm{id} + \hbar^n \sigma_j(\phi_j^n)).$$

Then modulo $\hbar^{n+1}$ we have that $(\mathrm{id} + \hbar^n \sigma_i(\phi_i^n))^{-1}$ is equivalent to $(\mathrm{id} - \hbar^n \sigma_i(\phi_i^n))$ and so G_{ij}^{n+1} is cohomologous to $(\mathrm{id} - \hbar^n \sigma_i(\phi_i^n) + (G_{ij}^n - \mathrm{id}) + \hbar^n \sigma_j(\phi_j^n))$ where $(G_{ij}^n - \mathrm{id}) \in \check{C}^1(\mathfrak{U}, \hbar^n \mathscr{E}nd_{\mathscr{A}}(\mathscr{A}^{\oplus m}))$. Clearly, G^{n+1} is cohomologous to G^n, $\rho(G^{n+1}) = \mathrm{id}$ and $\rho((G^{n+1} - \mathrm{id})/\hbar^k) = 0$ for $0 \leqslant k < n$. Finally,

$$\rho((G^{n+1} - \mathrm{id})/\hbar^n) = \rho((G_{ij}^n - \mathrm{id})/\hbar^n) - \phi_i^{(n)} + \phi_j^{(n)} = 0.$$

Therefore, we have produced the required element G^{n+1} completing the induction step and the proof of the lemma. □

The previous lemma shows that if there are no non-trivial rank m vector bundles on S and if the trivial rank m vector bundle has no infinitesimal deformations, then there are no non-trivial quantum vector bundles of rank m on the deformation quantization $\mathbb{S} := (S, \mathscr{A})$.

Next we discuss the obstructions for quantizing a rank m holomorphic vector bundle W on S along a given deformation quantization $\mathbb{S} = (S, \mathscr{A})$. We do this order by order in the formal parameter $\hbar$. One may consider these investigations as first steps in understanding $\mathscr{A} - \mathrm{mod}$ as the formal deformation of the abelian category $\mathcal{O}_S - \mathrm{mod}$. Hopefully, this will eventually lead to an interpretation in terms of a more systematic study of the deformation theory of abelian categories. This study has appeared, for instance, in [LV04].

As before, we write $\mathscr{A}ut_{\mathscr{A}/\hbar^q\mathscr{A}}((\mathscr{A}/\hbar^q\mathscr{A})^{\oplus m})$ instead of $\mathscr{A}ut_{\mathscr{A}/\hbar^q\mathscr{A}-\mathrm{mod}}((\mathscr{A}/\hbar^q\mathscr{A})^{\oplus m})$. Consider the projection map

$$\mathrm{proj}_{\mathrm{q}} : H^1(S, \mathscr{A}ut_{\mathscr{A}/\hbar^q\mathscr{A}}((\mathscr{A}/\hbar^q\mathscr{A})^{\oplus m})) \to H^1(S, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})).$$

We define the set of length q quantizations of W along $\mathbb{S}$ as

$$\mathrm{Quant}_q(W) := \mathrm{proj}_q^{-1}([W]).$$

Lemma B.2. *Let W be a holomorphic vector bundle on a Hausdorff analytic space S. Then there is a map*

$$\mathrm{ob}_{n+1} : \mathrm{Quant}_{n+1}(W) \to H^2(S, \mathscr{E}nd(W))$$

which measures the obstruction for a length $n+1$ quantization of W to prolong to a length $n+2$ quantization. The ambiguity in choosing such a prolongation is given by $H^1(S, \mathscr{E}nd(W))$. In other words, we have an exact sequence of sets

$$\mathrm{Quant}_{n+2}(W) \to \mathrm{Quant}_{n+1}(W) \overset{\mathrm{ob}_{n+1}}{\to} H^2(S, \mathscr{E}nd(W))$$

and a free action of the additive group $H^1(S, \mathscr{E}nd(W))$ on $\mathrm{Quant}_{n+2}(W)$, so that

$$\mathrm{Quant}_{n+2}(W)/H^1(S, \mathscr{E}nd(W)) = \mathrm{im}[\mathrm{Quant}_{n+2}(W) \to \mathrm{Quant}_{n+1}(W)].$$

Proof. We first define the map $\mathrm{ob}_{n+1} : \mathrm{Quant}_{n+1}(W) \to H^2(S, \mathscr{E}nd(W))$ and prove that $\mathrm{ob}_{n+1}^{-1}(0)$ is the image of $\mathrm{Quant}_{n+2}(W)$. Fix a fine enough open cover $\mathfrak{U} = \{U_i \mid i \in I\}$ of S. Represent $[W] \in H^1(S, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m}))$ by a cocycle $g \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m}))$ where $g_{ij} = \mu_i \circ \mu_j^{-1}$ for some trivializations $\mu_i : W_{|U_i} \to \mathcal{O}^{\oplus m}_{|U_i}$. Similarly, we represent $G \in \mathrm{Quant}_{n+1}(W)$ by an element $\{G_{ij}\} \in \check{Z}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}/\hbar^{n+1}\mathscr{A}}((\mathscr{A}/\hbar^{n+1}\mathscr{A})^{\oplus m}))$.

Define $\mathrm{ob}_{n+1}(G)$ to be the element of $H^2(S, \mathscr{E}nd(W))$ induced by taking the limit over all open covers of the elements $\{\mathrm{ob}_{n+1}(G)_{ijk}\} \in \check{Z}^1(S, \mathfrak{U}, \mathscr{E}nd_{\mathcal{O}}(W))$ where

$$\mathrm{ob}_{n+1}(G)_{ijk} = \mu_k^{-1} \circ \rho\left(\frac{\widetilde{G}_{ki} \circ \widetilde{G}_{ij} \circ \widetilde{G}_{jk} - \mathrm{id}}{\hbar^{n+1}}\right) \circ \mu_k = \mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ij} \circ \widetilde{G}_{jk} - \widetilde{G}_{ik}}{\hbar^{n+1}}\right) \circ \mu_k.$$

Here $\rho : \mathscr{A}ut_{\mathscr{A}/\hbar^{n+1}}(\mathscr{A}^{\oplus m}/\hbar^{n+1}) \to \mathscr{A}ut_{\mathcal{O}}(\mathcal{O}^{\oplus m})$ is the reduction modulo $\hbar$, and $\{\widetilde{G}_{ij}\}$ is a lift of $\{G_{ij}\}$ to $\check{C}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}/\hbar^{n+2}\mathscr{A}}((\mathscr{A}/\hbar^{n+2}\mathscr{A})^{\oplus m}))$. We check that $\{\mathrm{ob}_{n+1}(G)_{ijk}\}$ is closed. Checking that the above definition of ob_{n+1} is independent of all choices made in the construction is easy but tedious and is left to the reader:

$$\begin{aligned}(\delta\{\mathrm{ob}_{n+1}(G)_{abc}\})_{ijkl} &= \mathrm{ob}_{n+1}(G)_{jkl} - \mathrm{ob}_{n+1}(G)_{ikl} + \mathrm{ob}_{n+1}(G)_{ijl} - \mathrm{ob}_{n+1}(G)_{ijk} \\ &= \mu_j^{-1} \circ \rho\left(\frac{\widetilde{G}_{jk} \circ \widetilde{G}_{kl} - \widetilde{G}_{jl}}{\hbar^{n+1}}\right) \circ \mu_l - \mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ik} \circ \widetilde{G}_{kl} - \widetilde{G}_{il}}{\hbar^{n+1}}\right) \circ \mu_l \\ &\quad + \mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ij} \circ \widetilde{G}_{jl} - \widetilde{G}_{il}}{\hbar^{n+1}}\right) \circ \mu_l - \mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ij} \circ \widetilde{G}_{jk} - \widetilde{G}_{ik}}{\hbar^{n+1}}\right) \circ \mu_k.\end{aligned}$$

By making the substitution $\widetilde{G}_{ik} = \widetilde{G}_{ij} \circ \widetilde{G}_{jk} + (\widetilde{G}_{ik} - \widetilde{G}_{ij} \circ \widetilde{G}_{jk})$ in the second term, we can rewrite this term as

$$-\mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ij} \circ \widetilde{G}_{jk} \circ \widetilde{G}_{kl} - \widetilde{G}_{il}}{\hbar^{n+1}}\right) \circ \mu_l - \mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ik} - \widetilde{G}_{ij} \circ \widetilde{G}_{jk}}{\hbar^{n+1}}\right) \circ \mu_k.$$

Similarly, by making the substitution $\widetilde{G}_{jl} = \widetilde{G}_{jk} \circ \widetilde{G}_{kl} + (\widetilde{G}_{jl} - \widetilde{G}_{jk} \circ \widetilde{G}_{kl})$ into the third term, we can rewrite this term as

$$\mu_i^{-1} \circ \rho\left(\frac{\widetilde{G}_{ij} \circ \widetilde{G}_{jk} \circ \widetilde{G}_{kl} - \widetilde{G}_{il}}{\hbar^{n+1}}\right) \circ \mu_l + \mu_j^{-1} \circ \rho\left(\frac{\widetilde{G}_{jl} - \widetilde{G}_{jk} \circ \widetilde{G}_{kl}}{\hbar^{n+1}}\right) \circ \mu_l.$$

It is now clear that all of the terms cancel, and we have shown that $\delta\{\mathrm{ob}_{n+1}(G)_{abc}\} = 0$. Clearly if G comes from $\mathrm{Quant}_{n+1}(W)$ then we can choose the cochain $\{\widetilde{G}_{ij}\}$ to be a cocycle, and so the cohomology class $\mathrm{ob}_{n+1}(G)$ is 0 in $H^2(S, \mathscr{E}nd_{\mathcal{O}}(W))$. Conversely, if $\mathrm{ob}_{n+1}(G) = 0$ then for a fine enough $\mathfrak{U}$ we can find an element $C \in \check{C}^1(S, \mathfrak{U}, \mathscr{E}nd_{\mathcal{O}}(W))$, with $\delta(C)_{ijk} = \mathrm{ob}_{n+1}(G)_{ijk}$. Then we can define a new element $\widetilde{G}' \in \check{C}^1(S, \mathfrak{U}, \mathscr{A}ut_{\mathscr{A}/\hbar^{n+2}\mathscr{A}}((\mathscr{A}/\hbar^{n+2}\mathscr{A})^{\oplus m}))$ by the rule

$$\widetilde{G}'_{ij} = \widetilde{G}_{ij} - \hbar^{n+1}\sigma_i(\mu_i \circ C_{ij} \circ \mu_j^{-1}).$$

Here, σ_i is our chosen $\mathbb{C}$-module splitting of $\rho : \mathscr{A} \to \mathcal{O}$ over U_i. Note that $\widetilde{G}'$ maps to G modulo $\hbar^{n+1}$ and so in order to see that it is closed, we merely observe the following vanishing:

$$\begin{aligned}
&\rho(((\widetilde{G}_{ij} - \hbar^{n+1}\sigma_i(\mu_i \circ C_{ij} \circ \mu_j^{-1})) \circ (\widetilde{G}_{jk} - \hbar^{n+1}\sigma_j(\mu_j \circ C_{jk} \circ \mu_k^{-1}))\\
&\; - (\widetilde{G}_{ik} - \hbar^{n+1}\sigma_i(\mu_i \circ C_{ik} \circ \mu_k^{-1})))/\hbar^{n+1})\\
&\qquad = \mu_i \circ \mathrm{ob}_{n+1}(G)_{ijk} \circ \mu_k^{-1} - g_{ij} \circ \mu_j \circ C_{jk} \circ \mu_k^{-1} - \mu_i \circ C_{ij} \circ \mu_j^{-1} \circ g_{jk} + \mu_i \circ C_{ik} \circ \mu_k^{-1}\\
&\qquad = \mu_i \circ (\mathrm{ob}_{n+1}(G)_{ijk} - C_{jk} - C_{ij} + C_{ik}) \circ \mu_k^{-1} = \mu_i \circ ((\mathrm{ob}_{n+1}(G) - \delta(C))_{ijk}) \circ \mu_k^{-1}\\
&\qquad = 0.
\end{aligned}$$

The group $H^1(S, \mathscr{E}nd_{\mathcal{O}}(W))$ acts on $\mathrm{Quant}_{n+2}(W)$ by

$$\{K_{ij}\} \mapsto \{K_{ij} \circ (\mathrm{id} - \hbar^{n+1}\sigma_i(\mu_j \circ h_{ij} \circ \mu_j^{-1}))\} = \{K_{ij} - \hbar^{n+1}\sigma_i(\mu_i \circ h_{ij} \circ \mu_j^{-1}))\}$$

for $h \in H^1(S, \mathscr{E}nd_{\mathcal{O}}(W))$ and $K \in \mathrm{Quant}_{n+2}(W)$. This action is clearly free and preserves the fibers of the map $\mathrm{Quant}_{n+2}(W) \to \mathrm{Quant}_{n+1}(W)$. In order to see that it is transitive, consider two elements $K, K' \in \mathrm{Quant}_{n+2}(W)$ in the same fiber. They define a unique element of $h \in H^1(S, \mathscr{E}nd_{\mathcal{O}}(W))$ by the formula

$$h_{ij} = \mu_j^{-1} \circ \rho\left(\frac{\mathrm{id} - K_{ji} \circ (K'_{ji})^{-1}}{\hbar^{n+1}}\right) \circ \mu_j$$

and it is easily seen that h maps K to K'. In order to see that h is closed, we simply calculate

$$\begin{aligned}
h_{ij} &= \mu_j^{-1} \circ \rho\left(\frac{K_{jk} \circ (K_{kj} - K_{kj} \circ K_{ji} \circ (K'_{ji})^{-1})}{\hbar^{n+1}}\right) \circ \mu_j\\
&= \mu_k^{-1} \circ \rho\left(\frac{K_{kj} - K_{ki} \circ (K'_{ji})^{-1}}{\hbar^{n+1}}\right) \circ \mu_j\\
&= \mu_k^{-1} \circ \rho\left(\frac{(K_{kj} - K_{ki} \circ (K'_{ji})^{-1}) \circ K'_{jk}}{\hbar^{n+1}}\right) \circ \mu_k\\
&= \mu_k^{-1} \circ \rho\left(\frac{K_{kj} \circ (K'_{kj})^{-1} - K_{ki} \circ (K'_{ki})^{-1}}{\hbar^{n+1}}\right) \circ \mu_k\\
&= h_{ik} - h_{jk}.
\end{aligned}$$

This concludes the proof of the lemma. ☐

Remark B.3. The set $\mathrm{Quant}_q(W)$ can be naturally identified with the set of isomorphism classes of objects in a complex analytic stack. The map $\mathrm{Quant}_{n+2}(W) \to \mathrm{Quant}_{n+1}(W)$ is induced from a morphism of stacks and the action of $H^1(S, \mathscr{E}nd_{\mathcal{O}}(W))$ on $\mathrm{Quant}_{n+2}(W)$ can be refined to an analytic action on the stack corresponding to $\mathrm{Quant}_{n+2}(W)$.

Remark B.4. The above lemma implies that if W is a classical vector bundle for which $H^2(S, \mathscr{E}nd(W)) \cong \{0\}$, then W extends to a locally free left $\mathscr{A}$ module of the same rank. At each stage of extension, the ambiguity is precisely $H^1(S, \mathscr{E}nd(W))$. However, the vanishing of the obstruction space is not necessary for quantizability. There are many bundles with non-trivial obstruction spaces which quantize to all orders. For instance, any flat bundle does.

Appendix C. The quantum Appell–Humbert theorem

We now focus on the case of the Moyal deformation quantization of the sheaf of holomorphic functions on a complex torus. Here we can use factors of automorphy to obtain more precise formulas for the various obstruction maps. We begin by computing explicitly the obstruction $\mathrm{ob}_0(W)$ for a line bundle W on X to quantize to first order in $\hbar$. The assignment $W \mapsto \mathrm{ob}_0(W)$ gives rise to a short exact sequence of pointed cohomology sets

$$H^1(X, (\mathscr{A}/\hbar^2\mathscr{A})^\times) \longrightarrow H^1(X, \mathcal{O}^\times) \xrightarrow{\mathrm{ob}_0} H^2(X, \mathcal{O}). \tag{C.1}$$

If W is a line bundle on X, then the 'relative to W' part of this sequence is precisely the sequence appearing in the statement of Lemma B.2. Indeed $\mathrm{Quant}_0(W) = \{[W]\} \subset H^1(X, \mathcal{O}^\times)$, and $\mathrm{Quant}_1(W)$ is just the fiber of the map $H^1(X, (\mathscr{A}/\hbar^2\mathscr{A})^\times) \to H^1(X, \mathcal{O}^\times)$ over the point $[W]$.

We can understand the sequence (C.1) in terms of the group cohomology of Λ acting on functions on the universal cover V. Taking into account the fact that $H^1(V, (\mathscr{A}/\hbar^2\mathscr{A})^\times)) = 0$ (see the proof of Lemma B.1) we can rewrite (C.1) as the exact sequence of pointed group cohomology sets:

$$H^1(\Lambda, H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^\times)) \to H^1(\Lambda, H^0(V, \mathcal{O}^\times)) \to H^2(\Lambda, H^0(V, \mathcal{O})).$$

Now recall from (4.1) that $Z^1(\Lambda, H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^\times))$ consists of maps

$$\phi = \phi_0 + \hbar\phi_1 : \Lambda \to H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^\times)$$

satisfying

$$(\delta\phi)(\lambda_1, \lambda_2) = \phi(\lambda_1 + \lambda_2) - \phi(\lambda_2) \star (\phi(\lambda_1) \cdot \lambda_2). \tag{C.2}$$

Two cocycles ϕ and ψ are cohomologous if there exists $f \in H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^\times)$ which satisfies, for all $\lambda \in \Lambda$, the relationship

$$\psi(\lambda) = f^{-1} \star \phi(\lambda) \star (f \cdot \lambda). \tag{C.3}$$

In the following, we use the notation

$$f \star g = fg + \sum_{j=1}^{\infty} \hbar^j (f \star g)_j,$$

for the components of a star product.

Suppose now that we are given a holomorphic line bundle W on X represented by a particular cocycle $\phi_0 \in Z^1(\Lambda, H^0(V, \mathcal{O}^\times))$. By the classical Appell–Humbert theorem we can always replace ϕ_0 by a cohomologous cocycle which is given by the Appell–Humbert formula:

$$\mathsf{ah}_{(H,\chi)}(\lambda)(v) = \chi(\lambda) \exp\left(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\right). \tag{C.4}$$

Here H is an element in the Neron–Severi group of X thought of as a Hermitian form on V which satisfies $\mathrm{Im}\, H(\Lambda, \Lambda) \subset \mathbb{Z}$, and χ is an H-semi-character of Λ.

Denote by $\mathcal{P}(\Lambda)$ the group of all pairs (H, χ) where $H \in NS(X)$ and χ is a semicharacter for H (see [BL99, § 1.2]). By the Appell–Humbert theorem, the assignment

$$\begin{tikzcd} \mathcal{P}(\Lambda) \arrow[r] & Z^1(\Lambda, H^0(V, \mathcal{O}^{\times})) \\ (H,\chi) \arrow[r] & \mathsf{ah}_{(H,\chi)}, \end{tikzcd}$$

is an injective group homomorphism, which after a composition with the projection

$$Z^1(\Lambda, H^0(V, \mathcal{O}^{\times}) \twoheadrightarrow H^1(\Lambda, H^0(V, \mathcal{O}^{\times}) \cong \mathrm{Pic}(X)$$

becomes an isomorphism.

We now have the following lemma which computes the obstruction and ambiguity to extending ϕ_0 to a non-commutative cocycle $\phi = \phi_0 + \hbar\phi_1$.

Lemma C.1. (a) *The obstruction map*

$$\mathrm{ob}_0 : H^1(\Lambda, H^0(V, \mathcal{O}^{\times})) \to H^2(\Lambda, H^0(V, \mathcal{O}))$$

can be lifted to a map on Appell–Humbert data:

$$\begin{tikzcd} \mathcal{P}(\Lambda) \arrow[r, "\mathfrak{ob}_0"] \arrow[d, hook] & Z^2(\Lambda, H^0(V, \mathcal{O})) \arrow[d] \\ H^1(\Lambda, H^0(V, \mathcal{O}^{\times})) \arrow[r, "\mathrm{ob}_0"'] & H^2(\Lambda, H^0(V, \mathcal{O})) \end{tikzcd}$$

where

$$\mathfrak{ob}_0(\mathsf{ah}_{(H,\chi)})(\lambda_1, \lambda_2) = \{h_{\lambda_2}, h_{\lambda_1}\}.$$

(b) *Suppose that $W \in \mathrm{Pic}(X)$ is such that $\mathrm{ob}_0([W]) = 0$ in $H^2(X, \mathcal{O})$. Let (H, χ) be the Appell–Humbert data corresponding to $[W]$. Then $\mathfrak{ob}_0(\mathsf{ah}_{(H,\chi)}) = 0$ in $Z^2(\Lambda, H^0(V, \mathcal{O}))$.*

Proof. Suppose that we can find a $\phi \in Z^1(\Lambda, H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^{\times})$, so that $\phi = \phi_0$ modulo $\hbar$. If ψ_0 is a cocycle, cohomologous to ϕ_0, then we can find a cocycle $\psi \in Z^1(\Lambda, H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^{\times})$, so that $\psi = \psi_0$ modulo $\hbar$, and ψ is cohomologous to ϕ. Indeed, if $f \in H^0(V, \mathcal{O}^{\times})$ is a global holomorphic function for which $\psi_0(\lambda)/\phi_0(\lambda) = (f \cdot \lambda)/f$, then by viewing f as an element in $H^0(V, \mathscr{A})$ we can define a new cocycle ψ according to the rule (C.3) using ϕ and f. This new ψ clearly has the required properties.

Hence, without loss of generality, we may assume that $\phi = \mathsf{ah}_{(H,\chi)} + \hbar\phi_1$ for some appropriately chosen Appell–Humbert data (H, χ).

Let now $\delta : C^1(\Lambda, H^0(V, \mathcal{O})) \to C^2(\Lambda, H^0(V, \mathcal{O}))$ denote the group cohomology differential given by

$$\theta \mapsto [(\lambda_1, \lambda_2) \mapsto \theta(\lambda_1 + \lambda_2) - \theta(\lambda_2) - \theta(\lambda_1) \cdot \lambda_2].$$

A non-commutative cochain $\phi = \phi_0 + \hbar\phi_1 \in C^1(\Lambda, H^0(V, (\mathscr{A}/\hbar^2\mathscr{A})^{\times})$ is a cocycle if and only if $\phi_1' = \phi_1/\phi_0 \in C^1(\Lambda, H^0(V, \mathcal{O}))$ satisfies the condition

$$(\delta\phi_1')(\lambda_1, \lambda_2) = \frac{(\phi_0(\lambda_2) \star ((\phi_0(\lambda_1)) \cdot \lambda_2))_1}{(\phi_0(\lambda_2))(\phi_0(\lambda_1) \cdot \lambda_2)} = \frac{(\phi_0(\lambda_2) \star ((\phi_0(\lambda_1)) \cdot \lambda_2))_1}{\phi_0(\lambda_1 + \lambda_2)}.$$

After substituting $\phi_0 = \mathsf{ah}_{(H,\chi)}$ into this formula, several terms cancel and we get:

$$\begin{aligned}
(\delta\phi_1')(\lambda_1,\lambda_2) &= \frac{(\exp(\pi H(v,\lambda_2)) \star \exp(\pi H(v+\lambda_2,\lambda_1)))_1}{\exp(\pi H(v,\lambda_2) + \pi H(v+\lambda_2,\lambda_1))} \\
&= \frac{(\exp(\pi H(v,\lambda_2)) \star \exp(\pi H(v,\lambda_1)))_1}{\exp(\pi H(v,\lambda_1+\lambda_2))} \\
&= \frac{(\exp(h_{\lambda_2}) \star \exp(h_{\lambda_1}))_1}{\exp(h_{\lambda_1+\lambda_2})} \\
&= (\exp(\hbar\{h_{\lambda_2},h_{\lambda_1}\}))_1 \\
&= \{h_{\lambda_2},h_{\lambda_1}\}.
\end{aligned}$$

Here $h_\lambda \in V^\vee$ denotes the $\mathbb{C}$-linear function $v \mapsto \pi H(v,\lambda)$ and in the last equality we used the identity (4.9). Due to the equality $\delta\phi_1' = \{h_{\lambda_2},h_{\lambda_1}\}$ we conclude that we will be able to extend $\mathsf{ah}_{(H,\chi)}$ to a non-commutative cocycle $\phi = \mathsf{ah}_{(H,\chi)} + \hbar\phi_1$ if and only if the cocycle $[(\lambda_1,\lambda_2) \mapsto \{h_{\lambda_2},h_{\lambda_1}\}] \in Z^2(\Lambda, H^0(V,\mathcal{O}))$ is a coboundary. This shows that $[(\lambda_1,\lambda_2) \mapsto \{h_{\lambda_2},h_{\lambda_1}\}]$ represents the obstruction class $\mathrm{ob}_0([\mathsf{ah}_{(H,\chi)}])$ and proves part (a) of the lemma.

For the proof of part (b) note that, by construction, the cocycle $\mathfrak{ob}_0(\mathsf{ah}_{(H,\chi)})$ is actually in $Z^2(\Lambda,\mathbb{C}) \subset Z^2(\Lambda, H^0(V,\mathcal{O}))$. Furthermore, if we consider the canonical Hodge decomposition

$$\begin{aligned}
H^2(X,\mathbb{C}) &= H^2(X,\mathcal{O}) \oplus H^1(X,\Omega^1) \oplus H^0(X,\Omega^2) \\
&= \bigwedge^2 \overline{V}^\vee \oplus (\overline{V}^\vee \otimes V^\vee) \oplus \bigwedge^2 V^\vee,
\end{aligned}$$

then the image of $\mathfrak{ob}_0(\mathsf{ah}_{(H,\chi)})$ in $H^2(X,\mathbb{C})$ lands entirely in the piece $H^2(X,\mathcal{O}) = \bigwedge^2 \overline{V}^\vee$. Indeed, thinking of H as an element in $\overline{V}^\vee \otimes V^\vee$ we can rewrite the image $\mathfrak{ob}_0(\mathsf{ah}_{(H,\chi)})$ in purely linear algebraic terms as the contraction $H \lrcorner \mathbf{\Pi} \llcorner H$. Indeed, the additive map $\lambda_1 \wedge \lambda_2 \mapsto \{h_{\lambda_2},h_{\lambda_1}\}$ extends by linearity to a unique conjugate linear homomorphism $\bigwedge^2 V \to \mathbb{C}$ which equals $H \lrcorner \mathbf{\Pi} \llcorner H$ as an element in $\bigwedge^2 \overline{V}^\vee$.

Therefore, the obstruction $\mathrm{ob}_0([\mathsf{ah}_{(H,\chi)}])$ vanishes if and only if $H \lrcorner \mathbf{\Pi} \llcorner H = 0$ in $H^2(X,\mathbb{C})$. Since $H \lrcorner \mathbf{\Pi} \llcorner H$ was proportional to the anti-linear extension of the map $\mathfrak{ob}(\mathsf{ah}_{(H,\chi)}) : \Lambda \times \Lambda \to \mathbb{C}$ this concludes the proof of part (b). ☐

The formula for ob_0 given in the lemma can also be deduced from the first-order analysis carried out in Toda's paper [Tod05]. However, in our case, the specific geometry of the Moyal quantization of a complex torus allows us to push the analysis further. In fact, it turns out that for a line bundle W on X, the vanishing of $\mathrm{ob}_0([W])$ is both necessary and sufficient for W to quantize to all orders.

Lemma C.2. *A line bundle W on X can be extended to a line bundle on $\mathbb{X}_{\mathbf{\Pi}}$ if and only if*

$$\mathrm{ob}_0([W]) = c_1(W) \lrcorner \mathbf{\Pi} \llcorner c_1(W) = 0 \quad \textit{in } H^2(X,\mathcal{O}).$$

Proof. Let (H,χ) be the Appell–Humbert data for the isomorphism class of line bundles $[W]$. Consider the map

$$\begin{tikzcd}
\Lambda \arrow[rr, "\phi"] & & H^0(V, \mathscr{A}^\times_{V,\mathbf{\Pi}}) \\
\lambda \arrow[r, mapsto] & \left[v \mapsto \chi(\lambda)\exp\left(\pi H(v,\lambda) + \dfrac{\pi}{2}H(\lambda,\lambda)\right)\right] &
\end{tikzcd}$$

where the exponential now is the $\star$-exponential. By definition, $\phi \in C^1(\Lambda, H^0(V,\mathscr{A}^\times_{V,\mathbf{\Pi}}))$ is a non-commutative cochain. In order for ϕ to be a cocycle, we must have that

$$\phi(\lambda_1+\lambda_2)^{-1} \star \phi(\lambda_2) \star (\phi(\lambda_1)\cdot\lambda_2) = 1. \tag{C.5}$$

However, in the proof of Lemma C.1 we evaluated the left-hand side of (C.5) and showed that it is equal to $\exp(\hbar\{h_{\lambda_2}, h_{\lambda_1}\})$. This proves our assertion since the constant $\{h_{\lambda_2}, h_{\lambda_1}\}$ is equal to the value of $H \lrcorner \mathbf{\Pi} \llcorner H \in \bigwedge^2 \overline{V}^\vee$ on the element $\lambda_1 \wedge \lambda_2$. ☐

Suppose that $(H, \chi) \in \mathcal{P}(\Lambda)$ is some Appell–Humbert data and let $l(\hbar) = \sum_{i=1}^{\infty} \hbar^i l_i \in \hbar\overline{V}^\vee[[\hbar]]$. Consider the map $\mathsf{qah}_{((H,\chi),l(\hbar))} : \Lambda \to H^0(V, \mathscr{A}^\times_{V,\mathbf{\Pi}})$ given by

$$\mathsf{qah}_{((H,\chi),\,l(\hbar))}(\lambda)(v) = \chi(\lambda)\exp\left(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda) + \sum_{j=1}^{\infty}\hbar^j\pi\langle l_j, \lambda\rangle\right). \tag{C.6}$$

A straightforward check shows that if (H, χ) satisfies $H \lrcorner \mathbf{\Pi} \llcorner H = 0$, then the map $\mathsf{qah}_{((H,\chi),\,l(\hbar))}$ is a non-commutative 1-cocycle, i.e. $\mathsf{qah}_{((H,\chi),\,l(\hbar))} \in Z^1(\Lambda, H^0(V, \mathscr{A}^\times))$.

Consider now the subset $\mathcal{P}(\Lambda, \mathbf{\Pi}) \subset \mathcal{P}(\Lambda)$ defined by

$$\mathcal{P}(\Lambda, \mathbf{\Pi}) = \{(H, \chi) \in \mathcal{P}(\Lambda) \mid H \lrcorner \mathbf{\Pi} \llcorner H = 0\}.$$

With this notation we have the following quantum version of the Appell–Humbert theorem.

Proposition C.3. *The map*

$$\begin{tikzcd}
\mathcal{P}(\Lambda, \mathbf{\Pi}) \times \hbar\overline{V}^\vee[[\hbar]] \arrow[r] & H^1(X, \mathscr{A}^\times_{X,\mathbf{\Pi}}) \\
\left((H,\chi), \sum_{i=1}^{\infty}\hbar^i l_i\right) \arrow[r, mapsto] & \mathsf{qah}_{((H,\chi),\,l(\hbar))}(\lambda)
\end{tikzcd}$$

is a bijection of pointed sets.

Proof. It suffices to show that for each j the map qah induces a bijection

$$\mathcal{P}(\Lambda, \mathbf{\Pi}) \times \hbar\overline{V}^\vee[[\hbar]]/\hbar^j \to H^1(X, (\mathscr{A}_{X,\mathbf{\Pi}}/\hbar^j)^\times).$$

Note that the case $j = 1$ is the usual Appell–Humbert theorem. If we assume that this has been shown for $j \leqslant n$ then to show that it holds for $j = n + 1$ we can use the argument from the proof of Lemma C.1 where it was shown that the case $j = 1$ implies the case $j = 2$. The key point is that for $\psi \in C^1(\Lambda, H^0(V, \mathcal{O}))$ the cochain in $C^1(\Lambda, H^0(V, (\mathscr{A}_{\mathbf{\Pi}}/\hbar^{n+1})^\times)$ given by

$$\lambda \mapsto \chi(\lambda)\exp\left(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda) + \sum_{j=1}^{n}\hbar^j\pi\langle l_j, \lambda\rangle + \hbar^{n+1}\psi\right)$$

is a cocycle if and only if ψ is in $Z^1(\Lambda, H^0(V, \mathcal{O}))$. ☐

Remark C.4. The quantum Appell–Humbert theorem gives preferred group cocycle representatives for the isomorphism classes of quantum line bundles. For degree zero line bundles the representatives are given by

$$\lambda \mapsto \chi(\lambda)\exp\left(\sum_{j=1}^{\infty}\hbar^j\pi\langle l_j, \lambda\rangle\right).$$

Here $\chi \in \mathrm{Hom}(\Lambda, U(1))$ and $l_j \in \overline{V}^\times$. Thus, the connected component $H^1(X, \mathscr{A}^\times_{X,\mathbf{\Pi}})_o$ of the quantum Picard $H^1(X, \mathscr{A}^\times_{X,\mathbf{\Pi}})$ is in bijection with $X^\vee \times (\overline{V}^\vee)^{\mathbb{Z}_{>0}} \cong H^1(X, \mathcal{O}[[\hbar]]^\times)_o$.

Remark C.5. As an example consider the product of elliptic curves $E_1 \times E_2$ with coordinates (z_1, z_2) and Poisson structure $\partial/\partial z_1 \wedge \partial/\partial z_2$. Let L be the line bundle corresponding to the divisor $E_1 \times \{0\}$ and M the line bundle corresponding to the divisor $\{0\} \times E_2$. Then L and M are quantizable but $L \otimes M$ is not. Note also that $L \oplus M$ is a quantizable vector bundle yet has a non-zero second Chern class, given by the Poincaré dual to the intersection of the two divisors.

Appendix D. On the cohomology of left $\mathscr{A}$-modules

Let S be a Hausdorff analytic space and $\mathscr{A}$ be a sheaf of $\mathbb{C}[[\hbar]]$-algebras such that $\mathscr{A} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathcal{O}_S$. The usual arguments in [Har77] go through to show that if $\mathscr{L}$ is a left $\mathscr{A}$-module, then there are well-defined cohomology groups of $\mathscr{L}$ computed from the derived functors of the global sections functor in the categories of sheaves of abelian groups, sheaves of $\mathbb{C}$-vector spaces, sheaves of $\mathbb{C}[[\hbar]]$-modules, and sheaves of $\mathscr{A}$-modules. Furthermore, all of these cohomologies are naturally isomorphic to each other. If $\mathscr{L}$ is a locally free left $\mathscr{A}$-module of finite rank, then these cohomologies also agree with the Čech cohomology of $\mathscr{L}$. Indeed, choose a cover $\mathfrak{U} = \{U_i \mid i \in \mathcal{I}\}$ by contractible open sets such that for $U_I = \bigcap_{j\in I} U_j$ we have $H^j(S, U_I, \mathcal{O}) = \{0\}$ and $H^1(S, U_I, \mathcal{O}^\times) = \{1\}$ for all finite subsets $I \subset \mathcal{I}$ and all $j \geqslant 1$. Denote the inclusion maps by $\kappa_I : U_I \to S$. Then, by Lemma B.1 we have $H^j(S, U_I, \mathscr{L}) \cong H^j(S, U_I, \mathscr{A}) \cong H^j(S, U_I, \mathcal{O})[[\hbar]] = \{0\}$ for all I and $j \geqslant 1$. Therefore we can compute cohomology from the following acyclic resolution

$$0 \to \mathscr{L} \to \bigoplus_{i\in\mathcal{I}} \kappa_{i*}(\mathscr{L}|_{U_i}) \to \bigoplus_{\{i,j\}\in\mathcal{I}^{\times 2}} \kappa_{ij*}(\mathscr{L}|_{U_{i,j}}) \to \cdots .$$

This cohomology is precisely the Čech cohomology.

LEMMA D.1. *Let $\mathscr{L}$ be a degree zero line bundle on the non-commutative torus $\mathbb{X}_{\mathbf{\Pi}}$. View $\mathscr{L}$ as a sheaf of left $\mathscr{A}_{X,\mathbf{\Pi}}$ modules on the underlying torus X. Then:*

(a) *$\mathscr{L}$ is non-trivial if and only if $H^0(X, \mathscr{L}) = 0$;*

(b) *if $\mathscr{L}/\hbar\mathscr{L} \ncong \mathcal{O}$, then $H^i(X, \mathscr{L}) = 0$ for all $i \geqslant 0$.*

Proof. First we prove part (a). Since $\mathscr{L}$ is a translation invariant line bundle on $\mathbb{X}_{\mathbf{\Pi}}$, it is given by a constant factor of automorphy (see Remark C.4) and so the sheaf $\mathscr{L}$ has a preferred flat connection. Denote the corresponding local system of $\mathbb{C}[[\hbar]]$-modules by $\underline{\mathscr{L}}$. Furthermore, the sheaf $\mathscr{L}$ has a natural structure of an $\mathcal{O}[[\hbar]]$-module. This follows from the identification $\mathscr{L} \cong \underline{\mathscr{L}} \otimes_{\mathbb{C}} \mathcal{O} = \underline{\mathscr{L}} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{O}[[\hbar]]$. Now observe that

$$H^0(X, \mathscr{L}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{mod}}(\mathcal{O}, \mathscr{L}) = \varprojlim \mathrm{Hom}_{\mathcal{O}-\mathrm{mod}}(\mathcal{O}, \mathscr{L}/\hbar^k) = \varprojlim H^0(X, \mathscr{L}/\hbar^k).$$

Note that $\mathscr{L}/\hbar^k$ is a free $\mathcal{O}[[\hbar]]/\hbar^k$ module and so is a holomorphic vector bundle of rank k on X. The vector bundle $\mathscr{L}/\hbar^k$ has a preferred flat connection coming from the natural identification $\mathscr{L}/\hbar^k \cong (\underline{\mathscr{L}}/\hbar^k) \otimes_{\mathbb{C}} \mathcal{O}$. From the formula for the factor of automorphy for $\mathscr{L}$ (see Remark C.4) it is clear that the flat connection on $\mathscr{L}/\hbar^k$ is unitary. Since X is Kähler we can now use that the Hodge decomposition to compare the cohomology of the local system $\underline{\mathscr{L}}/\hbar^k$ and the holomorphic bundle $\mathscr{L}/\hbar^k$. In particular, we have that the natural map $\underline{\mathscr{L}}/\hbar^k \to \mathscr{L}/\hbar^k$ induces an isomorphism on H^0. Therefore,

$$H^0(X, \mathscr{L}) = \varprojlim H^0(X, \mathscr{L}/\hbar^k) = \varprojlim H^0(X, \underline{\mathscr{L}}/\hbar^k) = H^0(X, \underline{\mathscr{L}}),$$

and so a non-zero global section of $\mathscr{L}$ is nowhere vanishing. Since $\mathscr{L}$ is a locally free rank one $\mathscr{A}_{X,\mathbf{\Pi}}$-module this implies that $\mathscr{L}$ is trivial. The opposite implication is obvious. This completes the proof of part (a).

For part (b) recall the classical result that if L is a non-trivial degree zero line bundle on X, then $H^j(L) = 0$ for all $j \geqslant 0$. Suppose now that $\mathscr{L}$ is a quantum line bundle of degree zero for which $\mathscr{L}/\hbar\mathscr{L}$ is non-trivial. Consider the short exact sequence

$$0 \longrightarrow \mathscr{L} \xrightarrow{\hbar} \mathscr{L} \longrightarrow \mathscr{L}/\hbar\mathscr{L} \longrightarrow 0.$$

Since all cohomology groups of $\mathscr{L}/\hbar\mathscr{L}$ vanish, the long exact cohomology sequence implies that $\hbar$ induces an isomorphism of $\mathbb{C}[[\hbar]]$-modules $H^j(\mathscr{L}) \to H^j(\mathscr{L})$ for all j. This implies that $H^j(\mathscr{L}) = 0$

for all j. Indeed if not, choose the largest possible $p \in \{0, 1, 2, \dots\}$ such that $H^j(\mathscr{L})/\hbar^p H^j(\mathscr{L}) = (0)$, then applying the isomorphism gives that $H^j(\mathscr{L})/\hbar^{p+1} H^j(\mathscr{L}) = 0$, a contradiction. ☐

Suppose that $\mathscr{L}$ is a degree zero quantum line bundle with $\mathscr{L}/\hbar\mathscr{L} \cong \mathcal{O}$, yet $\mathscr{L}$ is non-trivial. In this case the higher cohomology groups of $\mathscr{L}$ need not vanish. The easiest way to see this is to note that

$$0 \longrightarrow \mathscr{L} \xrightarrow{\hbar} \mathscr{L} \longrightarrow \mathcal{O} \longrightarrow 0 \tag{D.1}$$

is a short exact sequence of $\mathcal{O}$-modules and so the extension class of this sequence lies in $H^1(X, \mathscr{L})$. Modulo $\hbar^2$ the sequence (D.1) induces a short exact sequence of $\mathcal{O}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}/\hbar \arrow[r] \arrow[d, equal] & \mathscr{L}/\hbar^2 \arrow[r] & \mathscr{L}/\hbar \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O} & & \mathcal{O} & \end{tikzcd} \tag{D.2}$$

whose extension class is in $H^1(X, \mathcal{O})$. If $H^1(X, \mathscr{L})$ was zero, then this sequence (D.1), and hence the sequence (D.2), will split. However, it is immediate to check that if $\mathscr{L}$ is represented by quantum Appell–Humbert data $((0,1), \sum_{i=1}^{\infty} \hbar^i l_i)$, then the extension class of (D.2) is given by the group cohomology class $[l_1] \in H^1(\Lambda, H^0(V, \mathcal{O}))$. Since we are completely free to choose the l_i this shows that (D.2), and hence (D.1), are non-split in general. This implies that a general $\mathscr{L}$ with $\mathscr{L}/\hbar \cong \mathcal{O}$ has non-trivial first cohomology.

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