Cyclic Homology of Filtered Algebras

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(Received: 2 October, 1987)

Abstract. We prove an analogue of a theorem of Quillen about the cyclic homology of filtered rings.

Key words. Cyclic homology, filtered rings.

1. Introduction

Quillen proves the following theorem about the algebraic K-theory of filtered rings:

THEOREM 1.1 ([4]), p. 112, Theorem 7. Let $A$ be a ring equipped with an increasing filtration $F_i A$, and such that $F_0 A$ is regular. Suppose that $B = \text{gr}(A)$ has finite Tor dimension as a right $F_0 A$ module and that $F_0 A$ has finite Tor dimension as a right $B$ module. Then the inclusion $F_0 A \subset A$ induces an isomorphism $K_i (F_0 A) \cong K_i (A)$.

This result implies for example that the natural inclusion $\mathcal{O}(V) \subset \mathcal{D}(V)$ of the coordinate ring of a smooth affine variety into its ring of differential operators induces an isomorphism $K_i (\mathcal{O}(V)) \cong K_i (\mathcal{D}(V))$ in algebraic K-theory.

Recently, this calculation has been carried out in cyclic homology. Thus Block [1], Brylinski [2], and Wodzicki [5] show that one has an isomorphism in cyclic homology $HC_q (\mathcal{O}(V)) \cong HC_q (\mathcal{D}(V))$ for $q \geq 2n - 1$ where $\dim V = n$. This suggests that a theorem analogous to that of Quillen should hold. The purpose of this note is to prove such a theorem.

2. Notation

For convenience, $A$ will always be an algebra over a field $k$ of characteristic zero. By a filtration on an algebra we will mean an increasing filtration

$$0 = F_{-1} A \subset F_0 A \subset F_1 A \subset \ldots$$

which satisfies $F_p A \cdot F_q A \subset F_{p+q} A$, $1 \in A$ and $A = \cup F_p A$. Hochschild homology of $A$ with coefficients in a bimodule $M$ will be denoted by $H_i (A; M)$. Cyclic homology of $A$ is denoted by $HC_i (A)$ and periodic cyclic homology by $HP_i (A)$.

3. Cyclic Homology of Filtered Rings

DEFINITION 3.1. Let $A$ be an algebra. Let $d(A) = \inf \{ n \in \mathbb{Z}^+ | H_i (A; A) = 0 \text{ for } i > n \}$. Call $d(A)$ the Hochschild dimension of $A$.

Remark 3.2. $d(A) = n$ implies that $S: HC_{i+2} (A) \to HC_i (A)$ is an isomorphism for all $i \geq n$, and an injection for $i = n - 1$. Thus $HP_i (A) \cong HC_i (A)$ for $i \geq n$. 

LEMMA 3.3. $d(A) \leq d(\text{gr}(A))$

Proof. This is just a spectral sequence argument.

We can now prove

THEOREM 3.4. Let $A$ be an algebra with an increasing filtration as above. Suppose that the Hochschild dimension of the associated graded algebra $d(\text{gr}(A)) = n < \infty$. Then the natural map

$$H_{C_i}(F_0 A) \to H_{C_i}(A)$$

is an isomorphism for all $i \geq n$. In particular $H_{C_i}(F_0 A) \cong H_{C_i}(A)$ for all $i$.

Proof. Let $C_* A$ be the standard cyclic vector space associated to an algebra $A$, [3], p. 189. Thus $C_n A = A^{\otimes n+1}$ for each $n \geq 0$ and the face and degeneracy maps are defined as [loc. cit.]. Filter $C_n A$ in the following manner:

$$F_p C_n A = \sum_{k_0 \cdots + k_n = p} F_{k_0} A \otimes \cdots \otimes F_{k_n} A.$$

It is easy to see that the face, degeneracy and cyclic maps preserve this filtration, hence, for each $p$, $F_p C_* A$ forms a cyclic vector space.

LEMMA 3.5. $\text{gr}(F_* C_* A) \cong C_* \text{gr}(A)$.

LEMMA 3.6

$$S: H_{C_i+2}(F_0 C_* A/F_{i-1} C_* A) \to H_{C_i}(F_i C_* A/F_{i-1} C_* A)$$

acts like zero.

Proof. As in [3], p. 197, claim 1, define on $\text{gr}(A)$ a derivation $\delta: \text{gr}(A) \to \text{gr}(A)$ by the following formula $\delta(a) = |a|a$ for homogeneous elements of degree $|a|$. Then define the endomorphism $\mathcal{L}_\delta$ of $C_* A$ by

$$\mathcal{L}_\delta(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} a_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n.$$

Then by [3] $\mathcal{L}_\delta \circ S = 0$ on $H_{C_*}(\text{gr}(A))$. On the other hand it is clear that $\mathcal{L}_\delta$ acts by multiplication by $l$ on $H_{C_*}(F_i C_* A/F_{i-1} C_* A)$. So that $S: H_{C_i+2}(F_0 C_* A/F_{i-1} C_* A) \to H_{C_i}(F_i C_* A/F_{i-1} C_* A)$ acts like zero, for $i > 0$.

To continue with the proof of the theorem consider the following short exact sequence of cyclic vector spaces, $0 \to F_0 C_* A \to F_1 C_* A \to F_1 C_* A/F_0 C_* A \to 0$. This induces a long exact sequence in cyclic homology

$$\cdots \to H_{C_i}(F_0 C_* A) \to H_{C_i}(F_1 C_* A) \to H_{C_i}(F_1 C_* A/F_0 C_* A) \to \cdots$$

By Lemma 3.6 above, $S$ acts like zero on $H_{C_i}(F_1 C_* A/F_0 C_* A)$ while by the hypothesis on the Hochschild dimension,

$$S: H_{C_i+2}(F_1 C_* A/F_0 C_* A) \to H_{C_i}(F_1 C_* A/F_0 C_* A)$$

is an isomorphism for $i \geq n$, hence $H_{C_i}(F_1 C_* A/F_0 C_* A) = 0$. Therefore by the long exact sequence above we know that $H_{C_i}(F_0 C_* A) \to H_{C_i}(F_1 C_* A)$ is an
isomorphism, for \( i \geq n \). Continuing in the same manner, it follows from the short exact sequence of cyclic vector spaces

\[
0 \to F_i C_\ast A \to F_{i+1} C_\ast A \to F_{i+1} C_\ast A/F_i C_\ast A \to 0
\]

that \( HC_i(F_i C_\ast A) \to HC_i(F_{i+1} C_\ast A) \) is an isomorphism for \( i \geq n \). We then have the chain of isomorphisms for \( i \geq n \),

\[
HC_i(F_0 A) = HC_i(F_0 C_\ast A) \cong HC_i(F_1 C_\ast A) \cong \cdots \cong HC_i(F_i C_\ast A) \cong \cdots
\]

and the theorem follows by observing that \( HC_i(A) = \lim HC_i(F_i C_\ast A) \).

\[ \square \]

**Examples:**

1. Let \( A = U(X) \) be the universal enveloping algebra of a Lie algebra \( X \) of dimension \( n \) over a field \( k \) of characteristic 0. Then \( \text{gr}(A) \cong S^\ast(X) \) and \( d(S^\ast(X)) = n \). For \( i \geq n \), \( HC_i(A) \cong HC_i(F_0 A) = HC_i(k) \).

2. Let \( \mathcal{D}(V) \) be the algebra of differential operators on a smooth affine variety \( V \). Then \( \text{gr}(\mathcal{D}(V)) \cong \mathcal{O}(T^*V) \) and \( d(\mathcal{O}(T^*V)) = 2n \) so that \( HC_i(\mathcal{O}(V)) \cong HC_i(\mathcal{D}(V)) \) for \( i \geq 2n \).

3. Let \( Y \to X \) be the Karoubi–Jouanolou resolution of a smooth projective variety \( X \), as in [2]. Thus, \( Y \) is a smooth affine variety and the fiber is affine space. Then Brylinski [2] shows that

\[
H_\ast(\mathcal{O}_{Y/X}; \mathcal{O}_{Y/X}) \cong \bigoplus H^{2n+i-\ast}(X, \Omega_X^i).
\]

is the Hodge cohomology. By the result above,

\[
HP_\ast(\mathcal{O}_{Y/X}) \cong \bigoplus H^{2i+i-\ast}(Y) \cong \bigoplus H^{2i+i-\ast}(X)
\]

is just the de Rham cohomology of \( X \).

**Remark 3.7**

1. It seems likely that one should be able to replace the hypothesis that \( d(A) < \infty \) which is a statement about the homological dimension of \( A \) over \( k \), by one about the homological dimension of \( A \) over \( F_0 A \) as in Quillen’s theorem.

2. The theorem is true in the situation where one considers a topological algebra as long as one makes appropriate compatibility conditions on the filtration and the topology.

In [1], [2] and [5], the Hochschild homology of \( \mathcal{D}(V) \) is computed and it is shown that

\[
H_i(\mathcal{D}(V); \mathcal{D}(V)) \cong H^{2n-i}(V)
\]

where this latter group is de Rham cohomology of \( V \). Now our theorem immediately implies

**COROLLARY 3.8.** The Hochschild to cyclic spectral sequence degenerates at the \( E^1 \) term. Thus

\[
HC_i(\mathcal{D}(V)) \cong H^{2n-i}(V) \oplus H^{2n-i+2}(V) \oplus \cdots
\]

**Proof.** By the theorem above and the computation of the Hochschild homology, it
follows that for $i \geq 2n$ that
\[
\Sigma_{j=1}^{i}E_{jl}^1 \cong \Sigma_{j=1}^{i}E_{jl}^2
\]

hence any differential must be zero when the target of the differential has total
degree greater than $2n$. But each differential, as for example, the one at the $E^1$ term
which is just Connes’ $B$ operator, is given by an explicit formula which does not
depend on where in the spectral sequence it lies. Hence, all the differentials must be
zero. \hfill \square

Remark 3.10. This shows that the theorem is best possible in one sense; in general,
one can not have an isomorphism until one reaches the stable range.

Acknowledgement

This note is Section 1.5 of the author’s Ph.D. thesis. The author would like to thank
Professors Bott, Goodwillie and Bernstein for much help and guidance.

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