

## On the generalized Nielsen realization problem

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**Abstract.** The main goal of this paper is to give the first examples of equivariant aspherical Poincaré complexes, that are not realized by group actions on closed aspherical manifolds  $M$ . These will also provide new counterexamples to the Nielsen realization problem about lifting homotopy actions of finite groups to honest group actions. Our examples show that one cannot guarantee that a given action of a finitely generated group  $\pi$  on Euclidean space extends to an action of  $\Pi$ , a group containing  $\pi$  as a subgroup of finite index, even when all the torsion of  $\Pi$  lives in  $\pi$ .

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### 1. Introduction

Consider a closed aspherical manifold  $M$ . Then  $\pi_0(\mathcal{H}(M))$ , where  $\mathcal{H}(M)$  is the space of self homotopy equivalences of  $M$ , is isomorphic to the group  $\text{Out}(\pi_1(M))$  of outer automorphisms of  $\pi_1(M)$ . The celebrated Borel conjecture ([15]) implies that any  $\phi \in \mathcal{H}(M)$  is homotopic to a homeomorphism. In general, it asserts that homotopy equivalences between homotopy equivalent compact aspherical manifolds are homotopic to homeomorphisms.

The generalized Nielsen realization problem is stated as follows.

**Problem 1.1.** Given a finite subgroup  $G$  of  $\text{Out}(\pi_1(M))$ , does there exist a group action of  $G$  on  $M$  realizing this outer action on  $\pi_1(M)$ .

Nielsen's original question was whether for a closed Riemann surface  $S$  of genus greater than one, any finite subgroup  $G$  of  $\text{Out}(\pi_1(S))$  could be lifted to a group of isometries for some hyperbolic structure on  $S$ . Nielsen solved this himself for  $G$  cyclic, [26]. Further partial results were obtained by others, e.g. Zieschang, [36] and the full problem was finally solved positively by Kerckhoff [21]. Subsequently,

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Tromba [31], Gabai [18] and Wolpert [35] gave new proofs of Kerckhoff's theorem. Gabai's proof is a consequence of his solution of the convergence subgroup problem.

In high dimensions, it is easy to give smooth counterexamples to the generalized Nielsen realization problem using exotic differential structures on the sphere. For example, consider  $M = (T^7 \# \Sigma^7) \times S^1$ , the product of a circle with the connected sum of the 7-torus with an exotic 7 sphere. Then,  $\pi_1(M) = \mathbb{Z}^8$  and there is no group of diffeomorphisms that lifts the permutation group action on  $\mathbb{Z}^8$ . Thus, it makes most sense to consider this problem in topological settings.

A first obstruction to  $G$  acting on  $M$  realizing a given outer action comes from the nonexistence of certain group extensions. More precisely, if the outer action lifts to an actual action, then there is an extension of groups

$$1 \rightarrow \pi \rightarrow \Pi \rightarrow G \rightarrow 1 \quad (1.1)$$

where  $\pi = \pi_1(M)$  and the outer action of  $G$  on  $\pi_1(M)$  arising from the extension is the given one. This condition can be nontrivial. Raymond and Scott, [27], produced examples where  $\pi$  is the fundamental group of a nilmanifold, and for some cyclic  $G$ , there exists no such extension (1.1). However, if the center  $\mathfrak{Z}(\pi) = 0$  there always exists a unique such an extension, up to isomorphism, [3], Corollary 6.8, page 106.

Positive results for the generalized Nielsen problem have been obtained by Farrell and Jones. See [13], page 282, where a survey of the problem and their positive results are described. In particular, they assume that the fundamental group is centerless and thus the obstruction of Raymond and Scott vanishes.

Henceforth we assume that the obstruction of Raymond and Scott vanishes, that is, there is an extension (1.1). Some of our examples will even produce centerless  $\pi$ . Thus one reformulates the Nielsen realization problem and asks if this is enough to guarantee the existence of an action of  $G$  on  $M$ . We note that, before us, there was no example of nonrealization even for infinite  $G$ . There are still no examples for torsion free  $G$ . (However, see [25] for the differentiable failure of this infinite "Nielsen problem" for surfaces.)

If  $\Pi$  is torsion free there is a good conjectural reason to expect the answer to be positive:

**Proposition 1.2.** *If  $\Pi$  is torsion free then it is a Poincaré duality group if and only if  $\pi$  is. If  $B\pi = M$  is a closed manifold of dimension at least 5, and the Borel conjecture holds for  $\Pi$  and  $\pi$ , then  $B\Pi$  is a manifold as well and the normal cover corresponding to  $G$  is  $M$ ; thus  $M$  has a free  $G$  action.*

**Remark 1.3.** We understand the Borel conjecture to assert that if  $B\Gamma$  is any compact manifold with boundary and

$$\phi: (M, \partial M) \rightarrow T^n \times (B\Gamma, \partial B\Gamma)$$

is a homotopy equivalence of pairs that is already a homeomorphism on the boundary, then  $\phi$  is homotopic rel boundary to a homeomorphism. When  $B\Gamma$  is a finite complex, this is well known to be equivalent to various vanishing statements of Whitehead groups and isomorphism statements of  $L$ -theory assembly maps. In particular, it does not matter which compact manifold with boundary model of  $B\Gamma$  one chooses.

*Proof.* The first statement is Proposition 10.2, page 224 of [3]. As for the second, first observe that  $B\Pi$  is a finite complex by the vanishing of the Wall finiteness obstruction that lies in the vanishing group  $\tilde{K}_0(\mathbb{Z}\Pi)$ . Now, the existence of the manifold structure on  $B\Pi$  follows from the theory of the total surgery obstruction: the obstruction to the existence of a homology manifold realizing  $B\Pi$  lies in a group which the Borel conjecture asserts is trivial (for this version, see [4]). This homology manifold is actually a manifold, because it is covered by one.  $\square$

**Remark 1.4.** We shall see that the analogue of this proposition for non-free actions is not true.

One can view the Nielsen problem as one of extending group actions as follows: If  $\pi$  is the fundamental group of  $M$ , then  $\pi$  naturally acts freely on  $\tilde{M}$ ; Assuming the extension  $\Pi$  exists, the Nielsen problem asks whether the original  $\pi$  action extends to a  $\Pi$  action<sup>1</sup>. (The  $\Pi$  action will be free, if and only if  $\Pi$  is torsion free, as in the proposition just discussed.) Modifying this somewhat, one can ask these extension questions wherein we demand more on the  $\Pi$  action, e.g. that all fixed sets are empty or contractible (we call this an *aspherical action*, and such an extension of a group action, an *aspherical extension*), cf. e.g. [23], [24]. On the way to giving our counterexample to Nielsen, we prove the following theorem which can be thought of as giving a counterexample to Nielsen realization of free actions on orbifolds.

**Theorem 1.5.** *There is a group extension*

$$1 \rightarrow \pi \rightarrow \Pi \rightarrow G \rightarrow 1$$

*satisfying the following properties.*

- (1) *Any torsion element in  $\Pi$  is in  $\pi$ , that is  $\Pi$  is relatively torsion free.*
- (2)  *$\pi$  is virtually torsion free.*
- (3) *We can guarantee either*
  - a)  *$\pi$  acts properly discontinuously and cocompactly on Euclidean space such that the fixed sets of all finite subgroups are Euclidean spaces, so  $\pi$  is acting aspherically;*

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<sup>1</sup>Unfortunately, standard mathematical terminology forces us to overuse the word “extension”.

or

- b)  $\pi$  acts properly discontinuously and cocompactly on a contractible manifold such that the fixed sets of all finite subgroups are contractible manifolds, so  $\pi$  is acting aspherically. In this case, one can construct our example to have centerless fundamental group.
- (4) The action of  $\pi$  does not extend to one of  $\Pi$ . In fact, there is no properly discontinuous action of  $\Pi$  on Euclidean space with only contractible fixed-point sets.
- (5) There is a properly discontinuous action of  $\Pi$  on a contractible space such that all of the fixed sets of all finite subgroups are contractible.

Point (4) above discusses both the statement about free actions on nonmanifolds and nonfree actions on manifolds. We give two constructions. They in fact give a cyclic group of prime order (of order two for the first construction),  $\mathbb{Z}/p$ , which does not act aspherically on a suitable aspherical manifold.

We also derive

**Theorem 1.6.** *There is a counter example to the Nielsen realization problem with group  $\mathbb{Z}/2$  and centerless fundamental group.*

For a finitely generated discrete group  $\Pi$  one can define the asymptotic homology  $HX_*(\Pi)$  of  $\Pi$  considered as a metric space. One has the following dichotomy.

**Proposition 1.7** ([1]). *If  $\Pi$  is a group of virtual finite type, then either  $HX_*(\Pi) = \mathbb{Z}$  for  $* = n$  and zero otherwise (which we will call simple) or  $HX_*(\Pi)$  is infinitely generated in some dimension.*

We warn the reader that there are finitely generated groups of infinite type whose asymptotic homology vanishes in all dimensions. For a discrete group  $\Pi$  there is a space  $E\Pi$ , which is universal for proper actions. which is unique up to equivariant homotopy equivalence, [23] and [24]. If there is a model for  $B\Pi = E\Pi/\Pi$  which is a compact manifold, then the asymptotic homology is simple. It is natural to ask if this is also sufficient. Our examples answer this as well.

**Theorem 1.8.** *There is a group  $\Pi$  of virtual finite type with  $HX_*(\Pi)$  simple and which has no proper cocompact action on a contractible manifold.*

## 2. The construction

For all the theorems above, the constructions are of the following sort. We will construct  $\Pi$  directly via a  $\mathbb{Z}/p$  action on an aspherical complex, so that properties (2),

(4) and (5) either hold directly by construction, or by computation of a relevant obstruction. Since this obstruction will vanish on passing to a finite cover one also obtains the finite index subgroup  $\pi$  as in (3).

We will give two different constructions of such  $\Pi$ . While they differ in some details, they both are of the following form. We will have two aspherical manifolds with boundary  $W_1$  and  $W_2$ , both boundaries being tori and so that the fundamental group of the boundary injects. (Or one manifold with two boundary components.) These manifolds possess  $\mathbb{Z}/p$  actions, but the key feature is that, while the action on  $\partial W_1$  is affine, the action on  $\partial W_2$  is not topologically equivalent to an affine one. However the actions on the boundaries are equivariantly homotopy equivalent. Gluing  $W_1$  and  $W_2$  together by a homeomorphism homotopic to the equivariant homotopy equivalence gives a closed manifold  $V$  with a homotopy action of  $\mathbb{Z}/p$  on it, and gluing them together by the equivariant homotopy equivalence gives the homotopy equivalent complex  $X$  with a genuine  $\mathbb{Z}/p$  action. Since the geometric actions on  $W_1$  and  $W_2$  are not conjugate, it would seem unlikely that there would be a corresponding action on the manifold  $V = W_1 \cup_{\partial} W_2$ , and showing that will be one of our tasks. Our debt to [19] and [20] for inspiration should be apparent.

Actions on tori with the properties asserted are counterexamples to the “equivariant Borel conjecture”. By now, many of these are known, [9], [10], [33], [34], [29]. We shall use two examples: one based on surgery theory (Cappell’s UNils) and another based on embedding theory. The exotic aspherical manifolds are built by Gromov’s hyperbolization, [11], [12].

**2.1. Surgery theory technique.** We will construct our example to satisfy condition (3) a) of Theorem 1.5, so that  $\pi$  acts on Euclidean space, but is not centerless. It will still have vanishing Raymond–Scott invariant. After the proofs of Theorem 1.5, we will describe how to make changes to handle condition (3) b).

Consider  $\mathbb{Z}/2$  acting on the torus

$$T = (S^1)^{4n} \times S^1$$

by complex conjugation on the first  $4n$  factors and trivially on the last. The orbifold fundamental group of  $T/(\mathbb{Z}/2)$  (i.e. the group of lifts of the action of  $\mathbb{Z}/2$  on the universal cover is

$$\Gamma = (\mathbb{Z}^{4n} \rtimes \mathbb{Z}/2) \times \mathbb{Z}.$$

Let  $a \in L_2(\mathbb{Z}/2 * \mathbb{Z}/2)$  be one of Cappell’s UNil elements, that do not lie in the image of

$$L_2(\mathbb{Z}/2) \oplus L_2(\mathbb{Z}/2).$$

[5], [6], [7]. Note that  $\mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$ .  $\Gamma$  retracts onto  $(\mathbb{Z} \rtimes \mathbb{Z}/2) \times \mathbb{Z}$  and so this class gives rise to a non-zero class  $\alpha \in L_2(\Gamma)$ . So far we have  $(T, \mathbb{Z}/2)$  with fixed

set  $F$  a disjoint union of circles. Let  $K$  be the complement of a tubular neighborhood  $\text{Nbd}(F)$ . Then  $\pi_1(K/\mathbb{Z}/2) \cong \Gamma$ . By Wall realization there is a structure

$$w(\alpha) \in \mathcal{S}(K/\mathbb{Z}/2 \text{ rel } \partial) \cong \mathcal{S}^{\mathbb{Z}/2}(K \text{ rel } \partial).$$

Now set

$$T' = \text{Nbd}(F) \cup w(\alpha).$$

We have thus obtained a new involution on the torus. Moreover  $T$  and  $T'$  are built equivariantly normally cobordant, call this normal cobordism  $W$ . It is not hard to see that the action is not topologically conjugate to the original affine action, although it is equivariantly homotopically equivalent to it. ([9], [10], [33]). This can be detected by an element of the isovariant (that is stratified) structure set in the sense of [33].

Now according to [12], we can relatively equivariantly hyperbolize this normal cobordism  $W$  relative to  $T \cup T'$  to get  $W_h$ , and furthermore, the fundamental groups of the boundaries still inject into the hyperbolization. The fixed sets on the boundaries are circles and so the fixed sets in the cobordism is a surface (of high genus). Now we glue the boundary components  $T$  and  $T'$  as described above to get a manifold  $V$  and a complex  $X$ .  $X$  is a  $\mathbb{Z}/2$ -isovariant aspherical Poincaré complex and  $V$  is a manifold with a  $\mathbb{Z}/2$ -homotopy action. Let

$$\Pi = \pi_1^{\text{orb}}(X)$$

be the orbifold fundamental group of  $X$ .

Now, elements of  $\text{UNil}$  die on passage to suitable finite covers. This follows from [?], Corollary 1. Indeed that corollary directly asserts the topological equivalence of suitable covers of homotopy affine  $G$ -tori, which is what we are asserting here. Thus our element  $\alpha$  dies when lifted to some finite cover of  $T$ . So over  $X$  or  $V$ , the corresponding cover  $\widehat{X}$  or  $\widehat{V}$  has an honest manifold structure with an honest  $\mathbb{Z}/2$ -action. Set

$$\pi = \pi_1^{\text{orb}}(\widehat{V}).$$

Then we get

$$1 \rightarrow \pi \rightarrow \Pi \rightarrow G \rightarrow 1$$

where  $G$  is the group of the finite cover.  $\pi$  is centerless since it is an HNN extension, where the big group comes from hyperbolization.

We now verify the properties (1)–(5) of Theorem 1.5.

(1) The conjugacy classes of finite order in  $\Pi$  correspond to fixed sets in  $X$  and thus occur already in  $\pi$ .

(2)  $\pi$  is virtually torsion free since  $\pi \rightarrow \mathbb{Z}/2$  has torsion free kernel  $\pi_1(\widehat{X})$  (and  $X$  is an aspherical finite complex).

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(3) We know that  $\tilde{X}$  and  $\tilde{V}$  are contractible. Moreover, so are all of their fixed sets. Up until now, the group  $\pi$  is centerless and therefore the Raymond–Scott obstruction vanishes. In order to ensure that the action is on Euclidean space, we can cross  $X$  and  $V$  with  $S^1$  (and change  $\pi$  to  $\pi \times \mathbb{Z}$  and  $\Pi$  to  $\Pi \times \mathbb{Z}$ ). This ensures that these universal covers are simply connected at infinity and are thus homeomorphic to Euclidean space, [30], Corollary 2. The Raymond–Scott obstruction still vanishes.

**Remark 2.1.** If we do not cross with  $S^1$ , we get a centerless fundamental group at the expense of the being on a contractible manifold, not necessarily Euclidean space.

(4) We show that  $\Pi$  can not act on  $\tilde{V}$ , as in the statement of the theorem, with contractible fixed point sets. If it did, then  $\tilde{V}$  is equivariantly homotopy equivalent to  $\tilde{X}$ , since  $\tilde{V}$  is a model for  $E\Pi$ , the classifying space for proper actions and such are unique up to equivariant homotopy equivalence, [23] and [24]. Thus  $V$  and its  $\mathbb{Z}/2$ -action is equivariantly homotopy equivalent to  $X$  with its action. Note that whenever a finite group acts on a manifold with manifold fixed sets, then it also admits such an action with homeomorphic fixed set which is locally flatly embedded. For a proof of taming theory which generalizes verbatim to the equivariant situation, see [14]. Now we can apply a theorem of Browder, [34], recently proved in print by Schultz, [28], which says that under suitable gap and tameness hypotheses, that isovariant and equivariant homotopy equivalence are the same. So we conclude that our tamed  $\Pi$ -space  $V$  would be isovariantly homotopy equivalent to  $X$ .

Hence it suffices to show that  $X$  is not isovariantly homotopy equivalent to a  $\mathbb{Z}/2$ -manifold. Further it therefore suffices to show that  $Y = (X - (X^{\mathbb{Z}/2})) / (\mathbb{Z}/2)$  does not have the proper homotopy type of a manifold. We thus calculate the proper total surgery obstruction of  $Y$ . Since the fundamental group at infinity of the proper Poincaré complex  $Y$  has vanishing  $K_0$  invariant, the proper surgery obstruction is equivalent to the more familiar version of the total surgery obstruction of a compact Poincaré pair. We have the following diagram:

$$\begin{array}{ccc}
 W_h & \xrightarrow{\phi} & W \\
 & \searrow_{\tilde{\phi}} & \downarrow \psi \\
 & & T \times I.
 \end{array}
 \tag{2.1}$$

All three maps are degree one normal maps. By [11],  $W_h$  is normally cobordant to  $W$  and hence  $\phi$  has zero surgery obstruction.  $\psi$  on the other hand has surgery obstruction the original element  $a \in L_2(\Gamma)$ .

Now set

$$W_b = W_h \cup_T \sqcup_{T'} (-W)$$

gluing the boundaries together as before. But this time we get a manifold. The surgery obstruction of  $W_b \rightarrow X$  is still the original  $a$ . This obstruction is an element

of  $L_2(\Gamma, \pi_1^\infty(Y))$  where of course  $\pi_1^\infty(Y)$  is a groupoid and not a group since  $Y$  is not connected at infinity. This maps to

$$L_2(\mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2), \coprod \mathbb{Z} \times \mathbb{Z}/2\text{'s}).$$

We can analyze this by looking at the exact sequence of a pair

$$\begin{aligned} \cdots \rightarrow L_n(\coprod \mathbb{Z} \times \mathbb{Z}/2\text{'s}) &\rightarrow L_n(\mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2)) \\ &\rightarrow L_n(\mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2), \coprod \mathbb{Z} \times \mathbb{Z}/2\text{'s}) \cdots \end{aligned}$$

According to Shaneson for any  $G$  (ignoring decorations which we can do since  $\mathbb{Z}/2 * \mathbb{Z}/2$  has vanishing  $K$ -theory)

$$L_n(\mathbb{Z} \times G) \cong L_n(G) \times L_{n-1}(G)$$

and according to Cappell for any  $G$  and  $H$

$$\tilde{L}(G * H) = \tilde{L}(G) \times \tilde{L}(H) \times \text{UNil}(e; G, H).$$

Hence the original element of  $\text{UNil}$  survives inclusion into the relative group. Therefore the surgery obstruction of this normal map is non-zero.

Of course for any other degree one normal map the same reasoning shows that the difference between its surgery obstruction and the one above lies in the image of the assembly map for

$$\begin{aligned} H_*(B(\mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2)), \coprod B(\mathbb{Z} \times \mathbb{Z}/2); \mathbb{L}(e)) \\ \rightarrow L_*(\mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2), \coprod \mathbb{Z} \times \mathbb{Z}/2\text{'s}). \end{aligned}$$

But now, as noted above, the image of this latter group in  $\text{UNil}$  is trivial, so we are done.  $\square$

**Remark 2.2.** Connolly–Davis ([8]) completed the computation of  $L_n(\mathbb{Z}/2 * \mathbb{Z}/2, \omega)$  for all  $n$  and all orientation characters  $\omega$ . As a result, one can modify the above construction using orientation reversing involutions on tori with isolated fixed sets, to produce different examples.

*Proof of Theorem 1.6.* We begin with the aspherical manifold  $V$  constructed above. In this case set  $\pi = \pi_1 V$ . If we use the example, satisfying condition (3) a) of Theorem 1.5 then we argue as follows.  $V$  also has its  $\mathbb{Z}/2$ -homotopy action and therefore acts on  $\pi$  and  $\Pi$  is the semi-direct product. We now argue that the  $\pi$ -action does not extend to  $\Pi$ . This is simply a matter of showing that any action of  $\Pi$  on  $V$  automatically has contractible manifold fixed sets so that we can appeal to the proof of Theorem 1.5.



Now, by Smith theory, the fixed set is a  $\mathbb{Z}/2$ -homology manifold homology equivalent mod 2 to  $\mathbb{R}^2$  (by comparison with the Poincaré model  $X$ .) By [2], Theorem 16.32, page 388, for any  $p$ , any second countable  $\mathbb{Z}/p$ -homology manifold of dimension less than or equal to two is a topological manifold. Thus, the fixed set is a 2-manifold which the classification of surfaces implies that any mod 2 acyclic surface is  $\mathbb{R}^2$ .

If instead, we want a centerless example and thus assume condition (3) b) holds, then given the calculations of Connolly and Davis, the proof that these examples work is even more elementary with regard to the verification of manifoldness of putative fixed sets: the characterization of the circle is much more straightforward.  $\square$

**2.2. Embedding theory technique.** We now give a construction, based on embedding theory, that suffices for an alternate proof of Theorem 1.5, which gives examples for  $\mathbb{Z}/p$  for  $p$  odd. These are insufficient for the Nielsen problem since the fixed sets will be of higher dimension and so we have no way of seeing that they are automatically manifolds, as in the proof of Theorem 1.6.

Let  $W_1 = T^n \times S^\circ$  where  $S^\circ$  is a punctured surface and  $n = 2p - 4$ . Now  $\mathbb{Z}/p$  acts on  $W_1$  by permuting the first  $p$  circles of  $T^n$  leaving the other factors fixed. Let  $F = W_1^{\mathbb{Z}/p}$ . Then  $\dim(W_1) = 2p - 2$  and  $\dim(F) = p - 1$ .

We now build a second manifold  $W_2$  with a group action by first producing a new embedding of the fixed set in the boundary torus  $T^n$  using the following general construction, called a finger move, [29]: Let  $M^k \subset N^{2k+1}$  be an embedding of

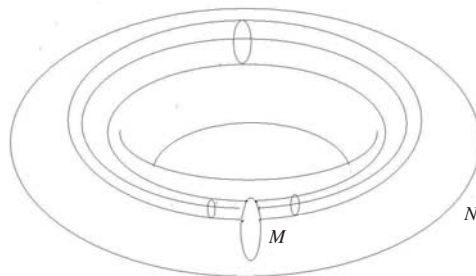


Figure 1. A submanifold  $M$  of  $N$  together with a curve, as data for a finger move.

manifolds. Let  $[\gamma] \in \pi_1(N)$  be a class represented by a path  $\gamma$  which intersects  $M$  only in its two distinct endpoints, which are assumed to lie in a little ball. Let  $R$  be a regular neighborhood of  $\gamma$ , a  $2k + 1$ -disk. Then  $R \cap M = D^k \cup D^k$ . Move one of the disks  $D^k$  along  $\gamma$  to have rel  $\partial$  linking number one with the other disk. Remove one disk of intersection and glue in the other one. We thus arrive at a new manifold pair  $(\text{Fing}(N, M, \gamma), M)$  where  $\text{Fing}(N, M, \gamma)$  is homeomorphic to  $N$  and  $M$  is embedded differently. We can perform the same construction relative to any

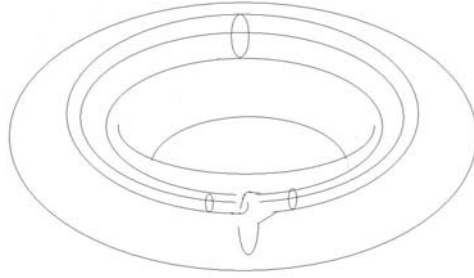


Figure 2. The result after the finger move.

finite collection of disjoint curves  $\gamma_1, \dots, \gamma_k$ .

Back to our manifold  $W_1$  with its  $\mathbb{Z}/p$ -action. Let  $\gamma$  be a curve in  $\partial W_1 = T$  a torus. We may arrange this curve so that it and all its translates  $\gamma, g\gamma, \dots, g^{p-1}\gamma$  are disjoint. Now perform the finger move  $W_2 = \text{Fing}(\partial W_1, \partial W_1 \cap F, \gamma, g\gamma, \dots, g^{p-1}\gamma)$ . We get a new embedding  $F' \subset W_2$  and moreover  $gF'$  is isotopic to  $F'$ .

By the main theorem of [33], at the cost of repeating all of these finger moves some number  $p^k$  of times, we can find an equivariantly homotopy equivalent group action on  $T$  with fixed point set  $F'$ . This action, while a priori only continuous, can be made PL locally linear (even smooth) and equivariantly cobordant to the original action on  $T$ . This is because equivariant smoothing theory [22] and cobordism theory reduces such problems to the tangent bundle, but [17] (see [16] shows that equivariantly homotopy equivalent  $G$ -tori have topologically equivalent tangent bundles. PL ??

Now we can do our relative hyperbolizations and equivariant glueing as before to obtain a  $\mathbb{Z}/p$ -CW complex unequivariantly homotopy equivalent to an aspherical manifold  $W$ . We claim that this  $\mathbb{Z}/p$ -CW complex is not equivariantly homotopy equivalent to a manifold. The reason is simple: the inclusion of the fixed set  $F$  in the  $\mathbb{Z}/p$ -CW complex homotopy equivalent to  $W$  is not homotopic to an embedding in  $W$ . To check this, we consider the self intersections of any immersion homotopic to this inclusion. Note that we are in a non-simply connected situation, so it is appropriate to use the  $\mathbb{Z}[\pi]$ -intersection numbers as in [32]; however, since the subobject  $F$  is non-simply connected, they are not as well defined as in Wall's situation, as explained in [29]. The indeterminacy replaces the  $\mathbb{Z}[\pi]$  by  $\mathbb{Z}[\pi' \backslash \pi / \pi']$  (double cosets) where  $\pi'$  is the fundamental group of  $F$ , because one can change the path from basepoint to intersection point either on the way there or on the way back.

Since we are in the middle dimension, there is a  $\mathbb{Z}$ 's worth of ambiguity, which is reflected in the coefficient of the trivial double coset  $\pi' e \pi' = \pi'$ , so we ignore this coefficient. Of course, the finger move construction gives us a nontrivial element of  $\mathbb{Z}[\pi_1(T)/\pi_1(F \cap T)]$ : this is the usual relation between linking numbers of chains in a boundary and the intersection number of bounding cycles. We only need to see that

nothing is lost on passing to the larger group. Here we have a trick available because  $\pi_1(F \cap T)$  is normal in  $\pi_1(T)$ : the double cosets of  $\pi_1(F)$  in  $\pi_1(F \cup T)$  (=the group!)  $\pi_1(T)/\pi_1(F \cap T)$ . Now, general nonsense about amalgamated free products tells us that  $\pi_1(F \cup T)$  injects into  $\pi_1(W)$ , so we lose no information at this stage of our formation of intersection numbers.

Thus,  $F$  does not embed in  $W$ , and therefore neither does any manifold homotopy equivalent to  $F$  in any manifold homotopy equivalent to  $W$  (see e.g. Wall, [32], chapter 11 on embeddings). A fortiori, the group action does not exist and our proof is complete.  $\square$

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[6] is identical with [5] (there seems to be no II!)

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Now, by Smith theory, the fixed set is a  $\mathbb{Z}/2$ -homology manifold homology equivalent mod 2 to  $\mathbb{R}^2$  (by comparison with the Poincaré model  $X$ .) By [2], Theorem 16.32, page 388, for any  $p$ , any second countable  $\mathbb{Z}/p$ -homology manifold of dimension less than or equal to two is a topological manifold. Thus, the fixed set is a 2-manifold which the classification of surfaces implies that any mod 2 acyclic surface is  $\mathbb{R}^2$ .

If instead, we want a centerless example and thus assume condition (3) b) holds, then given the calculations of Connolly and Davis, the proof that these examples work is even more elementary with regard to the verification of manifoldness of putative fixed sets: the characterization of the circle is much more straightforward.  $\square$

**2.3. Embedding theory technique.** We now give a construction, based on embedding theory, that suffices for an alternate proof of Theorem 1.5, which gives examples for  $\mathbb{Z}/p$  for  $p$  odd. These are insufficient for the Nielsen problem since the fixed sets will be of higher dimension and so we have no way of seeing that they are automatically manifolds, as in the proof of Theorem 1.6.

Let  $W_1 = T^n \times S^\circ$  where  $S^\circ$  is a punctured surface and  $n = 2p - 4$ . Now  $\mathbb{Z}/p$  acts on  $W_1$  by permuting the first  $p$  circles of  $T^n$  leaving the other factors fixed. Let  $F = W_1^{\mathbb{Z}/p}$ . Then  $\dim(W_1) = 2p - 2$  and  $\dim(F) = p - 1$ .

We now build a second manifold  $W_2$  with a group action by first producing a new embedding of the fixed set in the boundary torus  $T^n$  using the following general construction, called a finger move, [29]: Let  $M^k \subset N^{2k+1}$  be an embedding of

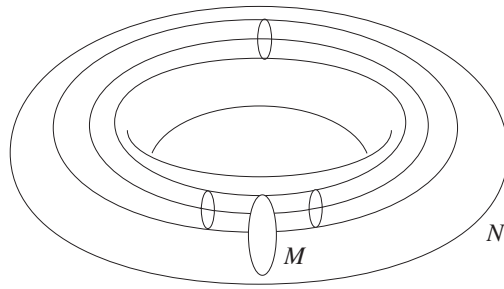


Figure 3. A submanifold  $M$  of  $N$  together with a curve, as data for a finger move.

manifolds. Let  $[\gamma] \in \pi_1(N)$  be a class represented by a path  $\gamma$  which intersects  $M$  only in its two distinct endpoints, which are assumed to lie in a little ball. Let  $R$  be a regular neighborhood of  $\gamma$ , a  $2k + 1$ -disk. Then  $R \cap M = D^k \cup D^k$ . Move one of the disks  $D^k$  along  $\gamma$  to have rel  $\partial$  linking number one with the other disk. Remove one disk of intersection and glue in the other one. We thus arrive at a new manifold pair  $(\text{Fing}(N, M, \gamma), M)$  where  $\text{Fing}(N, M, \gamma)$  is homeomorphic to  $N$  and

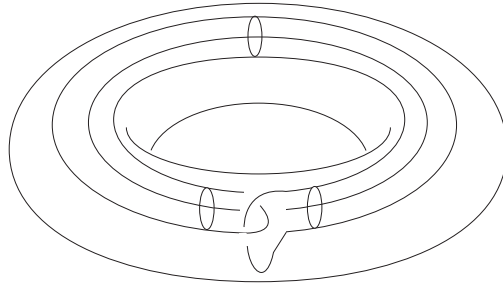


Figure 4. The result after the finger move.

$M$  is embedded differently. We can perform the same construction relative to any finite collection of disjoint curves  $\gamma_1, \dots, \gamma_k$ .

Back to our manifold  $W_1$  with its  $\mathbb{Z}/p$ -action. Let  $\gamma$  be a curve in  $\partial W_1 = T$  a torus. We may arrange this curve so that it and all its translates  $\gamma, g\gamma, \dots, g^{p-1}\gamma$  are disjoint. Now perform the finger move  $W_2 = \text{Fing}(\partial W_1, \partial W_1 \cap F, \gamma, g\gamma, \dots, g^{p-1}\gamma)$ . We get a new embedding  $F' \subset W_2$  and moreover  $gF'$  is isotopic to  $F'$ .

PL ??

By the main theorem of [33], at the cost of repeating all of these finger moves some number  $p^k$  of times, we can find an equivariantly homotopy equivalent group action on  $T$  with fixed point set  $F'$ . This action, while a priori only continuous, can be made PL locally linear (even smooth) and equivariantly cobordant to the original action on  $T$ . This is because equivariant smoothing theory [22] and cobordism theory reduces such problems to the tangent bundle, but [17] (see [16] shows that equivariantly homotopy equivalent  $G$ -tori have topologically equivalent tangent bundles.

Now we can do our relative hyperbolizations and equivariant glueing as before to obtain a  $\mathbb{Z}/p$ -CW complex unequivariantly homotopy equivalent to an aspherical manifold  $W$ . We claim that this  $\mathbb{Z}/p$ -CW complex is not equivariantly homotopy equivalent to a manifold. The reason is simple: the inclusion of the fixed set  $F$  in the  $\mathbb{Z}/p$ -CW complex homotopy equivalent to  $W$  is not homotopic to an embedding in  $W$ . To check this, we consider the self intersections of any immersion homotopic to this inclusion. Note that we are in a non-simply connected situation, so it is appropriate to use the  $\mathbb{Z}[\pi]$ -intersection numbers as in [32]; however, since the subobject  $F$  is non-simply connected, they are not as well defined as in Wall's situation, as explained in [29]. The indeterminacy replaces the  $\mathbb{Z}[\pi]$  by  $\mathbb{Z}[\pi' \backslash \pi / \pi']$  (double cosets) where  $\pi'$  is the fundamental group of  $F$ , because one can change the path from basepoint to intersection point either on the way there or on the way back.

Since we are in the middle dimension, there is a  $\mathbb{Z}$ 's worth of ambiguity, which is reflected in the coefficient of the trivial double coset  $\pi' e \pi' = \pi'$ , so we ignore this coefficient. Of course, the finger move construction gives us a nontrivial element of

$Z[\pi_1(T)/\pi_1(F \cap T)]$ : this is the usual relation between linking numbers of chains in a boundary and the intersection number of bounding cycles. We only need to see that nothing is lost on passing to the larger group. Here we have a trick available because  $\pi_1(F \cap T)$  is normal in  $\pi_1(T)$ : the double cosets of  $\pi_1(F)$  in  $\pi_1(F \cup T)$  = (the group!)  $\pi_1(T)/\pi_1(F \cap T)$ . Now, general nonsense about amalgamated free products tells us that  $\pi_1(F \cup T)$  injects into  $\pi_1(W)$ , so we lose no information at this stage of our formation of intersection numbers.

Thus,  $F$  does not embed in  $W$ , and therefore neither does any manifold homotopy equivalent to  $F$  in any manifold homotopy equivalent to  $W$  (see e.g. Wall, [32], chapter 11 on embeddings). A fortiori, the group action does not exist and our proof is complete.  $\square$

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