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LARGE SCALE HOMOLOGY THEORIES AND GEOMETRY

JONATHAN BLOCK AND SHMUEL WEINBERGER

In recent years there has been an explosion of interest in the application of "large scale" or "coarse" methods in geometry, topology, and group theory. The modest aim of this paper is to describe some "homological" methods that can be (and have been) applied to questions of this ilk, and to explain some of the geometry involved in these theories.

There are two seperate strands that oft intertwine in this subject. The first type of question is itself coarse: we want to understand how someone with blurry vision would view a discrete metric space. Thus, bounded metric spaces are entirely indistinguishable from points.

The second type of question is almost complementary in its force: What kinds of restrictions on conventional geometric questions are forced by large scale considerations. In other words, one studies smooth questions on manifolds while insisting that the large scale structure of the space, or a map, remain intact.

Our choice of material has been dictated by a few considerations. Firstly, we tried to avoid topics that seem to be very well known (excluding therefore, bounded cohomology and L^2 cohomology) or recently surveyed (thus excluding the theories of hyperbolic and automatic groups, or, much more generally, the contents of [58]). Secondly, we have tried to cover topics that have simple to understand geometric meanings or that can be used to answer questions posed by friends. And, finally, we have tried not to stray too far from things we actually understand.

The contents are as follows:

1. Definitions and examples.
2. HX_* and R_*X.
3. Metric rigidity phenomena I (packing).
4. Metric rigidity phenomena II (topological rigidity).
5. Connection to large scale index theory.

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1. Definitions and Examples

Let us start with some common definitions:

Definition. A *Lipschitz* map between metric spaces is a map $\phi : Y \longrightarrow Z$ such that $d(\phi x, \phi x') \leq Cd(x, x')$, for some constant C. A *coarse-Lipschitz* map satisfies $d(\phi x, \phi x') \leq Cd(x, x') + D$, for some C, D. Note that coarse Lipschitz maps need not be continuous. Note, too, that for *effectively* discrete metric spaces (e.g. metric spaces with a lower bound on the distance between distinct points) Lipschitz and coarse Lipschitz are the same. We use the term *bi-Lipschitz* to denote a coarse Lipschitz map with a coarse Lipschitz inverse.

Definition. A *quasilattice* is an effectively discrete subset of Z, which is *C-dense*[1], that is which is of some bounded distance of any point of Z. Thus $3n + (-)^{g(n)}$ is a quasilattice in $\mathbb{R}$ for any integer function g (e.g. the n-th base 2 digit of π).

Definition. Two metric spaces are *coarsely equivalent* (or *coarsely quasiisometric*) if they contain quasilattices which are homeomorphic by bi-Lipschitz homeomorphisms. There is a similar notion of *quasi-isometric embedding*.

The by now standard example of these notions is the relationship between a finitely generated group with its word metric and the as-

[1] Also called *syndectic* in the literature. Quasilattices are also called nets.

sociated cover of a compact manifold[2] (or polyhedron). This famous observation of Milnor and Svarc implies (as they noted) that, say, fundamental groups of negatively curved manifolds must have exponential growth.

The basic question to be asked under these circumstances is how can we distinguish coarse equivalence classes of spaces?

There are a number of subsidiaries to this question. Is there a "nicest" space associated to the quasiisometry type which captures the type, or at least which can be used effectively in constructing invariants? We have introduced the quasilattices, which of course, have the simplest structure in some sense in that they are discrete! Can one find other "best" spaces by focusing on other features?

Perhaps the most concrete way to think about this question is to ponder some individual metric spaces.

Example 1. *Finitely generated groups and coset spaces.*

Let π be a finitely generated group. Given a finite generating set, one can construct its *Cayley graph* which has a vertices the group elements, and as edges pairs of the form $(g, \Gamma g)$ where Γ or Γ^{-1} lies in the generating set. The associated distance between group elements is called the *word distance* or *length* or *metric.* If Δ is a subgroup of π then the coset space π/Δ also has a natural word metric. The study of groups (and to a lesser extent, coset spaces) via the coarse properties of the Cayley graph is the focus of the popular subject called "geometry of groups". The great theorems that started this subject and remain major sources of inspiration are probably Mostow's rigidity theorem (see [84, 85]) and Gromov's theorem on groups of polynomial growth [59], which we will state below.

Remark. It is natural to consider infinitely generated situtations like infinitely generated subgroups of finitely generated groups. One can build a Cayley graph from an infinite set of generators with weights on the generating set so that the resulting metric has a finitely many vertices in any ball. However, the quasiisometry class depends on the choice of generators and weights.

Definition. A metric space has *coarse bounded geometry* if it posesses a quasilattice with the property that the number of elements in arbitrary balls of a given radius is uniformly bounded. Such a quasilattice will be called a *bounded quasilattice.*

[2]This naturally generalizes to groups acting properly discontinuously and isometrically, provided there is a bound on the size of isotropy.

Groups and coset spaces are good examples of spaces of bounded geometry. A (simplicial) tree with vertices with growing numbers of neighbors is the prototypical example of a space with unbounded geometry.

Definition. The growth rate of a metric space is the "growth rate" of the number of points in a fixed quasilattice within distance R of a fixed base point. (Two nondecreasing functions f and $g : \mathbb{N} \longrightarrow \mathbb{R}$ have the same growth rate if there are constants such that $Af(Br) - C < g(r) < Df(Er) + F$.)

It is not hard to see that the growth rate of a metric space just depends on the coarse quasiisometry type, and that (for metric spaces of bounded geometry) it is at most exponential. In particular, the notion of polynomial growth makes sense.

Theorem. (Gromov) *A finitely generated group is virtually* [3] *nilpotent iff it has polynomial growth.*

Of course, one cannot do better than a virtual result because *commensurable* groups (i.e., virtually isomorphic groups) are quasiisometric. A central problem is to determine to what extent the converse holds, i.e., how far quasiisometry is from virtual isomorphism (and quasiisometries are from virtual isomorphisms). This is the core of one of the approaches to rigidity, but the "answer" is that there are many examples of noncommensurable quasiisomteric groups. For instance, Malcev's theorem on nilpotent groups together with Remark 2.15 in Raghunathan [91] gives examples of noncommensurable quasiisometric nilpotent groups. More explicitly, let Sol be the irreducible three dimensional solvable Lie group given as the semidirect product of $\mathbb{R}$ on $\mathbb{R}^2$ by the action $t \cdot (x, y) = (e^t x, e^{-t} y)$. Then any uniform lattice Γ intersects the normal subgroup in a free abelian subgroup of rank 2 and projects to a discrete subgroup of $\mathbb{R}$. Commensurable groups will have rationally dependent images and conversely. For any matrix A with $tr(A) > 2$ in $SL_2(\mathbb{Z})$ the semidirect product of $\mathbb{Z}$ acting by A on $\mathbb{Z}^2$ gives a lattice in Sol whose image in $\mathbb{R}$ is generated additively by

$$\log(\frac{tr(A) + \sqrt{tr(A)^2 - 4}}{2})$$

[3]For a property P, recall that a group is vitrually-P if it contains a finite index subgroup which is P. This is most convenient for properties P which are passed on from a group to its finite index subgroups.

It is an easy exercise that by varying A one obtains an infinite set of rationally independent images and therefore noncommensurable groups.[4]

Gersten [54], using a bounded cohomology trick, gave the first examples of uniform lattices in essentially different Lie which are quasi-isomteric. A "simple" question at the heart of rigidity is whether every group quasiisometric to a Lie group is a virtual uniform lattice. In the semi simple rank greater than one case, at least, this is a recent theorem of Kleiner and Lieb. For rank one nonuniform lattices Richard Schwartz has even shown that quasiisometry implies virtual isomorphism. (See also [48] for the Hilbert modular group.)

Other examples of noncommensurable groups can be obtained by using different lattices in the same Lie group and using invariants like volume.

Bass has given a nice formula for the growth of a virtually nilpotent group. (For a proof, see the appendix, by Tits, of [59].) If the dimensions of the successive free abelian quotients are denote by a_i, then growth $= \sum ia_i$. In a beautiful paper, Pansu [86] has shown that the individual a_i's are actually coarse invariants as well. For instance, we have the following consequence that is indeed much more elementary.[5]

Corollary. *"Virtually abelian" is a coarse notion.*

Corollary. *The length and cohomological dimension of nilpotent groups are coarse invariants.*

Problem. *When are virtually nilpotent groups quasiisometric?*

Problem. *Is virtual solvability coarse?*

Another important coarse algebraic notion is amenabilty which we will discuss in considerable detail in the Part II. We will also show that for a large class of groups virtual cohomological dimension[6] and the

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[4]For subgroups of a given group, there are two meanings to commensurable. One means that literally that they have a common finite index subgroup, while the other notion (the one we are using) means that they are virtually isomorphic. This difference is significant in the case of *Sol* because lattices have deformations, detected by covolume change.

[5]By the end of section 2 the reader should also be able to prove this.

[6]The corollary on being virtually abelian follows from coarseness of cohomological dimmension as follows: Virtually abelian groups are those with polynomial growth of degree equal to their virtual cohomological dimension. By Bass's theorem, if the nilpotent group is nonabelian, its growth rate is larger than this.

property of being a virtual duality group are coarse. (This had been independently observed by Gersten [55] as well.)

Now we introduce a class of spaces whose proper homotopy type reflect features of its quasiisometry type.

Definition. A space A is uniformly contractible[7] if there is a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that any ball of radius r, $B_x(r)$ is contractible in the ball of radius $\phi(r)$ $B_x(\phi(r))$. Note, ϕ need not be continuous or have $\phi(0) = 0$ or anything like that. We refer to ϕ as a contractibility function for A.

Proposition. *Finite dimensional coarsely equivalent uniformly contractible polyhedra are proper homotopy equivalent.*

This follows easily from the following proposition, which shows that if one thinks of the quasiisomtery type as being like the fundamental group, uniformly contractible spaces will then play the roles of $K(\pi, 1)$'s.

Proposition. (Connect the Dots) *If $f : Z \longrightarrow A$ is a coarse Lipschitz map from a finite dimensional polyhedron to uniformly contractible space, then f is of bounded distance[8] from a continuous coarse Lipschitz map $f' : Z \longrightarrow A$.*

The proof is by induction on the skeleta of Z, extending maps over simplices using the contractibility function.

Proposition. *The universal cover on any compact aspherical space is uniformly contractible.*

For many groups one knows compact $K(\pi, 1)$'s and so the above proposition gives very convenient models for the group as a metric space. The most basic examples of this to keep in mind are uniform lattices π of real connected Lie groups G, which act (freely, when π is torsion free) on the symmetric space G/K, K a maximal compact subgroup of G; G/K is uniformly contractible so it is coarsely quasiisometric to π. In the S-arithmetic case, one can use products of Tits buildings to replace the symmetric space. (See [23].) We will often call the universal

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[7]There is a similar notion of uniformly k-connected. It is also of some interest to examine the k-contracitiblity functions.

[8]The distance depends only on the dimension of Z, the coarse Lipschitz constants, the density of the quasilattice of Z that f is defined on, and the contractibility function of A.

cover of a compact aspherical space a *coarse-homogenous space* for the fundamental group[9].

The cohomological dimension of $\pi = \dim G/K$ which is of course a proper homotopy invariant (determined as the maximum dimension for which the cohomology with compact supports is nonzero.) In the next section we will make this reasoning more directly homological. In Part II we will also discuss how to build homology theories that take volume-type considerations and the like into account.

Remark. There are more delicate geometric properties of the "coarse homogenous" space implicated in the above discussion, if one looks at it more carefully. For instance, one can easily check that the growth types of contractibility functions are coarse. The 1-contractibility function has been called the *isodiametric function* by Gersten [56]; it is recursive iff the group has solvable word problem.

Example 2. *Cones on manifolds and polyhedra.*

A rather different class of examples comes about by taking cones on metric spaces. If (X, d) is a (typically compact) metric space, we can form its *(Euclidean) cone.* This is done (for finite dimensional X) by viewing X as isometrically embedied in the unit sphere of some Euclidean space (with say a box metric) and the looking at the open cone as a subspace of Euclidean space.

Other types of cones can be formed by replacing Euclidean space by other spaces. Alternatively, one can think of a process where one weights the levels of the cone by a function ϕ, and then forms the metric given by paths of shortest lengths (see [43] for some information about this construction, e.g. when it produces a metric!). This will be called the *ϕ-cone.* Using rapidly growing ϕ, one gets many examples of hyperbolic metric spaces in Gromov's sense [61].

Let us start with Euclidean coning. The basic question is to what extent one can recover (X, d) from the Euclidean cone?

Example. [10] *Consider $X = \{0\} \cup \{1/n^\alpha | n \in \mathbb{N}\}$. The growth rate of cX is $r^{(2\alpha+1)/(\alpha+1)}$ so that for varying α these metric spaces are not coarsely equivalent. As a consequence, the topological type of X does not determine the Lipschitz type of the cone.*

[9] The reader should realize immediately always that coarse-homogenous spaces are not at all spaces coarsely equivalent to homogenous spaces.

[10] This is a variant of an example shown to us by Greg Kuperberg which simplified our more complicated examples based on metrics of varying Hausdorff dimension on the circle.

In light of this example we will restrict attention to situations where X has some preferred class of metrics, e.g. X a polyhedron, or smooth compact manifold (all Riemannian metrics are biLipschitz) or even a topological manifold (by Sullivan's theorem [100][11].)

Proposition. *If M and N are smooth (or topological) closed manifolds (with Sullivan's metric, that is any metric compatible with Sullivan's Lipschitz structure, see* [101]*) of dimension at least four then cM and cN are coarsely equivalent iff the M and N are topologically h-cobordant.*

This is probably true in all dimensions, but we have not thought through the low dimensional case in enough detail to see how hard it would be to prove this. Using the ideas of [98] and [105] one can also extend this to "Lipschitz stratified spaces" with the suitable notion of stratified s-cobordism. For an application of this proposition, see [106]

Proof. We can restrict attention to quasiisometric quasilattices on which the map is genuinely bi-Lipschitz. Now rescale the metric spaces and maps by a factor of $1/n$. This gives a sequence of maps on denser and denser subsets which are *uniformly* bi-Lipschitz. The Arzela-Ascoli theorem (as extended to Gromov-Hausdorff space) gives a convergent subsequence of "bi-Lipeomorphisms" on the whole cone. Thus cM and cN are bi-Lipschitz homeomorphic in the non-coarse sense! This immediately implies that M and N are h-cobordant, since M embeds as a deformation retraction in $N \times \mathbb{R}$. □

Conversely, if they are topologically h-cobordant, then $M \times S^1 \cong N \times S^1$ by a bi-Lipschitz homeomorphism (according to Sullivan's uniqueness theorem). The infinite cyclic cover of this homeomorphism with the 0-points added gives a Bi-Lipschitz homeomorphism between these cones, as has been verified in [106].

Thus for Euclidean cones[12] one cannot hope for a well defined "space at infinity"; there is at most a well defined h-cobordism class. However for hyperbolic cones, there is the "Gromov boundary", made up

[11]Sullivan's theorem requires that the dimension of the manifold be other than 4. (See [101].) However, one can use the discussion below to show that homeomorphic 4-dimensional Lipschitz manifolds have biLipschitz cones. Even for 4-manifolds without Lipschitz structures, one can produce the Lipschitz structure on cX that would have come from its Lipschitz structure, had it had one.

[12]It seems that even with slower growing coning functions one still gets the h- cobordism class at infinity by a more complicated argument; if the coning function is bounded then the coarse type is of course always the same as that of a ray.

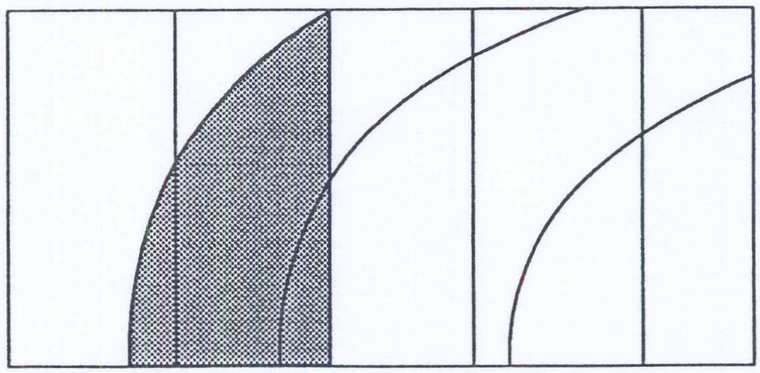

FIGURE 1. These represent two product structures on the same manifold, the shaded region an h-cobordism between them.

by equivalence classes of "coarse quasigeodesic rays" and the space at infinity not only includes the homeomorphism type of X, but also the conformal class of metric! (See [61, 57, 36, 97] for more on this.)

There is a weird type of coning construction which preserves exactly the homotopy type of a given polyhedron at infinity. Essentially one uses an iterated mapping cylinder of regular neighborhoods of embeddings of the polyhedron in higher and higher dimensional Euclidean spaces. This is like replacing the polyhedron with its product with the Hilbert cube and then takes the Euclidean cone. Then one can make use of regular neighborhood characterization of simple homotopy types and the previous arguments to prove the equivalence. (We note that what is produced on the ball of radius n in this coned metric space can be viewed as what would have been constructed from $P \times I^n$, so there is nothing "locally infinite dimensional" in this cone.)

Moral of the example: The conclusion is that there are various natural construcitons from compact metric spaces to coarse metric spaces which preserve more and more structure: from the mere homotopy type to h-cobordism type to homeomorphism and conformal type of the metric! It is not hard to see that there are manifolds (even Riemannian manifolds, see [50]) approximating these spaces. Thus "coarse considerations" can lead one to much more precise or fine problems in the geometry of complicated spaces than geometric topologists had been comfortable in studying in the past!

We note that it seems difficult to figure out what the overlap between the quasiisometry types described by cones and those described by groups (and coset spaces) are.

Example 3. *Leaves of foliations*

The leaves of a foliation of a compact manifold have natural quasiisometry classes of metrics. If Γ is a group of diffeomorphisms of a compact manifold M, then it is possible to get a foliation with many leaves quasiisometric to Γ, via the fiberred product construction $M \times_\Gamma \tilde{N}$, where N is a manifold with a surjection $\pi_1 N \longrightarrow \Gamma$, which produces the cover implicit in the notation $\tilde{N}$.

Problem. *Is every finitely generated group quasiisometric to a leaf?*[13]

There are similar questions that can be asked about coset spaces. (Their quasiisometry types would arise if Δ were the isotopy subgroup of some element for a Γ action on M.)

In any case, it is interesting to try to restrict the coarse quasiisometry type of leaves of foliations on manifolds. In Part II we will explain the method of Phillips-Sullivan-Janskiewicz that sometimes does this. There is also a very nice entropy method of Attie-Hurder that uses coarse/fine invariants of the sort to be discussed in the next section to produce leaves bounded-geometry homotopy equivalent to leaves that are not leaves (of at least smooth foliations).

Nonetheless, there are also many vary complicated leaves of foliations known that are not at all related to the previous examples. (As far as we know, the most detailed information on spaces at infinity of leaves is the realization of compact totally disconnected spaces as the space of ends [27].) It would be very interesting to understand how foliation invariants are related to the coarse geometry of the leaves. One beautiful result is Hurder's [76] that gives the vanishing of the Godbillon-Vey invariant of a foliation all of whose leaves have subexponential growth.

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[13]One can produce certain, fairly unnatural, groups that do not act by diffeomorphisms on any manifold; however, as far as we know, no such torsion free example is known.

2. HX_* AND R_*X

In the previous section, we explored the utility of having a uniformly contractible space within a given coarse quasiisometry type, especially its use in defining coarse invariants. Now we shall introduce invariants without having a uniformly contractible model around.

Examples. *If an increasing sequence $\{a_n\}$ of real numbers is viewed as a metric space, it is coarsely (equivalent to a) uniformly contractible (metric space) iff the successive differences are uniformly bounded.*

One can similarly string together spheres of dimension i and radii given by the $\{a_n\}$, and one obtains a uniformly i-connected space iff the sequence is bounded. If one varies the dimensions of the spheres one can get exmples that are coarsely uniformly k-connected for each k, but not uniformly contractible.

Depending on the metric on X, the open cone will or will not be uniformly contractible. Finite dimensional ANR's have metrics for which the warped open cones are uniformly contractible [43]. We do not know whether the Euclidean cone can always be taken UC.

A finitely generated group is coarsely equivalent to a uniformly k-connected space iff $B\pi$ has a finite $k+1$ skeleton. (Gersten, Gromov, Block-Weinberger, and probably others). Abels [1] has given many examples of groups with varying finiteness conditions, which are relevant here.

The basic method here is a variant of an idea of Rips (and has also been discovered on many occasions by different authors). Rips observed (see [61, 36, 57, 97]) that for X a ∂-hyperbolic metric space, for all sufficiently large k the complex obtained by considering the Cech nerve of the covering of X by k-balls is contractible[14].

However, for many non-word hyperbolic groups or yet more general metric spaces, even those posessing uniformly contractible models, there is no fixed size at which this Cech nerve is contractible. The solution to this problem is to consider the whole sequence of nerves and work with this.

More precisely, let Z be a metric space, which we assume to be (effectivly) discrete. Then one has the locally finite cover of X (a quasilattice

[14] Rips' application was to prove beautiful general facts regarding the cohomology of hyperbolic groups. Bestvina and Mess have given further properties of the compactified Rips complex [18]. These are applied in [52, 30] to give a geometric approach to the Novikov type theorems for hyperbolic groups, see [28, 39] and below.

in Z) by balls of radius n. We call $R_n(Z)$ the nerve of this cover[15]. Since these covers become coarser as we increase n, there are natural (proper) maps $R_n(Z) \longrightarrow R_{n+1}(Z)$, which are, in fact, coarse equivalences. The pro-homotopy type of this system is invariant of the choice of X.

The point of this construction is that topological structure at scale n is removed in $R_n(Z)$. We denote by R_*Z this system[16]. It is "pro-functorial" for proper Lipschitz maps ($f_* : R_n(Z) \longrightarrow R_{Cn+D}(Z)$ for a (C, D)-Lipschitz map) and its "pro-homotopy-type" is a well defined quasi-isometry invariant of the metric space. Note that if V is a simplicial complex with simplices of bounded diameter, and f is Lipschitz, one obtains a continuous map $f : V \longrightarrow R_n(Z)$ for n sufficiently large.

We can therefore define invariants of coarse metric spaces by taking invariants of R_*X.

Definition. $HX_*(Z) \cong H_*(R_*(Z))$ (or by the direct limit, if we are not interested in the details of the pro-system.)

Proposition. (Roe [92]**)** *If Z is uniformly contractible then the natural map $H_*^{lf}(Z) \longrightarrow HX_*(Z)$ is an isomorphism.*

This is not too hard. Now, if E is any homology theory (or homotopy functor!) one can now make:

Definition. $EX_*(Z) \cong E_*(R_*(Z))$.

Furthermore, there is a dual cohomological notion. These were in fact what were introduced in [58, 92]. Roe's interest in this cohomology was that it could be used to pair with elliptic operators on noncompact manifolds to form "exotic indices". We will return to this in a later section.

For us convenient choices of E are K-theory and K-theory with coefficients. It turns out that for nonconnective spectra there is no need for the proposition to hold, and there is different information in the EX-theory than the E-theory of a uniformly contractible model! Remarkably, it seems that EX-theory is actually more suitable for coarse geometric applications than the homology it was invented to model!

The following is the result of tracing definitions:

Proposition. *If $B\Gamma$ is a finite complex*[17]*, then $HX_*(\Gamma) \cong H_*(\Gamma; \mathbb{Z}\Gamma)$.*

[15] Metrized simplicially.

[16] The use of the system rather than taking the limit is suggested by shape theory. In this paper, though, we will only be studying limit type invariants.

[17] Under suitable finiteness condition on Γ, this proposition will be true in the appropriate range.

In particular the latter group is a coarse invariant, in this setting. Since, the cohomological dimension is the last degree in which the r.h.s. is nonzero, it is also coarse (as promised above). Finally, we have the following:

Proposition. *If $B\Gamma$ is a finite complex, then either*

$$(1) HX_*(\Gamma) \text{ is infinitely generated in some dimension}$$

$$\text{or } (2) HX_*(\Gamma) = 0 \text{ for } * \neq n, \text{ and } HX_*(\Gamma) = \mathbb{Z} \text{ for } * = n, \text{ for some } n.$$

In that case $B\Gamma$ is an n-dimensional Poincaré duality group.

Results similar to the proposition have been proven by Bieri-Eckmann, and Farrell (all though not with the coarse terminology); see Brown [23] for some discussion and references to their work.

Corollary. *No uniform lattice is coarsely equivalent to a nonuniform lattice.*

(Because $HX_*(\Gamma)$ is infinite dimensional in its top nonvanishing dimension for nonuniform lattices as one can see using the results of Borel and Serre [21].)

This proposition is probably known to those who work in cohomology of groups in light of the previous one [18]. We will give an alternate proof that has additional topological implications.

Proof. We shall make use of the following result: □

Theorem. (Quinn, see [31]) *Suppose $F \longrightarrow E \longrightarrow B$ is a fibration with all three space finite complexes, then E is a Poincaré complex iff F and B are.*

This follows from a consideration of boundaries of regular neighborhoods of embeddings in Euclidean space, and the following characterizations of Poincaré duality (which is essentially a special case[19] of the theorem, as well)

Theorem. (Mischenko and Quinn) *X is a Poincaré space iff when X is PL embedded in Euclidean space, the homotopy fiber of the map from the boundary of a regular neighborhood of X into X is a homotopy sphere.*

[18]The corollaries to the proposition and the coarseness of Poincaré (or Bieri-Eckmann) duality were observed by Gersten [55] without reference to HX.

[19]In light of Spivak's thesis [99].

Now we can prove the proposition. Let $N, \partial N$ be a regular neighborhood of $B\Gamma$. (We assume without loss of generality that the maps $\partial N \longrightarrow N \longrightarrow B\Gamma$ are isomorphisms on π_1.) The homotopy fiber of the map $\partial N \longrightarrow B\Gamma$ is the universal cover of ∂N. Thus, $B\Gamma$ is a Poincaré space iff this is a homotopy sphere. Since ∂N is a manifold, if this fiber is a finite complex, $B\Gamma$ is a Poincaré space, by the previous theorem. Thus, if the fiber (=cover) is finite, it is a homotopy sphere.

Since $B\Gamma$ is aspherical, N is contractible, and therefore $H_*^{lf}(N) \cong HX_*(B\Gamma)$. However $H_*^{lf}(\tilde{N}) \cong \tilde{H}^{n-*-1}(\partial\tilde{N})$ (where $n = \dim N$), so if HX is finite dimensional, it is the reduced cohomology (shifted) of a sphere, proving the proposition.

Corollary (of proof). *If the universal cover of a manifold with fundamental group of type virtual-FP has finitely generated homology, then the group is a virtual duality group.*

Remark. If $\mathbb{R}^n \times M$[20] has a proper discontinuous cocompact Γ action, M compact, then we have $HX^*(\Gamma) = 0$ for $* \neq n$, and $HX^*(\Gamma) = s$ for $* = n$, since Γ and a uniformly contractible copy of $\mathbb{R}^n$ have the same coarse type. Consequently, if Γ is torsion free, it is a duality group. For M a sphere, this is a theorem in [40].

This is nice because the coarse theory describes an analogue of the duality condition that is necessary even for non-residually torsion free groups (see [48] for such an example). It is quite easy to show that for torsion free groups a converse holds (with an extra condition[21]); for virtually torsion free groups the converse still holds if we are allowing arbitrary compact M—if we insist M is a sphere, there is at least an additional condition on periodicity of Farrell cohomology, and it is a difficult problem to characterize the groups that can act (see e.g. [68, 3]).

Problems. *Which countable groups (i.e. no sissy conditions) can act freely and cocompactly on $\mathbb{R}^n \times M$ for some compact M, on $\mathbb{R}^n \times S^k$ for some (specific?) sphere, or on some space of finite type? Which Γ act on a sphere in a way that is free and cocompact outside of its domain of discontinuity?*[22]

[20] Or even $X \times M$, for X a contractible manifold and M merely a compact space.

[21] Basically, one can define the fundamental group of the end of Γ in a coarsely invariant fashion; it must be trivial for the group to act cocompactly on a space that is coarsely simply connected at infinity. A contractible manifold that is simply connected at infinity is Euclidean space by a theorem of Stallings.

[22] For torsion free hyperbolic groups the answer is given entirely by the Poincaré duality condition.

3. Metric rigidity phenomena. I (Packing)

The coming sections contain a miscelleny of coarse rigidity theorems of a coarse /topological/homological nature. As we mentioned in the introduction, the literature is very large, (after all one of the main inspirations of coarse geometry was Mostow's work) and we would surely be remiss if we did not at least mention Gromov's bounded cohomological proof of Mostow rigidity for compact hyperbolic manifolds (see [66, 102]). However, our focus will be on phenomena that seem at least philosophically related to the types of considerations of the previous section[23].

Definition. A map $\phi : X \longrightarrow Y$ is *effectively proper* if there is a function real valued function f such that if $d(x, x') \geq f(r)$, then $d(\phi x, \phi x') \geq r$. A map which is both coarsely Lipschitz and effectively proper will be called EPL.

Remark. This condition has appeared in [1, 19] and a number of other investigations. However, this condition now seems too strong, as it encodes "asymptotic coarse injectivity". Probably one should modify the definition so that $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\phi(z) = z^2$ is effectively proper[24].

Problem 1 (A packing problem). *Suppose that $\phi : X \longrightarrow Y$ is EPL, what can one say about ϕ? Sometimes, we will use the (hopefully) evocative terminology that ϕ packs X into Y, or that ϕ is a packing.*

Of course, in this generality, nothing. We first thought about this question[25] in the special case where $X = Y = \mathbb{Z}^n$.

A metric space will be called overeuclidean if there is a (uniformly and coarsely) Lipschitz homeomorphism from it to Euclidean space. For instance any simply connected manifold of non-positive curvature is over-Euclidean as well as any simply connected nilpotent Lie group.

Theorem. [26] *Any packing of $\mathbb{Z}^n$ into itself is a quasiisometry. In particular, the image is always C-dense. In fact, any packing of a quasilattice in an over-Euclidean space into a uniformly contractible Riemanninan manifold of the same dimension is a quasiisometry.*

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[23]Indeed, these phenomena were our main motivations for the introduction of these homological techniques.

[24]The reader who insists on Lipschitz should modify ϕ to be $\phi(z) = z^2/|z|$.

[25]Hillel Furstenberg brought it to the attention of the second author in 1990.

[26]With some extra condition on ϕ, this was proved by Furstenberg as well, by ergodic methods, at least in low dimensions.

The proof we'll give implies that one can't assemble simplices in any coherent way using far apart vertices and get a PL homeomorphism:

The theorem can be restated that any of these packings necessarily has a linear effective propriety function.

Note however that it is critically important that the packings be quite tight: It's trivial to pack the line into the plane in many exotic ways. However, it seems possible that in the case of higher rank lattices there are relatively few packings of spaces even into rather larger ones.

We will prove the theorem geometrically. However, the argument (perhaps by being more perspicacious in applying the techniques of [49] coarsely—rather than just the theorem) could possibly be reorganized purely homologically. Intuitively, one wants to say that EPL maps are asymptotically injective, so that they have HX-degree one, which implies that they are coarsely onto.

We shall instead go back to uniformly contractible manifolds and use the following amazing theorem of Ferry [49] (See also [51] for the noncompact version we're using here):

Theorem. *For any manifold M^n, $n \geq 5$[27], there is a δ such that any continuous map $\phi : M \longrightarrow N^n$ with point inverses of diameter less than δ are within ϵ of a homeomorphism[28].*

Addendum. *If M is noncompact, δ might have to be taken to be a function which decays at infinity. However, if M has bounded geometry[29], δ can be chosen to be a small constant.*

However, many M have *shrinking self-homeomorphisms* [30] (e.g., Euclidean space has dilation), in which case obviously for any δ, any "δ-map" can be homotoped to a homeomorphism.

Remark. It is remarkable that one can use shrinking self homeomorphisms, in that their inverses are not Lipschitz.

In any case the theorem is now fairly direct. If one connects the dots on a packing one abtains a continuous-Lipschitz map between Euclidean spaces. Ferry's theorem puts it a finite distance from a homeomorphism. The image of the lattice under this homeomorphism is C-dense. One can also now define a coarse inverse map, g, which has a "bornology property", $d(gy, gy') \leq F(d(y, y'))$ for some function F (by the effective

[27]4 suffices according to [16, 51].

[28]This is true equivariantly as well for the locally linear case.

[29]The injectivity radius is bounded below and the curvature is bounded (see [11] for some discussion).

[30]This is related to "combing conditions" as studied in [45, 6, 5, 56] and elsewhere.

propriety). (The bornological terminology is due to Roe, [92]). However, since $\mathbb{R}^n$ is a coarse path space (i.e., the distance between points is proportional to the length of shortest paths connecting them), obviously, bornological implies Lipschitz, so the original map was coarse bi-Lipschitz.

The result on spaces comes from reflecting on Ferry's theorem, local contractibility of homeomorphism spaces [44], and Sullivan's theorem [100].

Regarding other uniform lattices, it is a little tricky to deduce the C-density result directly from Ferry's theorem because of the non-Lipschitzness of the inverse of the squeezing map (although one could directly try to prove a variant of Ferry's theorem for arbitrary δ on a symmetric space). Here homology theory enters. Let $\Gamma \longrightarrow \Delta$ be a packing and let X and Y be symmetric spaces for these groups. We also denote by $s : X \longrightarrow X$ a Lipschitz squeeze homeomorphism. Consider the diagram

$$\begin{array}{ccc}
HX_*(\Gamma) & \longrightarrow & HX_*(\Delta) \\
\downarrow & & \downarrow \\
HX_*(X) & \longrightarrow & HX_*(Y) \\
\uparrow & & \uparrow \\
H_*^{lf}(X) & \longrightarrow & H_*^{lf}(Y) \\
{\scriptstyle s^{-1}}\uparrow & & \\
H_*^{lf}(X) & &
\end{array}$$

where the bottom up arrow is induced by s^{-1}. This, and all vertical arrows are known to be isomorphism. The composite of the connect the dots map with s^{-1} is near a homeomorphism, by Ferry's theorem, so it is degree one, and we deduce that all the arrows are isomorphisms. The result then follows as before from the following lemma:

Lemma. *If X and Y are uniformly contractible manifolds and $f : X \longrightarrow Y$ is a coarsely proper Lipschitz map, and is nonzero on HX, then the image $f(X)$ is C-dense in Y.*

This is proven by a modification of the standard proof that a map of nonzero degree is onto There one assumes by way of contractiction that an open subset $\mathfrak{O}$ of Y is missed by f and then argues using the isomorphism $H_n^{lf}(Y) \longrightarrow H_n^{lf}(\mathfrak{O})$.

The trouble is that we only have compact balls in Y that are assumed missed by f, and HX_* of any such set vanishes. To deal with this, one uses the pro-system that defines HX. The idea is this, for each C, the balls of much greater size have "scale C homology" with the fundamental class of Y mapping nontrivially; that is, in the notation introduced before, $H_n^{lf}(Y) \longrightarrow H_n^{lf}(R_C(Y)) \longrightarrow H_n(R_C(B, \partial B))$ is injective. (In other words, for a fixed scale, the larger compact sets do not disappear; it is on changing scale that they get lost.) Having done this, the usual argument goes through.

Remark. Subsequent to the circulation of an early draft of this paper, Farb and Schwartz [48] applied packing theorems to gain control at infinity for maps between spaces that contain flat directions where the usual negative curvature ideas do not seem to directly apply. Their paper also contains a more direct approach to packing.

Exercise: Assuming the balls of Y to be convex, estimate how large the C in C-density is in terms of the diameter of the simplices in Δ needed to represent the fundamental class and the lipschitz constant of the packing.

We suspect that, at least overeuclidean groups, but probably Poincaré duality groups, and most other "nondegenrate" groups (maybe: fundamental groups of aspherical pseudomanifolds) cannot have exotic self-packings and cannot be packed into finite dimensional uniformly contractible metric spaces whose HX vanishes above the dimension of the group. (Indeed, some of this might follow from the proof of Ferry's theorem combined with the above argument in place of application of the theorem itself.) It would be interesting to know about other packing problems where the shapes of the spaces involved aren't as directly comparable. We note though that there are some "surprising" packings which raise rank, like horospherical packing of Euclidean space within Hyperbolic space.

4. Metric rigidity phenomena II (Topological rigidity)

In this section we will briefly discuss a statement which, through no fault of his, is referred to the Borel conjecture, and its metric analogue.

(Borel) Conjecture. *A group is the fundamental group of a closed aspherical manifold, which is unique up to homeomorphism iff it satisfies Poincaré duality.*

This conjecture can be viewed as one of the organizing problems of topology[31]. The uniqueness statement includes the three-dimensional Poincaré conjecture. Its analysis for tori [73, 103] was a key step in triangulation theory for manifolds [81]. It is known to be true for non-positively curved manifolds and other symmetric spaces [47].

As for the existence, there are hardly any nontrivial cases of aspherical manifolds produced via the group theory and proving the conjecture. (One can show that certain extensions of manifolds are realized by manifolds by producing geometric block bundles via a proof of the conjecture; superrigidity suggests that such examples rarely occur where not predicated by Lie theory.)

Our goals in this section are to (1) explain metric analogues of this conjecture (as in [52], (2) show that these lead to counterexamples to the analogue of existence for groups with torsion (but also provide evidence for a more reasonable version of existence discussed in [26]), and (3) give counterexamples to the metric rigidity conjecture following [43].

(Coarse Borel) Conjecture. *If M is a uniformly contractible manifold, then any bounded homotopy equivalence $M' \longrightarrow M$ is a bounded distance from a homeomorphism.*

If we view coarse types as the analogues of fundamental groups, then the uniformly contractible spaces are the analogues of the aspherical ones (they are both the terminal objects in the category of spaces with the given coarse type or fundamental group). In addition, the universal cover of a manifold to which the Borel conjecture applies is a metric space to which the coarse version applies. The Borel conjecture then asserts that in the absence of additional algebraic topology, there can be only unique geometric topology.

We note that Ferry's theorem (discussed in the previous section) implies that the Coarse Borel Conjecture is true for Euclidean space. In a suitable sense, it is true for weighted cones. It is also true for Hadamard manifolds, for instance. If it were true for groups then one could deduce the Novikov conjecture for all groups for which $B\Gamma$ is a finite complex (the principle of descent, see [52, 30])

To just indicate this direction of ideas, let us begin with an interesting theorem regarding a classical theme in smooth transformation groups.

Theorem. *Suppose a group G acts affinely on a torus T with a fixed point p, and that G acts continuously on T' as well. Let $\phi : T \longrightarrow T'$ be an equivariant map which is a homotopy equivalence. Then, there*

[31]There are, of course, many generalizations and analogues. We will just work with the closed manifold and metric versions here.

is an equivariant topological embedding of the tangent space TTp in a neighborhood of $\phi(p)$.

In short, if the action of G on T' were assumed smooth, one would be asserting the topological conjugacy of the tangential representations at the fixed points. One can modify the examples of [29] to show that even in the smooth category, linear conjugacy is not possible.

The point of view that this suggests to me is that if one thinks of the Novikov conjecture as a statement about tangentiality, one had best work in the topological category, and then one can enlarge the context some to obtain interesting information regarding even the restrictions of the tangent bundle to points! (In the classical (i.e. nonequivariant, manifold) case there is no information here except the dimension ... but we will soon see that here, too, there is more than meets the eye.)

Also, since this theorem is really part of the "Novikov package", one can go much beyond the torus case to at least, nonpositively curved spaces and groups (in some interpretation or other). *We will be using this extension shortly.* The following exercise describes one of the key ideas in this extension.

Exercise. *After reading the proof below, modify it to hold if the affine action on T is replaced by a G-aspherical manifold whose universal cover admits an equivariant homeomorphism that is Lipschitz to a G-representation (G-overeuclidean).*

Having increased the suspense by not giving the proof instantly, we give it now, for simplicity in the case that the action of G in a neighborhood of p has just p as a fixed point[32]. Consider $\phi : \tilde{T} \longrightarrow \tilde{T}'$. Clearly, the map is equivariant and has bounded size point inverses. This passes to the quotient. Apply Ferry's theorem near ∞, to produce isomorphisms outside of compact sets[33]. Now engulfing allows one to extend the homeomorphism to the complements of p and $\phi(p)$. These we extend by continuity! Now, by restricting to a small enough neighborhood of p, one still has an embedding of TTp on projection $\tilde{T}' \longrightarrow T'$. This completes the proof.

Now let's consider the existence part of the Borel conjecture for groups with torsion. We will see that not all reasonable candidates can be realized by aspherical orbifolds.

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[32]The general case only requires more equivariant or stratified machinery, but is no different in conception.

[33]This essentially gives a proof of the bounded Borel conjecture for open cones of manifolds, with the understanding that an isolated singularity is allowed, if one works relatice to a neighborhood of the singularity.

Proposition. *There are virtually PD^n groups that are not realized by aspherical orbifolds. Moreover, these groups can be constructed so that all torsion is of order p and all torsion elements generate their own normalizers.*

We emphasize the simple structure of the finite subgroup lattice to eliminate any homotopical obstructions. The groups considered are true analogs of PD^n groups with torsion if one studies the orbifold question.

To construct these groups, we start with a PL (topologically) nonlinear semifree action of $\mathbb{Z}_p$ on a sphere with, say, two fixed points. These can be obtained by suspending suitable free actions on the sphere. (The early ones produced by Milnor and Hirsch will not do; ones detected by the (ρ-invariant [103] are perfect—there are such examples constructible through the Brieskorn method of isolated hypersurface singularities that are invariant under a group action. For instance, for involutions, where every Brieskorn singularity is preserved by refelction through the origin, Hirzebruch had first proved the nonlinearity of the action on the link sphere. For a thorough discussion of several points of view on these involutions and the interrelationships among the invariants involved in their analysis, see [82].)

Since the action is PL, we can apply a hyperbolization technique [61, 41, 78]. This will produce an aspherical manifold with a PL $\mathbb{Z}_p$ action. The group of lifts of this action (the orbifold fundamental group!!) is not realized by an orbifold!

Firstly, how can this be? Orbifolds are spaces that are modelled on linear representations. Here the process of hyperbolozation will produce an action that is not locally linear at the fixed points; the action there will be modelled on the free PL action on the sphere we started with.

The proof is not so hard. Any other action will be equivariant homotopy equivalent to the action constructed. The hyperbolic analog of the previous proposition shows that the local structure must be (topologically) the same as the one we started with, yielding a contradiction.

Remark. In [107] this same strategy of using true rigidity theorems as a method for disproving false ones was applied to producing counterexamples to the equivariant Borel conjecture for affine tori which violate the gap hypothesis.

On reflection it's easy to understand the source of these examples. The definition of orbifold is too narrow. One really wanted to say there was an appropriate group action on a contractible manifold. The local linearity is just some extra feature of a group action that we should not have assumed.

The same is probably true for PD^n groups. In [26] it was shown that surgery theory holds better for the existence (and classification) of homology manifolds than it does for manifolds! (That is, s-cobordism classes of homology manifolds simple homotopy equivalent to a given simple Poincaré complex are in a 1-1 correspondence with the fiber of the 4-periodic surgery obstruction assembly map, and exist iff the corresponding "total surgery obstruction" vanishes.) For instance, all of the evidence for uniqueness Borel is equally evidence for existence Borel if we modify it to include homology manifolds (i.e. if we remove the assumption of local linearity from our homology manifolds), but not for the existence BC as we've phrased it.

More precisely, there is a $\mathbb{Z}$-obstruction to resolving homology manifolds, see [90]. Assuming Novikov, for $B\pi$ the $\mathbb{Z}$-obstruction for any realizing homology manifold is the same as that of any other one. (The Borel conjecture would say than any two realizations were s-cobordant.) It is unthematic, and in violation of the analogy between group actions and the theory of homology manifolds [108], to assume that the only realized homology manifolds will be resolvable.

Indeed, there is another generalization of the Borel conjecture where one can also see the failure of the existence half if one does not include homology manifolds. Following [105], say that X is haspherical if X has the same integral homology as $B\pi_1(X)$. The Borel conjecture for the fundamental group of X implies that if X is a topological manifold, it is rigid. However, it would only imply for Poincaré spaces realizability by homology manifolds.

Conjecture. *If Γ is a torsion free hyperbolic (or "nonpositively curved") group then there is an aspherical homology manifold with fundamental group Γ iff it is a PD^n group (iff $HX_i(\Gamma) = 0$ for $i \neq n$ and $HX_n(\Gamma) = \mathbb{Z}$). The realization is unique up to s-cobordism.*

There is some direct evidence for this in [52, 30, 47, 74] and probably elsewhere. More indirect evidence can also be adduced. However, most importantly, the all values of the realization obstruction should occur (at least for $n \geq 5$).

Now, let us return to the large scale Borel conjecture itself

Theorem. [43] *There is a uniformly contractible manifold which is not boundedly rigid, i.e., there are coarse q.i. between uniformly contractible manifolds M' and M, that are not a finite distance from a homeomorphism.*

The construction begins with a remarkable result of Dranishnikov [42] as refined in subsequent work of [43] (and depends importantly on other constructions of Edwards and Walsh [104]):

Theorem. *There is a cell-like map $S^n \longrightarrow X$ which is neither an injection nor surjection on mod p K-theory.*

This X cannot be a finite dimensional space. (CE maps induce isomorphisms on all connective homology theories, and on all theories if the target is finite dimensional.) The basic idea is to build a metric on $\mathbb{R}^{n+1}$ that asymptotically resembles a sufficiently quickly growing cone cX. Doing this one can achieve uniform contractibility of the metric on $\mathbb{R}^{n+1}$ but the natural map $K^{lf}(\mathbb{R}^{n+1}) \longrightarrow KX(\mathbb{R}^{n+1})$ is neither injective nor surjective[34]. This means that there are a different class of "large scale bundles" on this manifold than predicted by ordinary K-theory. These are used to construct invariants that surgery theory realizes. We will discuss this somewhat in the next section.

5. Connection to large scale index theory

As far as we can tell, the first major paper in "large scale index theory" and its application to the geometry of compact manifolds was the beautiful paper [GL] that contains within it many of the ideas that have been developed from any number of different points of view over the past decade. Let me start with one of their main theorems whose proof contains elements which should already resonate with reader.

Theorem. *No compact manifold of nonpositive sectional curvature possesses any Riemannian metric with positive scalar curvature.*

This is deduced from a "large scale" result:

Theorem. *If a Riemannian manifold M^n of bounded geometry[35] with positive scalar curvature, then M^n does not have a Lipschitz degree one map to $\mathbb{R}^n$.*

Remark. Gromov has asked [62] whether every uniformly contractible M^n has a coarse Lipschitz degree one map to $\mathbb{R}^n$. In [43] this is shown

[34]Nothing like this could happen for homology as we had seen earlier; the point is that for nonconnective theories the construction of large-scale homology theories produces a quite different theory than that of the (assuming it exists) uniformly contractible representative of the coarse type.

[35]We will discuss the significance of this condition in chapter 2.

not to be the case[36]. However, it is still conceivable (likely?) that no uniformly contractible manifold is coarsely equivalent to a manifold with positive scalar curvature.

The proof is a coarse variant of a theorem of Lichnerowicz (see [10]) which asserts that any spin manifold with positive scalar curvature has $\hat{A}$-genus. This theorem is proven using the index theorem and a Bochner-Weitzenbock formula

$$D^*D = \Delta + \kappa/4$$

where D is the Dirac operator, Δ is the Laplacian (which is positive semidefinite) and κ is the scalar curvature. If κ is positive, then neither D nor D^* can have kernel, so the index is 0, but the index is $\hat{A}$-genus, and Lichnerowicz's result follows.

Gromov and Lawson next prove an index theorem for open manifolds. The setting is that one has two open manifolds with elliptic operators on them, and a positivity condition on these operators at ∞. They also presume that the manifolds and operators are identified outside of some compact set. This situation leads them to a relative Fredholm index, which they compute as being the difference of the integral of the Atiyah-Singer integrand on the corresponding compact pieces of the two manifolds.

In the applications, there is only one manifold, but one will have two operators on it, which are identified near ∞. These operators will just be Dirac twisted by appropriate bundles (with connection). It is in the construction of these bundles that the coarse geometry enters.

Note now that if a bundle is *almost flat*, say, has sufficiently small curvature, then when we tensor the Dirac operator with it, the Weitzenbock formula still gives the same vanishing of the index. In this noncompact setting, we can define a bundle E with a family of connections ∇_t to be almost flat if (1) the connections are each flat outside of compact sets (depending on t) and (2) the curvatures of ∇_t are going to 0 as $t \longrightarrow \infty$.

If we apply the relative index theorem to (E, ∇_t) (relative to the trivial bundle trivially trivialized), we get the conclusion that $\int_M(\hat{A} \wedge ch(E))dVol = 0$.

Now we invoke the hypothesis on Lipschitz maps to construct the almost flat bundles. Suppose without loss of generality that the dimension is even. Then we can consider $\mathbb{R}^{2k} \cong \mathbb{C}^k$ which has the Bott bundle on it (pick some connection on it). By rescaling one gets a sequence of

[36]This follows simply from the construction outlined in the previous section. If there were such a Lipschitz map then one could use it to see that K^{lf} would at least split off KX, which it does not, by construction.

connections with decaying curvatures on $\mathbb{C}^k$, which when pulled back give the desired bundle on M. It is easy to see that for this bundle $\int_M(\hat{A} \wedge ch(E))dVol \neq 0$, and the theorem is proven.

Remark. In the nonpositive curvature case the inverse of the exponential map (at some base point) is a convenient choice of Lipschitz map to Euclidean space. The theory of almost flat bundles is further elaborated in [39, 70].

It is also worth pointing out here that in the last section of this same paper Gromov and Lawson show how to use the families version of the same index theorem (coupled with a family of Bott bundles) to show that if M is a compact spin manifold with the fundamental group that of a nonpositively curved manifold, then the $\hat{A}$-genus must be 0. One considers the family $\tilde{M} \times_\Gamma E\Gamma \longrightarrow B\Gamma$ and considers its family relative index in $K(B\Gamma)$ and obtains its vanishing.

This method of obtaining restrictions on invariants of compact manifolds by considering families of coarse problems on the universal cover is used, in one way or another, in [80, 39, 52, 30, 75]. We like to call it the method (or principle) of descent. It is quite remarkable that a good enough understanding of what is often a single large scale obstruction in $HX^n(M)$, is enough to get information on obstructions lying in a whole range of dimensions and of essentially unkown groups $H^i(B\Gamma)$.

In a very nice memoir, Roe [92], building on work of Higson conceptually clarified the situation greatly[37]. His idea is this: One considers on complete Riemannian manifolds elliptic operators of bounded propagation speed (see [35, 92] for this concept and some of its implications; essentially one wants approximations to the defining kernel of e^{itD} the wave group associated to the given elliptic operator to be zero outside a neighborhood of uniformly bounded girth around the diagonal in $X \times X$; it holds for "geometric" operators like laplacians and Dirac operators on manifolds).

Now, according to [11, 25, 80], an elliptic operator gives rise to an element of $K^{lf}(X)$. The business of constructing an index is finding a C^*-algebra over which the projections to kernel and cokernel live, so that the index can then be defined to be the difference of these projective modules in K_0 of this algebra.

This is only intuitively correct. There are several types of complications that can arise in this program. The simplest and perhaps most familiar example is the index theorem for families. Here the appropriate C^*-algebra is $C(B)$, the agebra of continuous complex valued functions

[37]It still is somewhat unclear what the precise connection between almost flat bundles and bundles on the corona is.

on the parameter space B. The kernel and cokernel of this family of operators do not themselves form vector bundles, i.e., projective modules over this algebra, because the ranks of these vector spaces can change from point to point. Nonetheless, a well known argument produces perturbations such that the kernel and cokernel become constant rank, and the difference between these projective modules (i.e. vector bundles) is a well-defined element in K-theory. In the general situation that we need, there is a simple abstract algebraic K-theoretic construction that gives us our index, see [92, 93].

Here what is relevant is the completion of the bounded propagation speed kernels, $C^*(X)$. Indices with values in C^*-algebras have all of the properties that you would expect from an index: one can get vanishing from a Bochner formula, or homotopy invariance from a signature operator[38].

The index theorem really has something to do with the map $K^{lf}(X) \longrightarrow K(C^*(X))$. $K(C^*(X))$ is essentially a coarse object. For X compact, it boils down to $K(\mathbb{C})$ which is the familiar alternating sequence of $\mathbb{Z}$'s and 0's. In general, one doesn't know that much about it[39], but one can get some information using an idea of Higson.

Definition. The corona of X is the set of ideal points in the completion defined as follows: Define the variation of a function f on X by $V_f(x,r) = sup|f(x) - f(y)|$ where $y \in B_x(r)$. We then consider the algebra of bounded functions with decaying variations, i.e., for all r, $V_f(x,r) \longrightarrow 0$ as $x \longrightarrow \infty$. By the Gelfand-Naimark theorem, this corresponds to a compact space, and on reflection one sees that this contains X as a dense open subset.

In practice one always uses quotients of this completion, but even before doing that, one does obtain a nice"space at infinity" by Higson's construction. If Z is a quotient of the corona and is a nice space, then there is a map $K_*(C^*(X)) \longrightarrow K_{*-1}(Z)$. The large scale index theorem states that the image of the elliptic operator is given by the topological pushforward of the "symbol cycle" under the ∂ map $K_*^{lf}(X) \longrightarrow K_{*-1}(Z)$.

Dually, thinking in terms of pairing, one gets to pair intelligently an operator with a bundle that's coming from $K^*(cZ)$. These are precisely Bott type bundles in the sense we just used them: they live on a compact

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[38] We are being a little glib here. See [87] for a discussion of what's involved and cases where this can currently be proven.

[39] See [71] for the current state of knowledge (which contains all Hadamard manifolds).

part of a cone, and rescaling them makes them "almost flat". The Gromov-Lawson argument is associated to compactifications using the sphere, but one can now systematically use other spaces (like, e.g., Tits boundaries).

Of course, the space at infinity, and cones on it, are ways at getting at a more basic invariant: $KX_*(X)$. One has an assembly map $\alpha : K_*^{lf}(X) \longrightarrow K_*(C^*(X))$, which we will will see factors through the large scale theory $K_*^{lf}(X) \longrightarrow KX_*(X) \longrightarrow K_*(C^*(X))$. α is constructed by a variant of the usual construction of the assembly map, see [17, 92, 80]. One chooses a quasilattice $Z \longrightarrow X$ and a Borel map $\phi : Z \longrightarrow X$ with $\phi^{-1}(z)$ of measure 1. Then $C^*(X)$ can be identified (not quite canonically) with compact operator valued kernels on Z of bounded propagation speed $C^*(Z)$. Then one constructs maps $K_*(R_n(Z)) \longrightarrow K_*(C^*(Z))$ by realizing elements of $K_*(R_n(Z))$ by "geometric operators" and taking the index of these operators. The maps commute with the system $R_n(Z) \longrightarrow R_{n+1}(Z)$ producing α and its factorization.

Those who are optimistic could conjecture that $KX_*(X) \longrightarrow K_*(C^*(X))$ is an isomorphism. This would be compatible with all known calculations and with the examples in [43].

Note that in the [43] example the map $K_*^{lf}(X) \longrightarrow KX_*(X)$ is not an isomorphism. It is a secondary invariant associated to the cokernel $K_{*+1}^{lf}(X) \longrightarrow KX_{*+1}(X)$ that distinguishes these manifolds from one another; it is morally like an η invariant associated to an almost flat bundle that is being used[40]. Unfortunately, one does not have enough spectral theory available for such a manifold and such bundles to make real mathematical sense out of this.

The geometry implicit in these examples, and in the theory of stratified space [105] suggests the extension of these ideas to a more sheaf theoretic context.

The idea is quite simple: Note that $R_*(X)$ is built by the same construction that produces Cech homology. We may reverse the usual arrows and construct the homology associated to a "cosheaf" (even of spectra, if we are discussing generalized cohomology theories). (See [105] for a quick treatment.) Since we are coarsening all the time, one doesn't need the cosheaf defined on all "open" sets (or balls), only sufficiently large ones. Thus, one can discuss "coarse cosheaves" if one liked.

The idea of these is that we can use them to record some part of

[40]This is parallel to the use of η-invariants and flat bundles in [9] to distinguish homotopy equivalent lens spaces from one another.

the information in the small, or in a semilocal fashion, that we do not want to lose as we go large scale. One useful example would be to take $B\pi \wedge K(\mathbb{Z}, 0)$ (where π is the fundamental groupoid). This lets one take higher signatures that keep track of both fundamental groups on some scale, and the coarse geometry. Indeed, using this, one can see that the punctured torus T^n does not have any complete metric of positive scalar curvature: Note though, that the metric could be quasiisometric to just the ray $[0, \infty)$, for which $KX \cong 0$, and the previous methods would not detect. Here we are keeping track of the fact that even by being coarse, the part at ∞ does not carry enough fundamental group for some of the compact obstruction to escape through. However, by taking a Lipschitz map ϕ to a ray, and giving it the coarse cosheaf described, there would be an obstruction in $KX_*([0, \infty); \phi * B\pi) \cong K_*^{lf}(T^{n'})$, (here $'$ denotes a puncture) which contains the nontrivial obstruction.

By choosing coarse cosheaves carefully one can do some more coarse analysis, in the same way that sheaves help out in complex manifold theory. The coarse cosheaves could, for instance, be things that keep track of growth of functions (or cycles) on chunks of the space. Some of these will be studied in the next chapter. But there seem to be many more possibilities, such as using Rips complexes of one growing scale in forming space fragments whose invariants are then assembled into a larger scale coarse theory.

6. Uniformly Finite Homology

Beginning with this section, we will put growth conditions of various sorts on the (co)cycles of our coarse (co)homology theories and show how these can be used to study more delicate coarse properties of metric spaces and manifolds. The theory we study in this section, H_*^{uf} was introduced in [19], and is useful for studying spaces with bounded geometry.

Let Z be a space with coarse bounded geometry. This means that Z contains a bounded quasilattice. Riemannian manifolds of bounded geometry (i.e., injectivity radius bounded from below and curvatures bounded in absolute value), or uniformly locally finite simplcial complexes (with simplicial metric) and finitely generated groups with the word metric all provide examples.

If Γ is a quasilattice in Z (Γ does not necessarily connote a group), then the Rips complexes $R_n(\Gamma)$ are uniformly[41] locally finite simplicial complex. Now consider simplicial chains on $R_n(\Gamma)$ which, while

[41] The uniformity is not in n.

perhaps having infinite support, have uniformly bounded coefficients[42], $c = \sum a_\sigma \sigma$, with $|a_\sigma| \leq K$. These form a chain complex $C^{uf}(R_n(\Gamma))$, and we can consider (as we did earlier for HX) the pro-system of the homology groups of these chain complexes. We will denote the direct limit of these by $H_i^{uf}(X;R)$; it is independent of the bounded quasilattice chosen[43]. We will concentrate on the cases $R = \mathbb{Z}$ or $\mathbb{Q}$.

It is also possible to define uniformly finite homology with coefficients in other spectra, see [12]. These are important in studying the geometry and the surgery theory of manifolds of bounded geometry.

To illustrate, let $X = \mathbb{N}$, and $R = \mathbb{Z}$. A typical chain is $\Sigma[n]$. In HX_0 this cycle is 0 since $\Sigma[n] = \partial(\Sigma n[n-1,n])$. Since the coefficients (n) are unbounded, this 1-chain is not uniformly finite. In fact one can easily check that

$$H_0^{uf}(\mathbb{N};\mathbb{Z}) \cong \{\phi : \mathbb{N} \longrightarrow \mathbb{Z} | \Delta\phi \text{ is bounded}\}/\{\text{bounded } \phi\},$$

where the isomorphism is given by summing up the coefficients of the chain from 0 to n. Note that this is a $\mathbb{R}$ vector space! This group tends to be very large if it is nonzero.

For another example consider the free group on 2 generators F_2. The 0-cycle $\Sigma[\gamma]$ vanishes in $H_0^{uf}(F_2;\mathbb{Z})$ as is demonstrated in the following figure.

The intuition involved in this group is that we are trying to produce geometric one chains whose boundaries are a given collection of points (with multiplicities), in such a way that there is only a bounded number of arcs passing through any ball. From this point of view $\Sigma[\gamma]$ is the hardest cycle to kill, since there is no room for cancellation. The "spaghetti" that this bounds can be used to produce escape routes for any 0-cycle.

Similarly, by partitioning the cosets of F_2 in the Cayley graph of a discrete group containing F_2, one has enough spaghetti to kill any 0-cycle. This, and the unfortunately disproved Von Neumann conjecture suggest parts of the following theorem:

Theorem. [19] *For a metric space X of coarse bounded geometry with bounded quasilattice Γ, the following are equivalent:*

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[42] We are envisioning a ring with norm. Other possibilities for the uniform finiteness are to allow only finitely many elements of the ring to appear as coefficients for any given chain. Note that for $\mathbb{Q}$, these chain complexes obtained in this way are quite different.

[43] These were first defined in [19]; similar cohomology groups were introduced in somewhat less generality in [54]. Obviously, this is an L^∞ notion and one can form coarse L^p groups as well.

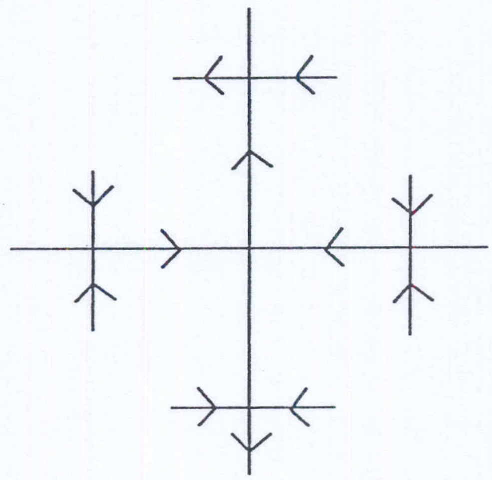

FIGURE 2. The pattern visible at the identity element is everywhere repeated. Two edges enter, one leaves, and one just has coefficient -Q. An easy induction show this is possible.

- $\Sigma[\gamma] = 0$ *in* $H_0^{uf}(X;\mathbb{Z})$
- $H_0^{uf}(X;\mathbb{Z}) \cong 0$
- $H_0^{uf}(X;\mathbb{R}) \cong 0$
- $\tilde{H}_0^{uf}(X;\mathbb{R}) \cong 0$
- X *is nonamenable.*

The group $\tilde{H}_0^{uf}(X;\mathbb{R})$ is defined similarly to the groups introduced before, except one kills the closure of ∂, not just the image of ∂. The condition of nonamenability requires a little preparation to state:

Definition. For a set S in Γ, we denote by $\partial_r S$ the points that are within r of both S and $\Gamma - S$.

Definition. If V is a finite set in Γ, then $\mathrm{vol}(V)$ = the cardinality of V.

Now, we say that X is nonamenable if for some r, and positive c, there is an inequality $\mathrm{vol}(S) < c\mathrm{vol}(\partial_r S)$ for all finite sets S.

The alternative is amenability: there is a sequence of subsets S_i for which $\mathrm{vol}(\partial_r S_i)/\mathrm{vol}(S_i) \longrightarrow 0$ for each r. Such a sequence will be called a regular sequence. It is more common, and, in fact, equivalent to suppose that $S_i \subset S_{i+1}$, and that $X = \cup S_i$. When these conditions hold, the sequence will be called a regular exhaustion. These arose in Folner's original characterisation of amenability of groups (see [88]) and in function theory (see [4]). Thus a discrete group is amenable iff in its word metric, it is an amenable metric space. In any case, as we will see, regular sequences are as easy to work with as the exhaustions, and have greater flexibility for studying certain geometric problems.

The original idea of using regular exhaustions was to be able to measure the average of a function on the manifold. One integrates a function[44] and rescales according to the volume of the region[45]. For this to be well defined in cohomology (i.e., that the boundary term in integration by parts to be negligable) one uses (boundedness of the function and) the regularity condition. This can be done equally well using regular sequences, except that one should then intuitively think of this as a process that samples the function in, perhaps carefully chosen, regions.

If the space is amenable then $\Sigma[\gamma] \neq 0$ in $H_0^{uf}(X;\mathbb{Z})$ as the average value of the coefficients is 1 on any region, and so for any regular exhaustion (or sequence).

It is also important to realize that although the vanishing of $H_0^{uf}(X;\mathbb{Z})$ is equivalent to that of $H_0^{uf}(X;\mathbb{R})$, the obvious map is far from injective. There seems to be a lot of subtle information located in this kernel, and we hope to unravel some of this. As an example, though, we point out that $\Sigma[n^2]$ vanishes in $H_0^{uf}(\mathbb{N};\mathbb{R})$ although it is nontrivial in $H_0^{uf}(\mathbb{N};\mathbb{R})$ (as is any infinite set of points with the same sign).

The following result is a more precise version of the relation of amenability to the vanishing of $H_0^{uf}(X;\mathbb{R})$.

Theorem. [20] *For a uniformly discrete metric space X and any cycle $c \in C_0^{uf}(X)$, $[c] = 0 \in \tilde{H}_0^{uf}(X;\mathbb{R})$ iff for any regular sequence S_i,*

$$\int_{S_i} c = o(vol(Si)),$$

where the integral of a cycle over a set is the sum of the coefficents the vertices in the cycle that lie in S.

The elements of the kernel of $H_0^{uf}(X;\mathbb{R}) \longrightarrow \tilde{H}_0^{uf}(X;\mathbb{R})$ are represented by very thinly sprinkled cycles. The relation of $\tilde{H}_0^{uf}(X;\mathbb{R})$ to integration over regular sequences means that the reduced theory is much more susceptible to direct analysis. The following theorem gives a complete hold on the meaning of 0-dimensional homology.

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[44]We view the coefficients of a 0-cycle as a function in this intuitive discussion.

[45]This should be done in a suitable function space, and a weak $*$ limit chosen for the process.

Theorem. *Given a uniformly discrete metric space X and a cycle $c \in C_0^{uf}(X)$, $[c] = 0 \in H_0^{uf}(X;\mathbb{R})$ iff for any regular sequence S_i,*

$$|\int_{S_i} c\,| = O(vol(\partial S_i)).$$

We have applied this to proving the bounded geometry homotopy invariance of the top Hirzebruch L-form.

7. Applications of H_0^{uf}

The results explained in the previous section have a number of interesting applications, especially to the geommetry of manifolds of bounded geometry. The details of some of these depend on the connection to index theory that we'll only get to in Section 9. Nonetheless, aspects of these can be apprehended immediately.

The most important application that provides the pattern for most of the others was pioneered early this century by the Italian American immigrant Ponzi in the area of financial services. The method was essentially to consider a set of clients each of which was trying to earn 1 unit and find a 1-cycle which can be viewed as giving instructions for how these clients should redistribute wealth. The idea is that if the boundary of the 1-cycle is the 0-cycle, then the redistribution will result in all points having gained, with no loss. Now Ponzi knew that this is impossible with a finite set, but he was dealing with *investors*, so he used time to allow the set to become infinite. However, the trouble was the one that occurred with the 1-chain on $\mathbb{N}$ whose boundary is $\Sigma[n]$: the coefficients were growing, and ultimately one couldn't find investors to provide for the large transfers required. Ponzi was jailed for this failure.

Now on the 4-valent tree, if each node would follow the instructions given in figure in the previous section, each would give a dollar to one neighbor and get two from two of its neighbors: universal profits for all![46] Moreover, this plan can be executed on any nonamenable space.

For the more mathematical applications, one is more often trying to remove resources, or actually, obstructions from all parts of a space, in a controlled fashion (which prevents the blow-up of the 1-cycle used for this purpose).

[46]As a moral lesson, we should point out that it is necessary for the agents to perform acts of kindness to neighbors who will not reciprocate it to them for the universal happiness to ensue.

Application 1. *We will follow* [93] *and* [19]. *Roe considers manifolds with bounded geometry*[47] *to to be complete Riemannian manifolds with positive injectivity radius and with uniformly bounded curvature tensor together with its its covariant derivatives. (Cheeger and Gromov* [32, 33], *see also Abresch* [2], *have shown that there is no need to worry about the covariant derivative condition. We will not discuss this and cognate points as they arise.) A strict quasi-isometry is a diffeomeorphism with upper and lower bounds on the lengths of images of unit vectors and for which the difference of the Levi-Civita connection of one and the pullback of the other is a bounded form.*

The PL version of these is that one is looking at PL manifolds of bounded combinatorial complexity, and the maps are allowed to only use uniformly finite subdivisions before being defined. See [11] for more on this and the connection to bounded distortion.

Covers of compact manifolds and leaves of foliations have natural bounded geometry structures and strict quasi-isometry classes of metrics; their study is a primary motivation for our work in this direction.

Roe used regular exhaustions and a modification of Atiyah's L^2 index theorem for regular covers to prove the following:

Theorem. [93] *If M is a spin manifold with $\int \hat{A}(M)dVol \neq 0$, then no amenable cover of M has a nonnegative scalar metric in its natural strict quasi-isometry class.*

We will discuss this result and refinements of it in Section 9. Using Ponzi schemes one can prove a sort of converse:

Theorem. [19] *For any nonamenable group Γ, there is a spin manifold M with $\int \hat{A}(M)dVol \neq 0$ and a regular Γ cover of M strictly quasi-isometric to manifold with positive scalar curvature.*

The method of proof is to use the fact that positive scalar curvature obstructions are known to be ultimately obstruction of spin bordism class (see [65]). One uses a Ponzi scheme to move all of these obstructions for the universal cover of a manifold $N\#W$, where N is a positive scalar curvature manifold with fundamental group Γ and W a simply connected spin manifold with $\hat{A}$-genus, off to infinity. (One can use the "spaghetti", i.e. the arcs that go off to Y that have the bounded "intersection and near-miss property", discussed last section, to build a bounded geometry cobordism from the cover of $N\#W$ to N.)

[47] This is intimately related to the uniform category considered by Januskiewicz [77].

Application 2. *A similar example occurs in [14], where the question of to what extent characteristic classes of a compact manifold obstruct the construction of strict quasi-isometries of for the universal cover.*

Theorem. [14] *For a group Γ, manifolds M and M', and a map $\phi\colon M' \longrightarrow M$ for which $\phi^*(p(M')) \neq p(M) \in H^*(M;\mathbb{Q})$, yet which is a finite distance from a strict quasi-isometry of Γ covers, exist iff Γ is nonamenable.*

In other words, for nonamenable groups characteristic classes downstairs do not necessarily prevent strict quasi-isometries upstairs, while for amenable groups they do. The obvious PL version of this theorem holds as well.

The construction of examples is an application of surgery together with a Ponzi scheme. That no examples exist in the amenable case is a reflection of properties of lower bounds on the higher H_i^{uf} for amenable groups. This will be discussed in the next section.

Application 3. *Another theorem that comes out of the Ponzi method is the following (which we call "Gerrymandering"):*

Theorem. *Let M be a manifold with a volume form, then there is a measurable ϕ a bounded distance from the identity, such that $\phi^*(Vol) = 2 \cdot Vol$ iff M is nonamenable.*

Application 4. *There is a pretty connection between tilings and H_0^{uf}. We consider a triangulated manifold M of bounded combinatorial complexity and tilings by codimension 0 submanifolds with oriented triangulated boundaries. A semibalanced tiling is one where weights are assigned to each oriented face, so that each tile has nonnegative total weight. The impurity set of a tiling, is a set of points in M taken one from each tile with positive total weight. An unbalanced tiling is one where all tiles have positive weight.*

Theorem. [19] *Any nonamenable manifold has an unbalanced tiling. A discrete set S of vertices of M is the impurity set of a semibalanced tiling iff $\Sigma[s] = 0 \in H_0^{uf}(M)$ (the sum taken over all $s \in S$).*

Thinking about tilings purely combinatorially leads to some insights about the nature of cycles in H_0. The following is a standard result, see Tilings and Patterns [67]:

Proposition. *Let M be an almost homogenous manifold (i.e., one which possesses a cocompact group of isometries). A finite set of prototiles which tiles arbitrarily large balls of M, tiles M.*

So, if one can tile, say, the upper half plane with a given set of tiles, one can tile the plane! (Note, tiles are oriented, so one can't reflect them!) Philosophically, this indicates that when H^{uf} is used to obstruct a structure, the violations of this property will have to be C-dense. In Section 9 such a refinement of Roe's theorem on the existence of negative scalar curvature will be given.

The proof of the proposition is an application of Konig's lemma, which finds an infinite path from a point through a finite valence directed graph provided that paths of arbitrarily large length exist (Exercise: verify this.). We build a tree with nodes collections of prototiles that fill up the ball of radius r, and such that no subset does, and connect two of these if one contains the other. This is a graph with finite valence, and the hypothesis gives paths of arbitrary length, and the conclusion of Konig's lemma is a tiling of M.

Application 5. *The previous example implicates H_0^{uf} in a geometric problem. The following results implicate H_0^{uf}.*

Theorem. *Let $\phi : M' \longrightarrow M$ be a bounded geometry homotopy equivalence, then $\phi^*(L(M') \cap [M']) = L(M) \cap [M] \in H_0^{uf}(M;\mathbb{Q})$.*

Theorem. *Suppose that M is a spin manifold with bounded geometry and positive scalar curvature, $\hat{A}(M) \cap [M] = 0 \in \tilde{H}_0^{uf}(M;\mathbb{Q})$.*

We conjecture these are true without the reduction, and for higher homology groups as well, but we have only been able to verify this in special cases related to where the Novikov conjecture can be verified by the principle of descent. The above theorems are proven using index theory, estimates on η-invariants by [32] and the criterion for vanishing of elements of $\tilde{H}_0^{uf}(M;\mathbb{Q})$ mentioned in the previous section.

Application 6. *The following is a reformulation of the mean ergodic theorem (again in light of the vanishing theorem):*

Theorem. *Suppose that Γ acts unitarily on a Hilbert space H, and that $P : H \longrightarrow H^\Gamma$ is orthogonal projection to the fixed point set. Then for $\xi, \nu \in H$, we have:*

$$\Sigma < \gamma\xi, \nu > [\gamma] =< P\xi, \nu > \Sigma[\gamma] \in \tilde{H}_0^{uf}(\Gamma;\mathbb{R}).$$

This does not hold if the unreduced version is used. (One can see this using the shift acting on l^2: for almost any pair of vectors the equality is false in $H_0^{uf}(\mathbb{Z};\mathbb{R})$.) We hope that there are higher homological ergodic theorems that are useful when Γ is nonamenable.

More conventional forms of the ergodic theorem can be obtained by assuming that the action on H comes from an ergodic action on a probability space. Then it is intelligent to take for ξ and ν characteristic functions, and the theorem directly encodes recurrence properties of the group action. One can quickly deduce (using adapted regular sequences as we will explain in another context in Section 9) that for an action of an amenable group, the "return times" of a set of nonzero measure is always a C-dense subset of the group (Khinchine's theorem).

8. Higher Homology

The higher homology groups are much more mysterious. However, a few cases have been somewhat analyzed.

Theorem. ([14, 54, 60]) *If Γ is amenable, then there is an injection $H_*(B\Gamma;\mathbb{R}) \longrightarrow \tilde{H}_*^{uf}(\Gamma;\mathbb{R})$.*

There is a useful variant of H_*^{uf}. Define $H_*^\infty(Z)$ for a simplicial complex of bounded geometry Z by using uniformly finite chains on Z (i.e. just like uf, but not coarsified using Rips complexes)[48].

Theorem. ([14]) *If Γ is amenable acting proper discontinuously, isometrically, and simplicially[49] on Z, a polyhedron of bounded geometry, then there is an injection $H_*(Z/\Gamma) \longrightarrow H_*^\infty(Z)$.*

This can be proven most directly[50] by defining a reverse map $H_*^\infty(Z) \longrightarrow H_*(Z/\Gamma)$: Given a chain c, assign to the simplex k in Z/Γ the average value (using the invariant mean on $l^\infty(\Gamma)$) of the coefficients of c of the simplices that lie above k. Clearly the composite $H_*(Z/\Gamma) \longrightarrow H_*^\infty(Z) \longrightarrow H_*(Z/\Gamma)$ is the identity.

Problem. *How is the image of $H_*(Z/\Gamma)$ related to $H_*^\infty(Z)^\Gamma$?*

Now using (Poincaré duals of) characteristic classes for manifolds of bounded geometry (or using the related cohomology theory defined using bounded differential forms, see [77, 60, 93] or below), and observing that they are invariant under strict quasiisometry, one obtains the other half of Example 2.

For symmetric spaces, one can get a certain amount of information on $H_*^{uf}(\Gamma;\mathbb{R})$. This leads to the statement that rank is a quasiisometry

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[48]For uniformly contractible Z, these theories coincide.

[49]This includes the proper discontinuity, but that hypothesis is included for clarity.

[50]This is an oblique criticism of [14], although this argument is implicit inthe more complicated one given there.

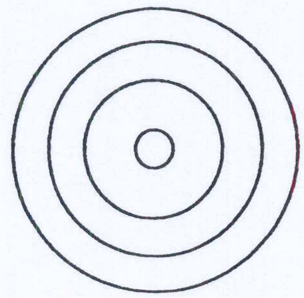

FIGURE 3. These concentric circles extending to the boundary give a nontrivial 1-cycle on the hyperbolic plane.

invariant, but to be honest, the proof depends on a key estimate of Mostow's that captures a critical aspect of the rank.

Theorem. ([20, 60]) *For X^n a symmetric space of real rank k, $H_*^{uf}(\Gamma;\mathbb{R}) = 0$ for $* \leq n-k-1$ and $* > n$; it is uncountable for $* = n-k$ and $\cong \mathbb{Z}$ in dimension n. (The same hold for $\tilde{H}_*^{uf}(\Gamma;\mathbb{R})$ as well.)*

So for $\mathbb{H}^2$, the hyperbolic plane, H_1^{uf} is very large. For later reference, we give some (undoubtedly incomplete) invariants. (Since the X's we are considering are uniformly contractible, one can use conventional cycles to represent the homology.)

A first, simplest, invariant is to take a geodesic and intersect the one cycle with it. A little thought shows that transversality enables us to take this intersection to be a uniformly finite cycle in the geodesic, i.e., an element of $H_0^{uf}(\mathbb{R})$. Using regular sequences on these one gets copies of the enormous $H_0^{uf}(\mathbb{R})$ in $H_1^{uf}(\mathbb{H}^2)$. It isn't hard to get different copies of this using different geodesics. As in Figure 3. Note that for the obvious geodesic, and regular exhaustions symmetric around the origin the concentric circle cycle is nontrivial.

One can do more creative things than just use a single geodesic. Using a collection of geodesic fragments with a regular sequence in this union one has greater flexibility. One way to do this is to find a bunch of balls of larger and larger size and use diametric (or other) geodesic chords in these balls. Then one can hopefully analyze cycles in terms of positions in hyperbolic space where one already has information.

The nonvanishing result for symmetric space comes via intersection with flats (or unions of fragments of these).

The vanishing result, as we've already mentioned, is a consequence of a basic estimate of Mostow [85] regarding projection from a symmetric

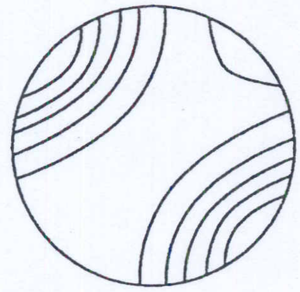

FIGURE 4. This collection of arcs gives a rather different looking nontrivial 1-cycle.

space down to a maximal flat. One can see that given a low dimensional cycle, the "shadow" of the cycle based on projection from a generic flat is a uniformly finite cycle with boundary the given cycle. Alternatively, one can argue cohomologically and use the proof of homotopy invariance of DeRham cohomology together with the exponential deacy of the deformation retraction to the flat (along the projection line) to see that for high degree bounded differential form ω the form ν with $d\nu = \omega$ is bounded.

The cognate cohomology theory to $H_*^\infty(Z)$ just applied is used in [77, 60, 93, 54, 14]. This is defined using cochains that are uniformly bounded on the simplices of Z or differential k-forms that are uniformly defined on k-tuples of unit vectors. We shall denote this by $H^*_\infty(Z)$ and by $H^*_\beta(Z)$ in the coarse version (using *inverse* limits of theories for Rips complexes). One can redo all of the arguments made for the homology theory for these groups as well. For n-manifolds one has the following useful relation:

Theorem. [13] *If Z is an n-manifold of bounded geometry, then* $H^*_\infty(Z) \cong H^\infty_{n-*}(Z)$.

In any case our last two applications in this section are most easily framed in terms of $H^*_\infty(Z)$.

The main advantage formally of $H^*_\infty(Z)$ is that it has a cup product structure. One also can put characteristic classes directly in it.

Proposition. [77] *If L is the leaf of a foliation of M and c is a characteristic class of L, then* $c \in imr^* : H^*(M) \longrightarrow H^*_\infty(L)$.

Now for $c =$ Euler class, Januskiewicz recovers the result of Phillips-Sullivan [PhS] that the average Euler characteristic of a(n amenable) leaf must vanish if it is a codimension one leaf in a simply connected manifold. This excludes Jacob's ladder or the infinite jungle gym. Using the previous results, one can exclude some exponentially growing quasiisometry classes from arising as leaves by taking universal covers of suitable manifolds with fundamental group a lattice. (We remark that there is a pretty paper of Attie and Hurder [15] that excludes certain quasi-isometry types as being leaves of C^1 foliations in any compact manifold based on entropic ideas.)

The final application of these methods that we'd like to mention is due to Gersten and Gromov [54, 58]. The Dehn function for a finitely presented group is the number of conjugates of relators one must multiply together to obtain any word of length n that represents the trivial element. Basically (but not exactly, as one can see quickly for the free group) its growth is an invariant of the group. It is recursive iff the group has a solvable word problem and is a first measure of the logical complexity of a group. In the riemannian setting, the parallel idea is to try to estimate the area of a bounding disk for nullhomotopic curves in terms of the length of the curve.

Gromov showed that a group is hyperbolic iff the Dehn function is linear. In [45], it is shown that automatic groups have quadratic Dehn functions. Interestingly, one can sometimes give lower bounds on the Dehn function (and the higher "filling functions" of Gromov) using $H^2_\beta(\Gamma)$ (and $H^i_\beta(\Gamma)$).

Elementary Illustration: The idea is quite simple. We try to find closed curves of small length which have explicit bounding disks. Integrating a representative from $H^2_\infty(\Gamma)$ over this coboundary gives a lower bound to its area.

The simplest case is $\mathbb{Z}^2$. One can use the 2-form $dx \wedge dy$ and the boundary of the square with size n to see that the Dehn function is quadratic.

More interesting is the Heisenberg group $H = 3 \times 3$ integral matrices with 0's below the diagonal and 1's on the diagonal. It has a presentation $\langle x, y | [x,[x,y]] = [y,[x,y]] = e\rangle$. ($x$, y, and $[x,y] = z$ are all elementary matrices.) It can also be viewed as $\mathbb{Z}^2 \times \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by the shear map $(z,x) \longrightarrow (z, z+x)$. This is volume preserving, so one obtains a 2-form on Γ, which lifts to the (the universal cover of the nilmanifold, the) contractible Nilpotent group. Note that the word of linear length $[x^n,[x^n,y^n]]$ is trivial in the group because $[x^n,y^m] = [x,y]^{nm}$, and thus lies in the center. This reasoning produces a bounding 2-disk, for which one easily computes the integral to be cubic. It is easy to then check Gersten's assertion that for the Heisenberg group the Dehn function has precisely cubic growth.

Remark. The cohomological method gives a good first approximation to Dehn functions, but us far from the last word. For instance [22] give examples where one can compute the Dehn function despite vanishing $H^2(\Gamma)$; I do not know whether $H^2_\beta(\Gamma)$ can be better applied in those cases, but in any case, there do not seem to be obvious cocycles to use. Gromov [58] has a lengthy discussion of both deeper upper and lower bounds, especially in the case of nilpotent groups.

9. Index Theory Again

Very similar to the manner in which HX and KX entered in the study of large scale index theory, the uniformly finite homology arises in the analysis of elliptic operators on manifolds of bounded geometry. This relation is necessary for some of the applications already mentioned (e.g., Example 1 in Section 7) and is the topic of this section.

This topic was initiated in Roe's thesis [93], which itself was motivated by the wish to remove equivariance from Atiyah's L^2 index theorem for coverings. It also shares some in spirit with Connes' foliation index theorem (see [38, 85]); our interest was spurred by the philosophical issue of whether Conne's theorem was one about foliations or really one about leaves. Our current view is that Connes' theorem really measures average phenomena about leaves, but requiring more of a "recurrence structure" than is merely encoded by the bounded geometry. In any case, the theorems here well fit into the Connes program wherein geometry is encoded by an operator algebra and indices take value in its K-theory, with geometric characteristic class theory living in the cyclic cohomology of dense subalgebras of the algebra. (See [37].)

To put the matter very bluntly, the connection roughly goes like this: On a compact connected manifold one assigns an integer to each elliptic operator, namely its index. For a disconnected compact manifold, one still has the index, but there is certainly more information in the pair of integers, namely the index of the restriction of the operator to each component. This is an element of H_0. For manifolds with bounded geometry, we have an index in H_0^{uf} and interesting information contained in the higher homology as well (the "fine theory" containing "symbol" information and the coarse theory containng "higher indices" a la the Novikov philosophy and that discussed in Section 5).

The relevant algebra for this problem is the algebra $\mathfrak{U}_{-\infty}(X)$ of smoothing operators of bounded propagation speed that are also unifirmly bounded. (N.B. A smoothing kernel need not be uniformly bounded to define a bounded operator.) The usual geometric elliptic operators will have inverses modulo the C^*-completion of this algebra, and thus define indices in the K-theory of that C^*-algebra.

Rather than pair cyclic cocycles of $\mathfrak{U}_{-\infty}(X)$ with K-theory, we take a naturally defined Chern character into HC_*, and this has a natural map into $H_*^{uf}(X;\mathbb{C})$. The computation of this class is the index theorem. Formally, the index theorem looks identical to, say, the original one in [10]. For the Dirac operator, it says:

$$A(\hat{X}) \cap [X] = \mathrm{ind} D.$$

In degree zero this is essentially a restatement of the index theorem in [93]. However, even here there are advantages to the restatement in terms of uniformly finite homology.

Corollary. *Suppose M is a spin manifold with amenable fundamental group and $\int A(M) dVol \neq 0$, then any metric g on $\tilde{M}$ strict quasi-isometric to the pullback metric has C-dense negative scalar curvature set, i.e., $N = x | s_g(x) < 0$ is C-dense.*

Proof. Suppose not. Then we can find balls of arbitrarily large size missing N. We can translate using the covering group copies of Folner (=regular) sequence into these balls. Now pair the index class with the average associated with this modified Folner sequence. The Bochner formula shows this must be 0, but, after a moments reflection one sees that this quantity is precisely $\int A(M) dVol$ and the conclusion follows. □

Remark. Even if the original Folner sequence was an exhaustion, the modified one introduced in the course of the proof was not. It is true that the index associated to any regular sequence is the same as the index associated to some other regular exhaustion; however, that exhaustion will contain parts of N, which would complicate the argument.

The higher classes enter as well in the problem of positive scalar curvature on universal covers. (Indeed it would not be hard to describe a "Gromov-Lawson-Rosenberg" conjecture for this problem.) The following theorem is interesting in this context.

Theorem. *Suppose M^n is a spin manifold with fundamental group that of a closed hyperbolic k-manifold $k > 2$. Then*

- *$\tilde{M}$ admits a complete metric of positive scalar curvature.*
- *$\tilde{M}$ has the coarse quasiisometric metric of positive scalar curvature iff ind $D = 0 \in KOX_n(M) = KO_{n-k}(*)$.*
- *$\tilde{M}$ has a strict quasiisometric metric of positive scalar curvature iff, in addition $\int \hat{A}(M) \cup \alpha\, dVol = 0$ for all closed 1-forms α. (What??)*

Addendum. *If $\int \hat{A}(M) \cup \alpha \, dVol \neq 0$, then the negative scalar curvature set of any strictly quasiisometric manifold is C-dense.*

Let us ignore the constructive issue. The exotic condition follows from the type of machinery discussed in Section 5. (See also [64], and combine the basic idea with [72].) The necessity is a consequence of pairing the index class with the cycle arising from the lift of appropriate closed geodesics from the hyperbolic manifold.

The C-density follows the same pattern as the case of amenable fundamental group. The geodesic used for detection of the index class has translates that have large intersections with the balls that would putatively be missing N.

10. Final Remarks

This paper has dealt with some of the geometry and index theory related to two particular large scale homology theories. There are many variants of these constructions, which have yet to be explored . . . .

The most obvious variant of uniformly finite homology would allow growth conditions. That is one could examine i-cycles with the property that the sum of the coefficients of the cycles that touch $B_x(r)$ is $\leq f(d(x,*))\phi(r)$ for some ϕ and function f of the distance from x to an arbitrary base point. There are different versions according to whether or not ϕ's growth type is specified.

Furthermore, nothing requires that the same function be used for different dimensional chains. One restricts attention to i-chains whose boundaries are $i-1$ allowable in defining homology.

Yet another variant just keeps track of balls around some base point, and insists that the amount of chain that one sees in $B(r)$ is bounded by $D(r)$.

We believe that all of these growth homologies, say even for polynomials, are an important invariant. It would be interesting to do the calculations even in the case of nilpotent groups. Presumably, by examining the growths for which the homologies of nilpotent groups vanish, one could gain some insight into Pansu's theorem that the dimensions of the associated graded pieces are coarse invariants.

What is nice about the growth homology (and cohomology) theories is that they seem likely to have nice connections to geometry, and to index and surgery theories. They can be used to give information to the questions raised by H^{uf}: If it is not possible to achieve a quasiisometric metric with some property, how must distortion must there in fact be in producing metrics with the given property.

In addition to these, one can, of course, develop Lp theories, and again with various weights, but here the geometric intuition is more remote. (Analytially $p = 2$ has many advantages: for some beautiful applications of this, see, e.g. [34].)

More possiblities for yet more theories arise for any quantifiable adjective. For instance the entropic method of [15] suggests building homology theories where one insists that the entropy of the cycles are not too large. Many of these possible theories will be impossible to compute; it seems that one usually needs some kind of functional analytic device for these to be useful.

Another theme that has arisen, most clearly in Sections 1,4,5 is the interconnection between the microscopic scale and the largest scale. The tangential representation of a group action is reflected in the large scale motion of a very thin quasilattice in the universal cover. For hyperbolic groups with homology spheres at ∞, one discover the local structure of any aspherical homology manifold with that fundamental group.

In the future one would hope for a large scale variant of Newtonian calculus, so that Euclidean lattices produce the conventional subject, but that other structures would yield different nonlinear best approximating functions.

Less speculatively, there are the techniques of variable conings. These can be thought of as being the microscopes that are (simultaneously) magnifying a space to many different scales, and encoding (partially) how these scales fit together to be indeed the different scales of one object. It seems certain that the small scale structure of fractals can be studied by using the growth homologies of their cones. As we saw, depending on the growth of the cone, more or less of the metric geometry of the space will be reflected in the coarse type.

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