~~1ec~~

B. Morrissey

Betti Geometric Langlands pg 1.

(Surveys Ben-Zvi-Nadler etc.).

What does GL say:

At the level of sets: (true for e.g. $GL(n)$).

$$\left\{\begin{matrix} {}^LG \text{ Local Systems} \\ \text{on } X \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{Hecke eigensheaves} \\ \text{on } Bun_G \end{matrix}\right\}.$$

For $G = \mathbb{C}^\times$:

$$\begin{tikzcd}
\left\{\begin{matrix} \text{Local system on} \\ \cancel{\text{Jacobian}}\ X \end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \left\{\begin{matrix} \text{Loc syst} \\ \text{on } Jac(X) \end{matrix}\right\} \arrow[d, phantom, "\cong"] \\
\left\{\begin{matrix} \text{Rep} \\ \pi_1(X) \to \mathbb{C}^\times \end{matrix}\right\} \arrow[r, squiggly, "\text{pushforward}"'] & \{ H^1(X,\mathbb{Z}) \to \mathbb{C}^\times \}
\end{tikzcd}$$

$X \rightleftarrows \cancel{J(X)}$

factors $\pi_1(X) \to \mathbb{C}^\times$

$\hookrightarrow H_1(X,\mathbb{Z}) \xrightarrow{\cong} H^1(X,\mathbb{Z}) = \pi_1(Jac(X))$

as $Jac(X) = \mathbb{C}^g / H^1(X,\mathbb{Z})$.

(Note: a local system on the Jacobian extends uniquely to a Hecke eigensheaf on $Pic(X)$).

~~At~~ ~~But~~ These objects fit into various natural categories.

aim: upgrade equivalence of sets to equivalence of categories commuting with natural functors.

~~De-Rham.~~	DeRham.	Betti	Physics KW.
Spectral.	$\mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{LG})$	~~Shv_N~~ $QC^!_N(\mathcal{L}oc_{LG}(X))$	B-branes on ~~LocSys~~ LocSys_{LG} (w/ [illegible] [illegible]).
automorphic	$D\text{-mod}(\mathrm{Bun}_G)$	$Shv_N(\mathrm{Bun}_G(X))$	~~A-branes~~ A-brane on $\mathrm{Hitch} \cong T^*\mathrm{Bun}_G$.
Wilson Loops, μ - highest weight	$(- \otimes (L_G \times_{G} V_\mu))$ L_G canonical global object over LocSys_{LG}.	$QC^!(\mathcal{L}oc_{LG}(X))$ $\simeq \mathrm{Ind}(\mathrm{Coh}(\mathfrak{g}^\vee[-1]/G^\vee))$ Perf at [illegible] $\cong \mathrm{Perf}(BG^\vee) \cong \mathrm{Rep}(G^\vee)$. $\simeq Z(S^2)$.	~~Add $L_G \times_G V_\mu$ as Chan-Paton bdle~~ Wilson loop $\times$ $[0,1]\times\mathbb{R}^+\times X$ at [illegible] $\mathbb{R}^+$ [illegible] $a \to 0$ [illegible] bdy cdts
Hecke operators	$\mathrm{Bun}_G \xleftarrow{p_1} \mathrm{Hecke} \xrightarrow{p_2} \mathrm{Bun}_G \times X$ $\Rightarrow p_{2!}(p_1^*(\mathcal{F}) \otimes IC_\mu)$	$\mathrm{Bun}_G(X_-) \leftarrow \mathrm{Bun}_G(X(x)) \rightarrow \mathrm{Bun}_G(X_+)$.	't - Hooft op in same situation.
Ramification.	$\mathrm{Bun}_G \rightsquigarrow \mathrm{Bun}_{G,P}$ $\mathrm{LocSys}_{LG} \rightsquigarrow \mathrm{LocSys}_{G,P}$	• Surface operators/ pt operators on Surfaces $\int_{S^2} \mathrm{Rep}(U_q(\mathfrak{g}))$-mod & same on other side.	Surface operators $[0,1]\times\mathbb{R}^+\times x$ require HE sol to YM eqs to have singularity at specified locus.
"Functoriality" (partial)	Geom Eisenstein series $M \subset P \subset G$ levi parabolic $\mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \rightarrow \mathrm{Bun}_M$ $\mathrm{LocSys}_{LG} \leftarrow \mathrm{LocSys}_{LP} \rightarrow \mathrm{LocSys}_{LM}$	Domain Walls between thy for M & thy for G. ? $\mathrm{Rep}\, U_q(\mathfrak{g})$ ⟲ tensor & [illegible] [illegible] $\mathrm{Rep}(U_q(\mathfrak{g}))$ $\mathrm{Rep}(U_q(\mathfrak{m}))$ & ? on other side $\mathcal{L}oc_{LG} \leftarrow \mathcal{L}oc_{LP} \rightarrow \mathcal{L}oc_{LM}$.	Domain walls.

Betti Geometric Langlands pg 2.

Spectral

$$\mathrm{Loc}_{G^\vee}(S) = R\mathrm{Hom}(S, BG^\vee)$$

- We consider S, $BG^\vee$ as top spaces & then as derived stacks. ($R \mapsto S$ $\forall R \in$ cdga$^{\leq 0}$).
- We take Hom in derived stacks
  - Has act of $\pi_0(\mathrm{Diff}(S))$.

Coboundary gen compounds

$$\mathrm{Loc}_{G^\vee}(S) \simeq (\underbrace{\mathrm{Rep}_{G^\vee}(S \setminus s)}_{= \mathrm{Hom}(\pi_1(S\setminus s), G)} \times_{G^\vee} \{e\}) / G^\vee$$

[then a stack with $G^\vee$-loc systs trivialized at a base pt, then nearly average $s = e$, & forget trivialisation by quotient]

→ With bdry:

$$\mathrm{Loc}_{G^\vee}(S, \partial S) = [(S, \partial S), (BG^\vee, BB^\vee)].$$

→ Twisted gps $1 \to G^\vee \to G' \to \mathbb{Z}/2 \to 1$.

$$\mathrm{loc}_{G^\vee, \sigma}(S) = \mathrm{loc}_{G'}(S) \times_{\mathrm{loc}_{\mathbb{Z}/2}(S)} (\{\tilde{S}\}).$$

Example

- (abelian case) $\mathcal{L}oc_{T^\vee}(S) \simeq \mathbb{B}T^\vee \times (T^\vee \otimes_{\mathbb{Z}} H^1(S, \mathbb{Z})) \times \mathfrak{t}^\vee[-1]$

$(\mathfrak{t}^\vee[-1] \simeq \{e\} \times_{T^\vee} \{e\}$
$= \mathrm{Spec}\,\mathrm{Sym}(\mathfrak{t}[1]))$

- $S = S^2 \simeq D^2 \coprod_{S^1} D^2$

$$\mathcal{L}oc_{G^\vee}(S^2) \simeq \{e\}/G^\vee \times_{G^\vee/G^\vee} \{e\}/G^\vee \simeq \mathfrak{g}^\vee[-1]/G^\vee.$$

- $\mathcal{L}oc_{G^\vee}(S^3) \simeq \mathfrak{g}^\vee[-2]/G^\vee$

- $\mathrm{Cyl} = S^1 \times [0,1]$.

$$\mathcal{L}oc_{G^\vee}(\mathrm{Cyl}, \partial \mathrm{Cyl}) \simeq St_{G^\vee} = B^\vee/B^\vee \times_{G^\vee/G^\vee} B^\vee/B^\vee$$

(Ben-Zvi–Nadler–Steinberg)

$$\mathcal{L}oc_{G^\vee}(T^2) = \{g, h \in G^\vee : gh = hg\}/G^\vee.$$

$\mathrm{Diff}(T^2) \curvearrowleft$

$T^2 \rtimes \underbrace{SL_2(\mathbb{Z})}_{\text{pairs products of powers of } g, h}$ is

-  $\mathcal{L}oc_{G^\vee}(S, \partial S) = \{g, h \in G^\vee, B_1, B_2, B_3 \in G^\vee/B^\vee : g \in B_1, h \in B_2, gh \in B_3\}/G^\vee$.

Möbius strip $\rightsquigarrow$ Langlands param space of "Loop Spaces & Representations".

$$QC^! = \mathrm{Ind}(\mathrm{Coh}(\mathcal{L}oc_{G^\vee}(S)))$$

[cf with ☆.
BBJ gives
$QCoh(\mathcal{L}oc_{G^\vee}(S))^\heartsuit$
$= (QC^!_{\mathrm{log}}(\mathcal{L}oc_{G^\vee}(S)))^\heartsuit$
$\Rightarrow$ What to modify to $\mathrm{Rep}(G_q)^\heartsuit$ to $\mathrm{Rep}(U_q(\mathfrak{g}))$ [ ].
$\Rightarrow$ Q. Does its gen cannot tys

Arinkin–Gaitsgory:

$\mathcal{L}oc_{G^\vee}(S)$ is a global complete intersection.

☆ Why.

$$A = \mathrm{Sym}(\mathfrak{g}^\vee[-2])^{G^\vee} \simeq \mathrm{Sym}(\mathfrak{h}^\vee[-2])^W$$

acts by endomorphisms on $\mathrm{Id} : QC^!(\mathcal{L}oc_{G^\vee}(S)) \to QC^!_{\ldots}(\mathcal{L}oc_{G^\vee}(S))$

$\Rightarrow$ endomorphisms are an A-algebra

$\Longrightarrow$ Sing support $\mathrm{supp}_A(\mathcal{F}) \subset \mathfrak{g}^\vee /\!/ G^\vee \simeq \mathfrak{h}^\vee /\!/ W$

Can take the nilpotent cone in both.

Betti GL (pg 3.).

Automorphic side

Notation: $Shv(Z)$ = dg cat of shvs on Z_{an} – complex analytic stack.

$Shv^c(Z)$ = cstable shvs.

$\Lambda \subset T^* Z$.

$Shv_\Lambda(Z)$ = singular support $\subset \Lambda$. (Z smooth).

$$T^* Bun_G(X) \cong Hitch(X)$$
$$\cong \{(E, \varphi) \mid E \in Bun_G(X), \ \varphi \in \Gamma(X, ad(E) \otimes K_X)\}$$
$$\downarrow h_X$$
$$A_G(X) = \{\varphi \in \Gamma(\mathfrak{g}^*/\!/G \otimes K_X)\}$$

Nilp cone $\mathcal{N}_{X,G} = h_X^{-1}(\{0\}) \subset T^* Bun_G(X)$.

Conic, Lagrangian.

$Shv_{\mathcal{N}}(Bun_G(X))$ is the dg cat. of shvs with singular support in global nilpotent cone.

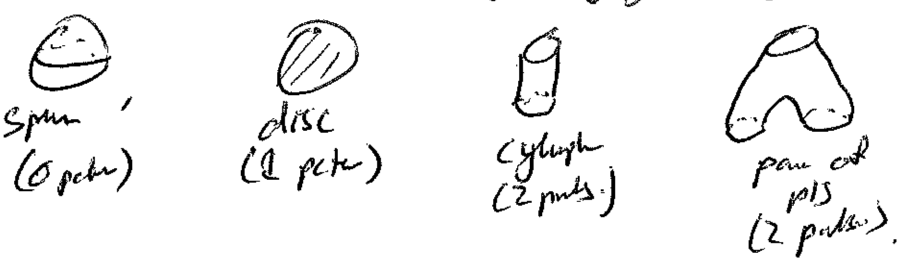
Betti GL

pg 4.

Betti Spectral Gluing

We want to describe $QC^!_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(S))$, $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(X))$ as what a some TFT assigns to a surface.

We do this by gluing together



(Aim: prove GL on these & glue.)

Again 2: We will define
$$(X, \partial X) \longmapsto \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(X, \partial X))$$
as a (2,1) TFT! (valid in all cases),

Conjecture: $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(X))$ depends only on topological structure of X.

(true for an elliptic curve).

Conjecture: Further we can degenerate to nodal curves and compute:

($T^2 \rightsquigarrow 2\ S^2$'s w/ marked pts).

(Nearby gluing at the pts)

Conjecture: Can define a ($\cancel{3,2,1}$ - fully extended 3dTFT) assigning ? to a pt

<u>Note</u>: ~~For~~
Betti GL holds for disc & cylinder.
[Bezrukavnikov]
& holds for $\mathbb{P}^1$. (Gurge?)
& for GL_2, SL_2, PGL_2 for 
[Bezrukavnikov]

Other side.

$$Z(S^1) := \mathcal{H}_{{}^L G} \quad \text{affine Hecke category.} = \overbrace{DCoh(St_{{}^L G})}^{[\text{Bezrukavnikov}]} \; [\text{on cylinder}] = \mathrm{End}_{\mathrm{Perf}(St_G)}(\mathrm{Perf}(B_G)) \; [\text{BNP}]$$

$= \cancel{D(G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x))}$ (not the an Cpt object in cat. natural enduring)

<u>Thm.</u> [1.1. in BZN [Betti Spectral gluing]].

Let. $\tilde{S} = S/\sim$ where $\sim$ identifies two circles in body of S.

Thm: $$D_{coh_N}(\mathcal{L}oc_G(\tilde{S}, \partial\tilde{S})) \simeq \mathcal{H}_G \otimes_{\mathcal{H}_G \otimes \mathcal{H}_G^{op}} DCoh_N(\mathcal{L}oc_G(S, \partial S))$$

[respecting commuting Hecke symmetries].

<u>Sketch of proof.</u> [Betti Spectral Gluing], Ben-Zvi–Nadler,

$$\star \quad \mathcal{L}oc_G(\tilde{S}, \partial\tilde{S}) \simeq \mathcal{L}oc_G(S, \partial S) \underset{G \times G}{\times} G. \quad \left(\text{taking monodromy of both } S^1\text{'s}\right).$$

~~D_{Coh_N}~~ ~~$DCoh_N$~~ $QC^!_N(\mathcal{L}oc_G(\tilde{S}, \partial\tilde{S}))$ is naturally a module

on $\mathrm{Perf}(\mathcal{L}oc_G(\tilde{S}, \partial\tilde{S})) \simeq \mathrm{Perf}(\mathcal{L}oc_G(S, \partial S)) \underset{\mathrm{Perf}(G \times G)}{\times} \mathrm{Perf}(G)$.

affine Hecke cat $\mathcal{H}_G \simeq DCoh(St_G) \simeq DCoh_N(\mathcal{L}oc_G(Cyl, \partial Cyl))$.
has natural struct.

$\mathcal{H}_G \simeq \mathrm{End}_{\mathrm{Perf}(G)}\mathrm{Perf}(B)$.

A naturally gets $\circ$ $QC^!_N(\mathcal{L}oc_G(S, \partial S))$ a $\mathcal{H}_G^{\otimes}$-modules (?)

$$\mathrm{Tr}(\mathcal{H}_G, \phi QC^!) = QC^!(\mathcal{L}oc(Cyl)) \otimes_{\mathcal{H}_G \otimes \mathcal{H}_G^{op}} \mathcal{H}_G.$$

is a $\mathrm{Perf}(\mathcal{L}oc_G(\tilde{S}))$ module.

We identify them as modules. $\star$.

Betti GL pg 5

Wilson Loops & Hecke Modifications

$$S^2 \simeq D^2 \coprod_{S^1} D^2$$

line operators
(4d $\Rightarrow$ around a line op there is a sphere).

$$\mathrm{Loc}_{G^\vee}(S^2) \simeq \mathfrak{g}^\vee[-1]/G^\vee$$

$$\mathrm{Loc}_{G^\vee}(S^2)_{cl} \simeq pt/G^\vee \simeq BG^\vee$$

$$\Rightarrow \mathrm{Rep}(G^\vee) \simeq \mathrm{Perf}(BG^\vee) \longleftrightarrow \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S^2))$$

$$\searrow^{\simeq} \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S^2))^{\heartsuit} \nearrow$$

why.

The full $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S^2))$ is the category in <u>derived Geometric Satake</u> [Bezrukavnikov-Finkelberg, Arinkin-Gaitsgory].

<u>State</u>: ~~line operators should form an E_3 algebra as we~~

<u>Alg structure</u>: project down one dims $\Rightarrow$ Line ops have an E_3-algebra structure.

Wilson line <u>at a pt.</u> have an E_1-algebra structure.

$\Rightarrow \mathrm{Loc}_{G^\vee}(S)$ has E_1-action of $\mathrm{Loc}_{G^\vee}(S^2)$ in corresponding context of derived stacks.

$\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S))$ is E_1-module over $\mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S^2))$

Also local ops.. derived functions $\mathcal{O}(\mathrm{Loc}_{G^\vee}(S^3)) \simeq \mathrm{Sym}((\mathfrak{g}^\vee)^*[2])^{G^\vee}$
$\leadsto E_4$-algebra. $\simeq \mathrm{Sym}(\mathfrak{h}[2])^{W}$,

Note $\mathrm{Spec}(\mathfrak{h}^*/\!/W)$ is Coulomb branch $\simeq$ moduli space vacua of $N=4$ SYM.
Higgs supt $\sim \mathcal{O}_C\, \mathfrak{h}^\vee/\!/W$.

Hecke modification:

$x \in X$. $\quad D_x = \mathrm{Spec}\ \mathcal{O}_x$
$D_x^\times = \mathrm{Spec}\ \mathcal{K}_x$.

$X_\pm = X$,
$X(x) = X_- \coprod_{X \setminus \{x\}} X_+$.

$$\mathrm{Bun}_G(X_-) \leftarrow \mathrm{Bun}_G(X(x)) \rightarrow \mathrm{Bun}_G(X_+)$$

~~Kernels for the composition~~
Objs are equivalent to $\mathrm{Gr}_G = \{(E, \beta)\ E$ G-bundle on D_x with β triv on $D_x^\times\}$

$\Rightarrow$ Kernels for transfer of sheaves are $G(\mathcal{O})$ equivt sheaves on Gr_G.

<u>Recall</u>: Geometric Satake.
$$\mathrm{PervShf}(\mathrm{Gr}_G)^{G(\mathcal{O})} \cong \mathrm{Rep}(G^\vee).$$

<u>Derived enhancement</u>:
$$\mathrm{Sh}_{\nu_c}(G(\mathcal{O}) \backslash \mathrm{Gr}_G) \simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(S^2)).$$

[subtlety: spin structure].

<u>Cptn</u>: This preserves nilp sheaves & is locally const upon varying $x \in X$.

Betti GL pg 6.

Vershik (type) Grps:

pair of pants $\times S^1$ gives E_2-alg structure on $\mathcal{Z}(T^2)$.

$$\mathcal{Z}^{\mathrm{gr}}(T^2) \simeq \underbrace{HH}_{\text{Hochschild homology}}(\mathcal{H}_{G^\vee}) \simeq \mathrm{Coh}_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(T^2)).$$

Sketch proof: [BNP]?

Closed from S^1 compactification

One or knows that $\mathcal{Z}(\mathrm{Cyl}) = \mathcal{H}_{G^\vee}$.

§ S^1-Localization

$S^1 \curvearrowright$ Cyl, Mob, T^2.

Equivariant localization. [BN3, Loop spaces & representations]

- $\mathrm{Coh}(St^{\hat{u}}_{G^\vee}) \rightsquigarrow D_{\mathrm{coh}}(B^\vee \backslash G^\vee / B^\vee)$.
- $\mathrm{Coh}(\mathcal{L}a^{\hat{u}, \theta^\vee}_{G^\vee}) \rightsquigarrow \bigoplus_{\sigma \in \Sigma} D_{\mathrm{coh}}(K^\vee_\sigma \backslash G^\vee / B^\vee)$
- $\mathrm{Coh}(\mathrm{Loc}^{\hat{u}}_{G^\vee}(T^2)) \rightsquigarrow D_{\mathrm{coh},\mathcal{N}}(G^\vee / G^\vee)$.

3d exten

Cjctr. :

rigorous,

$$\begin{cases} pt \rightsquigarrow (QC(BG^\vee)\text{-mod})\text{-mod}. \\ M^1 \rightsquigarrow QC(Loc_{G^\vee}(M^1))\text{-mod}. \\ M^2 \rightsquigarrow QC(Loc_{G^\vee}(M^2)). \end{cases}$$

$$M^3 \rightsquigarrow \mathcal{O}(Loc_{G^\vee}(M^3)).$$

Analogue: This geometric structure comes from Ben-Zvi–Francis–Nadler; [Integral Transforms & Centers in DAG].

- For a perfect stack X, Z_X (i.e. $QC(X) = \mathrm{IndPerf}(X)$ & diagonal is affine). $QC(X)$ is an E_∞-monoidal cat so we get $\simeq$ 2-cat

or $QC(X)$-mod fully dualizable.

$$Z_X(pt) = QC(X).$$
$$Z_X(S^1) = HH(QC(X)) = QC(\mathcal{L}X).$$
$$Z_X(\Sigma) = \Gamma(X^\Sigma, \mathcal{O}_{X^\Sigma}).$$
$$X^\Sigma = \mathrm{Map}(\Sigma, X).$$

Rmk (Conjecture) 3.2.9. To fully construct 2d TFT.

$Z(S^1) = \mathcal{H}_{G^\vee}$-mod to a circle: should be equivalent to $2\mathrm{Coh}_N(G^\vee/G^\vee)$.

3 cat on a pt conjecture: $3\mathrm{Coh}_N(BG^\vee)$: cats w/ action dig, props on $BG^\vee$, w/ nilp sing support,

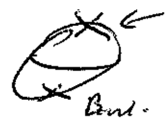
Betti GL   pg ~~6~~ 7

Ramification operators

Spectral: On a surface ⇝ classified by $Z(S^1)$.

for an annular disk

S with marked pt (in Borel).

glue on  ← other body cdt..

$= Z($$) = \mathcal{H}_{G^\vee} = \mathrm{Coh}(St_{G^\vee})$

unipotent Borel reduction:

$$Loc^u_{G^\vee}(Cyl, \partial Cyl) \simeq St^u_{G^\vee} = \tilde{N}^\vee/G^\vee \times_{G^\vee/G^\vee} \tilde{N}^\vee/B^\vee,$$

$$\mathcal{H}^u_{G^\vee} = \mathrm{Coh}(St^u_{G^\vee}).$$

Given $\mathcal{H}_{G^\vee}$-module $\mathcal{M}$.

$$Spec_{G^\vee}(S, \partial S, \mathcal{M}) = \mathrm{Coh}(Loc_{G^\vee}(S, \partial S)) \otimes_{\mathcal{H}_{G^\vee}} \mathcal{M}.$$

for $Z \to G^\vee/G^\vee$ stack (nice).

$$Loc_{G^\vee}(S, \partial S, Z) = Loc_{G^\vee}(S) \times_{Loc_{G^\vee}(S^1)} Z.$$

Can specify nilp sing support.

Corresponds to $\mathcal{M} = \mathrm{Coh}(Z)$.

Note: Could use Z a wild character variety on a cylinder.

[Stokes data].

Automorphic:

Consider $Bun_{G,}(X, x)$

G-bundles on X, B-reduction at x.

Corresponds

$Bun_G(X_-, x_-) \leftarrow Bun_G(X(x), x_- \cup x_+) \rightarrow Bun_G(X_+, x_+)$

now have fibers as $Fl_G = G(\mathcal{K})/I$

$I \subset G(\mathcal{O})$ the Iwahori subgrp.

Betti Geometric Langlands

~~pg~~ 8: Betti GL for , , 
(&  for SL_2, PGL_2.) [skip this].

Doc.

 or 

From Bezrukavnikov, Arkhipov–Bezrukavnikov

Stated module for Hecke cat.

$$\mathrm{Shv}^{u}_{\mathcal{N}}\big(\underbrace{\mathrm{Bun}_G(\mathbb{P}^1, 0)}_{\substack{\text{for semisimple} \\ = G(\mathcal{O})\backslash G(K)/I}}\big) \simeq QC^!_{\mathcal{N}}\big(\underbrace{\mathrm{Loc}_{G^\vee}(D, S^1)}\big)$$

I Iwahori.

$$= D^2 \amalg_{S^1} \text{stacky}$$
$$= \{e\}/G^\vee \times_{G^\vee/G^\vee} B^\vee/B^\vee$$

For  or 

Added module for Hecke cat.

$$\mathrm{Shv}^{u}_{\mathcal{N}}\big(\underbrace{\mathrm{Bun}_G(\mathbb{P}^1, 0, \infty)}_{\substack{\text{for semisimple} \\ = I\backslash G(K)/I}}\big) \simeq QC^!_{\mathcal{N}}\big(\underbrace{\mathrm{Loc}_{G^\vee}(Cyl, \partial Cyl)}\big)$$

$$= St_{G^\vee}$$
$$= B^\vee/B^\vee \times_{G^\vee/G^\vee} B^\vee/B^\vee$$
$$= \{g \in G^\vee,\ B_1, B_2 \in G^\vee/B^\vee,\ g \in B_1 \cap B_2\}/G^\vee$$

$\exists$ proofs: see ~~Ha~~ Nadler–Yun.

This is Bezrukavnikov's ramified geometric Satake

monodromic nearby cycles & Springer theory.

Betti GL pg 9.

Betti Class field theory.

(§4.3) of Betti GL.

$G = T \qquad G^\vee = T^\vee.$

$\Lambda = \mathrm{Hom}(\mathbb{C}^\times, T)$. cocharacters.

Aut: $\mathrm{Bun}_T(X) \simeq \mathrm{Pic}_T(X)^0 \times BT \times \Lambda$.

nilpotent cone $\mathcal{N} = \{0\} \subset T^*\mathrm{Bun}_T(X)$ is the zero section.

$\Rightarrow$ automorphic category is local systems.

?

Spectral: $\mathcal{L}oc_{T^\vee}(X) = \mathrm{Hom}(\pi_1(X), T^\vee) \times BT^\vee \times \mathrm{Spec}\,\mathrm{Sym}\,\mathfrak{t}[1]$.

$QC^!(\mathcal{L}oc_{T^\vee}(X)) = QC(\mathcal{L}oc_{T^\vee}(X))$

Aut cat graded by $\Lambda = \pi_0(\mathrm{Bun}_T(X))$.

Spectral cat graded by $\Lambda = K_0(BT^\vee)$.

$\{\text{Local systs on } BT\} \cong \{\text{Complete Modules on } H^*(BT)\} \underset{\text{Koszul}}{\cong} \{\text{QCoh Mod } \mathrm{Sym}\,\mathfrak{t}[1]\}$

$H^*(BT) = \mathrm{Sym}\,\mathfrak{t}^*[-2]$

$\{\mathrm{Loc\,Sys}\ \mathrm{Pic}_T(X)^0\} \cong \{k[\pi_1(\mathrm{Pic}_T(X)^0)]\text{-mod}\}$

$\cong \{k[H_1(X) \otimes \Lambda]\text{-mod}\}$

$\cong \mathrm{QCoh}(\mathrm{Hom}(\pi_1(X), T^\vee))$

$\cong \mathrm{QCoh}(T^\vee \otimes H^1(X))$

$\cong \mathrm{QCoh}(\cancel{\mathrm{Hom}\,k}\ \mathrm{Spec}(k[H_1(X) \otimes \Lambda]))$

[i.e. as in deRham the two match up).

ue

KVS — sketch. (pg 10.)

(Kapustin's talk at ICM). KRS, KR, KVS.

RW thy: 3d Σ-model, target X

RW on $S^1 \times \Sigma$.
gs (twisted?) B-model w/ target X.

$$RW(S^1) = D_{\mathbb{Z}_2}(Coh(X))$$
$\mathbb{Z}$-graded cplxs of coherent sheaves.

braided monoidal structure (for TFT) on $RW(S^1)$ is a deformation of the normal one.

$RW(pt)$ complicated but some morphisms are matrix factorizations.

deformation described by $\bigoplus_{p=2}^{\infty} \Gamma(Sym^p(TY))$.

~~$RW(pt)$~~ Target T^*Y:
$$RW(pt) = D^b(Coh(Y))\text{-Mod}.$$

Reduce GL twist on S^1.

$F(G,t,X)$ family of cats. of surface ops on X.

$t = i$: $G_{\mathbb{C}}$-equiv $\mathbb{Z}$-graded RW thy
target: $T^*G_{\mathbb{C}} \circlearrowleft_{\text{conjugation}} G_{\mathbb{C}}$

$$Bran = D^b(Coh(G_{\mathbb{C}}/G_{\mathbb{C}}))\text{-Mod}.$$

$t = 1$: Same sort of module cats for Fukaya–Floer category.

~~Gener~~

Betti GL   pg 11.

- Generalizations: Real Betti Langlands & Quantum Geom Langlands.

1. Real Betti Langlands:

(X, α) real form of X.

(G, θ) (quasi-split) real form of G.

$Bun_{G,\theta}(X, \alpha)$. G bdl on X identified with pullbacks under α, θ.

$Y \subset X$ α-inv.

ramified: $Bun_{G,\theta}(X, Y, \alpha)$.

(G, θ) defs L-grp $G^L \cong G^\vee \rtimes \underbrace{Gal(\mathbb{C}/\mathbb{R})}_{\cong \mathbb{Z}/2}$.

acts by $\theta^\vee$ alg inv. assoc. to conjugation.

Conjecture (Thm cpt grp, comm. real, Torus, SL_2, PGL_2) Affine Langlands-Vogan-Soergel Duality

$$Shv_N(Bun_{G,\theta}(\mathbb{P}^1, 0, \infty), \alpha)) \simeq QC^!_N(Loc_{G^L, \sigma}(\text{sub}, S^2))$$

(including Hecke symms).

Note: S^2-loc. gives a fam of Soergel cycls in local GL

$\longrightarrow$ Qtzn.: The are quantum versions.

$Shv_N \rightsquigarrow$ nonabelian shs = shs twisted by $\mathbb{C}^\times$ gerbe det$^{\log q}$.

Same with ~~Shv~~ . $QC^!_{q,N}$.

Supplement: BBBJ $\longrightarrow$ QCoh(Rep(G))

S° surface w/ 1 distinguished ∂ boundary component with $p \in \partial S^\circ$.

rep variety: $R_G(S^\circ) = \{\rho : \pi_1(S,p) \to G\}$

G-char stack: $\underline{Ch}_G(S^\circ) = R_G(S^\circ)/G$

$$\mu : R_G(S^\circ) \longrightarrow R_G(Ann) \quad (G \text{ equiv.})$$

$C \subset G$ conjugation invariant subvariety of G.

$$\underline{Ch}_G(S, C) = \mu^{-1}(C)/G$$

= "moduli stack G-local systems w/ monodromy in C".

Thm 6.1 of BBJ4.

$$QCoh(\underline{Ch}_G(S)) \simeq \int_S Rep(G). \quad (S \text{ punctured surface})$$

Mention: $\underline{Ch}_G(S) \cong Map(S, BG)$.

Because:

$$\underline{Ch}_G(S) \cong [G^{2g+n-1}/G]$$

$\Rightarrow n > 0$ punctures, $\Rightarrow g$ - genus. quot = diag. adj action.

$$\Rightarrow QCoh(\underline{Ch}_G(S)) \cong \cancel{\mathcal{O}\, Rep(G)}\ \cancel{QCoh}\ QCoh(G^{2g+n-1} \times BG)$$

$$\simeq \mathcal{O}(G)^{2g+n-1}\text{-mod}(Rep(G)).$$

So ~~St~~ $\int_S Rep(U_q(\mathfrak{g}))$ should be seen as sheaves on a "quantum char variety".

Now. $\mathrm{QCoh}(\underset{\substack{C \subset G \\ \text{adjt inv.}}}{\underset{\sim}{C}}/G)$ is braided module cat for $\mathrm{Rep}(G)$.

$$\mathrm{QCoh}(\mu^{-1}(G)/G) = \int_{(S, \not{\mathbb{C}} x)} (\mathrm{Rep}(G), \mathrm{QCoh}(C/G)).$$

Here the link to Quantum Langlands.

Singular Support Supplement.

~~Recall~~

Let Z be a locally complete intersection.

Recall that we have various categories of sheaves on Z.

$\mathrm{IndCoh}(Z)$ cptly gen by $\mathrm{Coh}(Z)$

$\mathrm{QCoh}(Z)$ cptly gen by $\mathrm{Perf}(Z)$.

- Idea of ~~cpt supp~~ Sing support.

  " In which directions can one resolve kinds of coherent E as cplx of vect bdles.

- $$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "\phi"] \\ pt \arrow[r] & V \end{tikzcd}$$ ($\mathcal{U}, V$ smooth).

  of a pt can replace V with tangent

  $\Rightarrow$ assume $V = V^{sp}$.

$$V^* \times \mathcal{U} \underset{\mathbb{A}^1}{\times} pt$$
$$(v,u) \mapsto \langle \phi(u), v\rangle$$

$$\mathrm{sing}(H) = \mathrm{Sing}(Z).$$
$$\mathcal{H} = H/\underbrace{\mathbb{G}_m}_{\text{acts by dilation}}$$

$\mathrm{IndCoh}(Z) \simeq \mathrm{IndCoh}(\mathcal{H})/\mathrm{QCoh}(\mathcal{H})$.

$$\mathcal{F} \in \mathrm{IndCoh}(Z) \rightsquigarrow \mathcal{F}' \in \mathrm{IndCoh}(\mathcal{H})/\mathrm{QCoh}(\mathcal{H})$$

$$\mathrm{SingSupp}(\mathcal{F}) = \mathrm{supp}(\mathcal{F}') \subset \mathrm{Sing}(Z) \subset V^* \times \mathcal{U}.$$

Note $\mathrm{Sing}(Z) = {}^{cl}\left(\mathrm{Spec}_Z(\mathrm{Sym}_{\mathcal{O}_Z}(T(Z)[1]))\right)$
$$= \{(z, \xi) \mid z \in Z(k),\ \xi \in H^{-1}(T_z^*(Z))\}$$

pt2.

What is singular support of $\mathcal{L}oc$?

~~Note.~~

Note. $\{e\} \times_{G^\vee} \{e\} = \mathrm{Spec}\ \mathrm{Sym}(\mathfrak{g}[1])$.

$\Rightarrow \cancel{\{e\} \times_{G^\vee} \{e\} / G^\vee \leq}$

This gives the result.