BASE CHANGE

BENEDICT MORRISSEY

Contents

1. Base Change Matrices

A *Basis* $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_n\}$ of a (real) vector space V gives us a map

$$\Phi_{\mathcal{B}} : \mathbb{R}^n_{\mathcal{B}} \to V$$

$$\Phi((b_1, b_2, ..., b_n)^T) = b_1\vec{v_1} + b_2\vec{v_2} + ... + b_n\vec{v_n}.$$

Exercise 1.1. Show that $\Phi_{\mathcal{B}}$ is a linear map.

Remark 1.2. The footnoted $\mathcal{B}$ is merely a piece of notation to allow ourselves to keep track of multiple copies of $\mathbb{R}^n$

Note that $\Phi^{-1}(\vec{0}) = 0$ because $\mathcal{B}$ is linearly independent. Furthermore as $\mathcal{B}$ spans V we have that the image of Φ (sometimes called the range or column space) is all of V.

Remark 1.3. We often call $\Phi^{-1}(\vec{0}) = 0$ the *Kernel* of the map $\Phi_{\mathcal{B}}$

We can hence define an inverse map $\Psi_{\mathcal{B}}$ by defining $\Psi_{\mathcal{B}}(\vec{v})$ to be the unique vector $(b_1, ..., b_n)$ such that $\vec{v} = b_1\vec{v}_1 + ... + b_n\vec{v}_n$.

Note: This is well defined because as the Kernel of $\Phi_{\mathcal{B}}$ was $\vec{0}$, and the image was V.

Furthermore this map is linear, because if $\Psi_{\mathcal{B}}(\vec{w_1}) = (b_{11}, b_{12}, ..., b_{1n})^T$, and $\Psi_{\mathcal{B}}(w_2) = (b_{21}, \vec{b}_{22}, ..., b_{2n})^T$, then

$$(b_{11} + b_{21})\vec{v_1} + ... + (b1n + b_{2n})\vec{v_n} = (b_{11}\vec{v_1} + ... + b_{1n}\vec{v_n}) + (b_{21}\vec{v_1} + ... + b_{2n}\vec{v_n}) = \vec{w}_1 + \vec{w}_2.$$

That is to say $\Psi_{\mathcal{B}}(\vec{w}_1 + \vec{w}_2) = \Psi_{\mathcal{B}}(\vec{w_1}) + \Psi_{\mathcal{B}}(\vec{w_2})$.

Exercise 1.4. Show that $\Psi_{\mathcal{B}}(a\vec{w}) = a\Psi_{\mathcal{B}}(\vec{w})$, or $\vec{w} \in V$, and $a \in \mathbb{R}$.

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Proposition 1.5. $\Psi_{\mathcal{B}}$ *is a Linear map.*

Furthermore we have that $\Psi_{\mathcal{B}}$ and $\Phi_{\mathcal{B}}$, are inverses, that is to say:

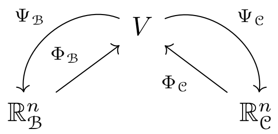
Exercise 1.6.

$$\Psi_{\mathcal{B}} \circ \Phi_{\mathcal{B}} = Id_V, \quad \Phi_{\mathcal{B}} \circ \Psi_{\mathcal{B}} = Id_{\mathbb{R}^n},$$

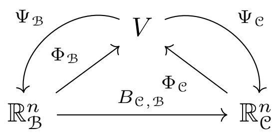
where Id_V, and $Id_{\mathbb{R}^n}$ are the identity functions on V and on $\mathbb{R}^n$ (which we will sometimes denote $\mathbb{R}^n_{\mathcal{B}}$) respectively.

Now suppose that we have two Bases $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_n\}$, and $\mathcal{C} = \{\vec{u}_1, ..., \vec{u}_n\}$, applying the above process to both $\mathcal{B}$ and $\mathcal{C}$.

We hence have a diagram



The base change map from $\mathcal{B}$ to $\mathcal{C}$ is the map $B_{\mathcal{C},\mathcal{B}}$ such that the below diagram commutes (that is to say, such that if we take any two paths along arrows in the diagram, which take us between two points, then composing the functions along each path gives us the same function, e.g. in the below graph we must have $B_{\mathcal{C},\mathcal{B}} = \Psi_{\mathcal{C}} \circ \Phi_{\mathcal{B}}$. In particular this shows us there is a base change map, and it is linear [as composition of linear maps are linear]).



Exercise 1.7. The above condition means that $B_{\mathcal{B},\mathcal{C}} = B^{-1}_{\mathcal{C},\mathcal{B}}$.

The *base change matrix* from basis $\mathcal{B}$ to $\mathcal{C}$ is the matrix corresponding to the linear map $B_{\mathcal{C},\mathcal{B}}$, and the standard bases of $\mathbb{R}^n_{\mathcal{B}}$ and $\mathbb{R}^n_{\mathcal{C}}$.

Example 1.8. Let V be the vector space of polynomials of degree less than or equal to 1, and let $\mathcal{B}$ be the basis $(1, x)$, and $\mathcal{C}$ be the basis $(1, (x-1))$.

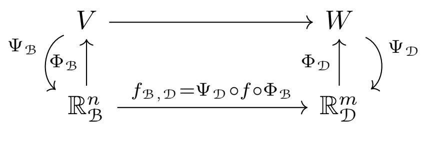
We then have that as a matrix $B_{\mathcal{C},\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Note that the i^{th} *columns* of this matrix correspond to what one gets by writing the i^{th} element of the basis $(1, x)$ in terms of the basis $(1, x-1)$, and taking the coefficients. E.g. The 2nd columnn is $(1, 1)^T$, because $x = 1 \cdot 1 + 1 \cdot (x-1)$.

Viewpoint 2: The Base change matrix $B_{\mathcal{C},\mathcal{B}}$ is the matrix whose i^{th} column is $(b_{1i}, ..., b_{ni})$, where

$$\vec{v_i} = b_{1i}\vec{u}_1 + ... + b_{ni}\vec{u_n}.$$

1.1. **Writing a matrix in a different Basis.** We now suppose that we have a linear map $f : V \to W$. Furthermore we have bases of $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_n\}$ of $\mathcal{B}$, and $\mathcal{D} = \{\vec{w}_1, ..., \vec{w}_m\}$ of W. The *matrix* of f with respect to the bases $\mathcal{B}$, and $\mathcal{D}$ is the matrix $A_{f,\mathcal{B},\mathcal{D}}$ corresponding to the linear map $f_{\mathcal{B},\mathcal{D}}$ making the below diagram commute:



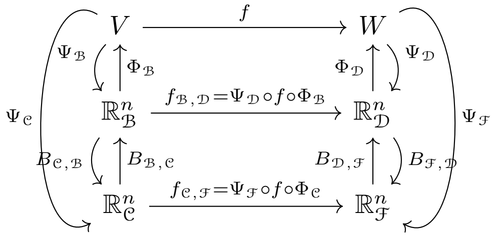
Or (viewpoint 2) the matrix of f with respect to the bases $\mathcal{B}$ and $\mathcal{D}$ is the matrix corresponding to the map $\Psi_{\mathcal{D}} \circ f \circ \Phi_{\mathcal{B}}$ (with respect to standard bases of $\mathbb{R}^n_{\mathcal{B}}$ and $\mathbb{R}^m_{\mathcal{D}}$).

Or (viewpoint 3) the matrix of f is the matrix formed from the columns $(a_{1i}, ...a_{1m})^T$, where

$$f(\vec{v_i}) = a_{1i}\vec{w}_1 + ... + a_{mi}\vec{w}_m.$$

Let us now consider the case where we have bases $\mathcal{B} = \{\vec{v_1}, ..., \vec{v_n}\}$, and $\mathcal{C} = \{\vec{u_1}, ..., \vec{u_n}\}$ of V, and bases $\mathcal{D} = \{\vec{w}_1, ..., \vec{w}_m\}$, and $\mathcal{F} = \{\vec{f_1}, ..., \vec{f_m}\}$of W.

We then have a commutative diagram



This tells us that

$$A_{f,\mathcal{B},\mathcal{D}} = B_{\mathcal{D}\leftarrow\mathcal{F}} A_{f,\mathcal{C},\mathcal{F}} B_{\mathcal{C}\leftarrow\mathcal{B}}.$$

Example 1.9. Consider the case where $V = W = \mathbb{R}^2$, and our bases are $\mathcal{B} = \mathcal{D} = \{\vec{e}_1, \vec{e}_2\}$, the ordinary coordinate basis, and $\mathcal{C} = \mathcal{F} = \{\vec{w}_1 := (1,0)^T, \vec{w}_2 := (1,1)^T\}$.

Let f be the linear transformation corresponding to the matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ with respect to the standard basis $\mathcal{B} = \mathcal{D} = \{\vec{e}_1, \vec{e}_2\}$.

The base change matrix $B_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Hence the *matrix* of f with respect to the basis $\mathcal{C} = \mathcal{F} = \{\vec{e}_1, \vec{e}_2\}$ of $\mathbb{R}^2$ is

$$A_{f,\mathcal{C},\mathcal{C}} = B^{-1}_{\mathcal{B}\leftarrow\mathcal{C}} A B_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 3 & 7 \end{pmatrix}$$

One can check by inspection that this agrees with viewpoint 2 above, because

$$A\vec{w}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $(-1, 3)^T$ is indeed the first column of the matrix.

Similarly

$$A\vec{w}_2 = \begin{pmatrix} 3 \\ 7 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and $(-4, 7)^T$ is indeed the second column of the matrix.