Talk (TCFT & CY - A_∞-cat)

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① Review

. Extended TQFT / classification.

(thm) $\mathcal{C}$ has duals (∞, n) Cat.

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \cong \tilde{\mathcal{C}}$$

(cf) $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{G}, \mathcal{C}) \cong \mathcal{C}^{hG} \cong \mathrm{Hom}_G(EG, \tilde{\mathcal{C}})$

$\cong \mathrm{Hom}_{O(n)}(\widetilde{BG}, \tilde{\mathcal{C}})$.

- $(\infty, 0)$ Cat equivalence
- $\mathcal{C}$ has duals ; fully dualizable .

  $\to$ Naively speaking, dualizable in all n.

- Bord_n^{fr}

  - X^m has n-framing f. $TX \oplus \mathbb{R}^{n-m} \cong \mathbb{R}^n$. stably trivial.
  - $\tilde{\mathcal{C}}$ has $O(n)$ action.

    (ex) $O(1) \curvearrowright \tilde{\mathcal{C}}$ given by $X \longmapsto X^{\vee}$

(ex) $\mathcal{C}$ $(\infty, 1)$ Category with duals

By thm, $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{fr}, \mathcal{C}) \cong \tilde{\mathcal{C}}$

Here, $\mathrm{Bord}_1^{fr} = \mathrm{Bord}_1^{or}$. obj : points (w/ orientation).

Mor : 1-manifold with boundary

$\uparrow$ orientation.

$\phi \in \mathrm{Bord}_1^{or}$

$\mathrm{Map}_{\mathrm{Bord}_1^{or}}(\phi, \phi)$: not just a circle, but classifying space for oriented ~~cl~~ 1-manifold

$Z \in \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{or}, \mathcal{C})$. By thm. Z is determined by $Z(pt) = X$

$Z(\emptyset) = 1$ in $\mathcal{C}$.

$$\begin{tikzcd}
Z : \mathrm{Map}(\emptyset, \emptyset) \arrow[d] & \supseteq & BSO(2) \simeq \mathbb{CP}^\infty \arrow[dll, "Z|_{\mathbb{CP}^\infty}"] \\
\mathrm{Map}_{\mathcal{C}}(1, 1) & &
\end{tikzcd}$$

$\Rightarrow$ a connected component.

So. $Z(S^1) = Z(\bigcirc) = Z(\underset{coev\ ev}{\ldots}) : \underline{\underline{\dim X}} \in \mathrm{Map}_{\mathcal{C}}(1, 1)$.

However, this does not determine. $Z|_{\mathbb{CP}^\infty}$.

because. $Z(S^1)$ should carry an action of $SO(2)$.

② Cobordism - Hypothesis : Non-compact version. (Lurie 4.2)

- $(\infty, 2)$. Category $\mathcal{C}$.

(thm) (CH. NC-version) TFAE

(1) Sym-monoidal functors $Z : \mathrm{Bord}_2^{nc} \to \mathcal{C}$

(2) Calabi-Yau objects of $\mathcal{C}$.

(Def) $\underset{(\infty, 2)}{\mathrm{Bord}_2^{nc}}$: obj : oriented 0-manifolds.

$\mathrm{Mor}(X, Y)$: Oriented bordism.

2-$\mathrm{Mor}(B, B')$: " from B to B'; Σ.

" every connected comp. of Σ has non-empty intersection with B.

As a name says, no. compactness in the definition.

Higher. as usual

(Def). Calabi-Yau. object of $\mathcal{C}$.

(Case 1).

(Case 1). $\mathcal{C}$ has dual.

By CH., $O(2) \curvearrowright \mathcal{C}^{\sim}$ so $\exists$ natural action of $SO(2) \curvearrowright \mathcal{C}^{\sim}$.

Then we call X a Calabi-Yau object if X is a $SO(2)$ fixed point.

(Rmk) • Given X, $SO(2) \cong S^1$ action gives an auto. of X.

we call it a Serre automorphism.

• (Rmk. 4.2.4).

(Case 2). $\mathcal{C}$ without dual assumption.

A CY obj of $\mathcal{C}$ consists of the following data.

(1) A dualizable obj. $X \in \mathcal{C}$

(2). A morphism $\eta : \dim(X) = ev_X \circ coev_X \longrightarrow 1$ in $\Omega\mathcal{C}$.

which is equivariant w.r.t the action of $SO(2)$ on $\dim X$.

and is counit for an adjunction bet. ev_X and $coev_X$.

(check) Case 2 $\Rightarrow$ Case 1.

(ex). S : "good" $(\infty, 1)$ $\quad \mathrm{Alg}_{(1)}S$: obj : E_1-alg obj of S.

$\mathrm{Mor}(A,B)$ = A–B bimodules.

$\uparrow$

$(\infty, 2)$ Category.

Note that $A \in \mathrm{Alg}_{(1)}S$ always fully dualizable.

By def. CY obj of $\mathrm{Alg}_{(1)}S$ is.

(1) A : dualizable

(2) $SO(2)$ eq. $tr : \int_{S^1} A \longrightarrow 1$ satisfying the following condition

$SO(2)$ fixed

why? $A \otimes A^{op} \cong \int_{S^0} A \longrightarrow \int_{S^1} A \xrightarrow{tr} 1$

induces the identification of A with dual $A^{\vee}$ in S.

counit get from

Unwinding. CH. for non-compact

(prop. 3.3.28) given. B_1. a. sym. monoidal $(\infty.1)$ – Cat w/ duals.

cartesian fibration. $(F : B_1 \longrightarrow B_2) \mapsto (\pi : C(F) \longrightarrow B_1)$

inner fibration determines. an equiv. the following types of data.

$\exists \quad \tilde{y}$
$\quad\downarrow$
$x \longrightarrow y$

(1) $F : B_1 \longrightarrow B_2$ ess. surj. B_2 : sym. $(\infty\ 2)$ Cat.

(2). coCartesian. fib. $C \longrightarrow B_1$

here. given $B_1 \longrightarrow B_2$.

$\Big(C(F)$: obj $(X.\eta)$ $X \in B_1$ $\eta : 1 \longrightarrow F(X)$. 1–mor. in B_2. $\Big)$

So, $(X.\eta)$
$\downarrow$
X

(ex). B_2: $(\infty.2)$ Cat. $\rightsquigarrow$ B_1 : $(\infty.1)$ Cat.

$f : B_1 \hookrightarrow B_2$. $\Longleftrightarrow$. $B_1^{N} \longrightarrow B_1$.

where B_1 has data of discarded. 2-morphism.

(ex) $\mathrm{Bord}_1^{or} \hookrightarrow \mathrm{Bord}_2^{nc}$ $\Longleftrightarrow$ $OC \longrightarrow B_1$ $(\infty.1)$ Cat

OC : obj = Oriented 1-m with boundary. $\longleftrightarrow$ $(*.\eta)$ $\eta : \phi \to *$ (ex). C_x^t

Mor = Satisfying similar thing $\longleftrightarrow$

$J \longmapsto \partial J$.

So,

$$\begin{tikzcd} OC \arrow[r, "Z_2"] \arrow[d] & \tilde{C_1} \arrow[d] \\ \mathrm{Bord}_1^{or} \arrow[r, "Z_1"] & C_1 \end{tikzcd}$$

preserves co-cartesian morphism.

$\Longleftrightarrow$ η : $\dim(x) \longrightarrow 1$ in ΩC.

counit. ——

(statement)

Let $\mathcal{O} \subset \mathcal{OC}$ whose objects are finite union of intervals. S: $(\infty,1)$ Cat.

(Thm) ① $Z : \mathcal{O} \longrightarrow S \iff$ CY-algebra in S.

② Given $Z_0 : \mathcal{O} \longrightarrow S$. ($S$: good), we get $Z : \mathcal{OC} \longrightarrow S$ by left Kan extension.

(Rmk) · Given ①.② $\Rightarrow$ Easy to prove. NC-unfolded version.

· Costello proved this theorem when $S = \mathrm{Cpx}_k$. (char $k = 0$)

③ TCFT + Main Theorem.

· CFT : $\mathcal{M} \longrightarrow \mathrm{Vect}$.

$\mathcal{M}$: Segal cat of Riemann Surface. $(\infty,1)$!

ob : sets "finite".

Mor. $\mathcal{M}(I.J)$.

· TCFT (Twisted version)

$\mathcal{M}$: topological Cat $\xrightarrow[\text{takes}]{} C_*(\mathcal{M})$ dg-cat

$\mathrm{Mor}_{C_*(\mathcal{M})}(a.b) = C_*(\mathrm{Mor}_{\mathcal{M}}(a.b)) = \mathcal{C}$.

$\mathcal{C} \longrightarrow$ Chain complex.

(Rmk) Twisting by local system.

$$\begin{array}{ccc} \det & : \ \text{l.s} & \det(H^*(\Sigma))\,[-\chi(\Sigma)] \\ | & & \downarrow \\ \mathcal{M} & & \Sigma \end{array}$$

$\Rightarrow \mathcal{C}^d = C_*(\mathcal{M}.\det^d)$) going to be trivialized. shifted up to degree.

$\det^d \cong \det^{\otimes d}$

Our main objects are the followings

Define M_Λ obj (C, O, s, t) $C, O \in \mathbb{Z}_{\geq 0}$ D-branes

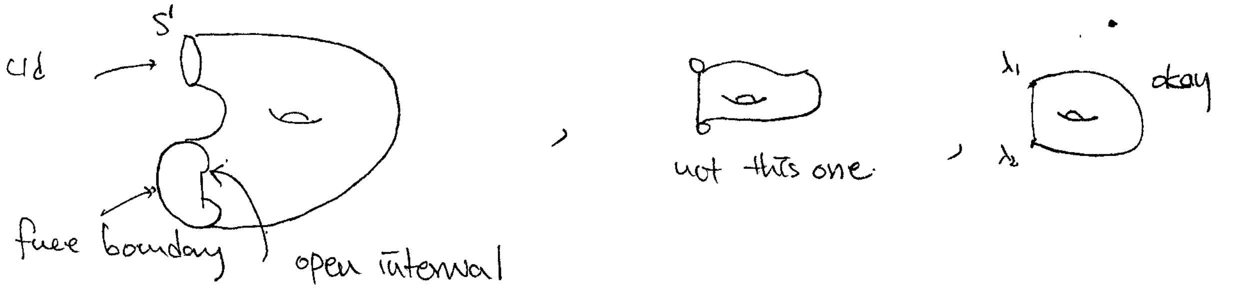
(topological) $s, t : O \longrightarrow \Lambda$. fun maps

Cat

$M_\Lambda((C_+, O_+, s_+, t_+)(C_-, O_-, s_-, t_-))$

= moduli sp of Riemann surfaces Σ with

open-cld boundary, with free boundaries labelled by D-branes



Similarly we can define $C_*(M_\lambda)$.

(Def) $\mathcal{OC}_\Lambda = C_*(M_\Lambda) \supset \mathcal{O}_\Lambda$. full sub objs are $(0, O, s, t)$ no-cld parts.

$\cup$ $\mathcal{C}$. sub cat objs have no open parts.

: dg-cat.

(Def) • Open-cld TCFT

$\mathcal{OC}_\Lambda \longrightarrow$ Complx

$\mathcal{C} \longrightarrow \mathcal{H}(C)$

$O \longrightarrow \mathrm{Hom}(\lambda, \lambda')$) chains

other definitions can be obtained in a similar way.

• Open-cld CFT

$\mathcal{OC}$ Vect

$C \longrightarrow \mathcal{H}(C)$ cld states.

$O \longrightarrow \mathrm{Hom}(\lambda, \lambda')$ some vector space.

(Convention)

$$\begin{tikzcd} \mathcal{O}^d_\Lambda \arrow[r, "i"] & \mathcal{OC}^d_\Lambda & \mathcal{C}^d \arrow[l, "j"'] \end{tikzcd}$$

Given, $\Phi \in \mathrm{Fun}(\mathcal{O}^d_\Lambda, \mathrm{Comp}_k)$. we can get $i_*\Phi \in \mathrm{Fun}(\mathcal{OC}^d_\Lambda, \mathrm{Comp}_k)$

where i_* is left adjoint to the pull-back functor.

If we think category as a algebra, Fun cat. is left-module.

we can write $i_*\Phi = \mathcal{OC}^d \otimes_{\mathcal{O}^d_\Lambda} \Phi$.

Not exact in the sense that does not preserve quasi-iso.

$$\mathbb{L} i_* \Phi = \mathcal{OC}^d_\Lambda \otimes^{\mathbb{L}}_{\mathcal{O}^d_\Lambda} \Phi .$$

obtained by replacing Φ by a flat resolution. given by

Just a bar construction

(ex) $A \otimes B$. $\mathrm{obj} = \mathrm{obj}\, A \times \mathrm{obj}\, B$

$\mathrm{Mor} = \mathrm{Mor}_A \otimes_k \mathrm{Mor}_B$

A–B bimodule is a monoidal functor $A \otimes B^{op} \to \mathrm{Comp}_k$.

Main Thm.

① Cat of OC_Λ^d-module is quasi-eq$_{(a)}$ to the category of (unital) Calabi-Yau extended A_∞-categories$_{(c)}$.

② Given any open TCFT, (Λ, Φ), we can push-forward the functor $\Phi: O_\Lambda^d \to Comp_k$ to $\mathbb{L}i_* \Phi : OC_\Lambda^d \to Comp_k$ then $(\Lambda, \mathbb{L}i_*\Phi)$ h-split & open-cld TCFT.

and this is homotopically universal.

③ $H_*(j^* \mathbb{L}i_* \Phi)_{(b)} \cong HH_*(A)_{(d)}$ where A is the A_∞-cat corresponding to (Λ, Φ)

$~~$e$~~$ q.e

(a) $F: C \rightleftarrows D : G$ $\quad F \circ G \simeq Id_D$. $G \circ F \simeq Id_C$. quasi isomorphism

natural trans. φ is quasi-iso. if. $F \circ G(c) \to c$ is quasi-iso.

φ

(b) Since H-split Ψ ; cld TCFT

$\Rightarrow H_*(\Psi(C)) = H_*(\Psi(1))^{\otimes C}$; $H_*(\Psi)$

$$H_*(\Phi(Z, O, s, t)) = \bigotimes_{o=0}^{O-1} H_*(\Phi(\{s(o), t(o)\}) \otimes H_*(T^*\Phi)^{\otimes Z}$$

(c) A_∞-categories. D

obj. : A, B $\quad Mor(A, B)$ chain-complex

{ homology grading convention. ; m_n $\quad n-2$
| Cohomology grading convention ; m_n $\quad 2-n$

For sequence of objects $A_0, A_1, \ldots, A_n$ of objects $n \geq 1$

$m_n : Hom(A_0, A_1) \otimes \cdots \otimes Hom(A_{n-1}, A_n) \to Hom(A_0, A_n)$

m_1 = ~~gra~~ differential.

" m_2 = some structure." $\to$ composition / multiplication

· CY-category $\mathcal{C}$ of dim d is a linear category w/ a trace map.

$\mathrm{Tr}_A : \mathrm{Hom}(A, A) \longrightarrow \mathbb{K}[d]$

for each obj A of $\mathcal{C}$.

Ass pairing $\langle \cdot, \cdot \rangle_{A \cdot B} : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, A) \longrightarrow \mathbb{K}[d]$.

is given by $\mathrm{tr}(\alpha \cdot \beta)$ to be sym. & non-degenerate.

* Calabi–Yau A_∞-categories is a duality

$\mathrm{HH}_i(D) \cong \mathrm{HH}^{d+i}(D^{op})^\vee$.

· Hochschild homo. HH_* HH^*