(Main Thm.)

① $\left\{\begin{array}{l} d\text{-dimensional} \\ \text{open-cl TCFT} \\ \text{with } \Lambda : \text{D-branes} \end{array}\right\} \xleftrightarrow{\text{q.e}} \left\{\begin{array}{l} \text{(Unital) CY.} \\ \underset{\sim\sim\sim}{\text{extended}}\ A_\infty\text{-cat} \\ \text{with the set of obj } \Lambda \end{array}\right\}$

② Given any open TCFT (Λ, Φ) $\Phi : \mathcal{O}_\Lambda^d \longrightarrow \mathrm{Comp}_K$,
we can pushforward it to $\mathbb{L}i_*\Phi : \mathcal{OC}_\Lambda^d \longrightarrow \mathrm{Comp}_K$.
& it is h-split. & open-cld TCFT. & homotopically universal

③ $H_*(j^*\mathbb{L}i_*\Phi) \cong HH_*(A)$ where A is the A_∞-cat
corresponding to (Λ, Φ)

<u>Notation</u> (Assume h-split) $H_*(\Psi) := H_*(\Psi(1))$

· $H_*(\Psi(c)) := H_*(\Psi(1))^{\otimes c}$ for Ψ ; cld TCFT

· similarly. $H_*(\Phi(C.O.s.t)) = \bigotimes_{\sigma=0}^{O-1} H_*(\Phi(\{s(\sigma), t(\sigma)\}) \otimes H_*(j^*\Phi)^{\otimes C}$

Also, $H_*(\Psi)$ carries operations from the homology of moduli-spaces of curves
(i.e) $H_*(\mathcal{C}^d(I.J)) \longrightarrow \mathrm{Hom}(H_*(\Psi)^{\otimes I}, H_*(\Psi)^{\otimes J})$

<u>Cor</u> : The homology of moduli-space acts on the Hochschild homology of any CY-A_∞ category D.
(i.e $\exists$ operation

$$H_*(\mathcal{M}(I.J), \det^d) \otimes HH_*(D)^{\otimes I} \longrightarrow HH_*(D)^{\otimes J}$$

$(\because)$ $D \sim (\Lambda\ \Phi) \Rightarrow (\Lambda\ \mathbb{L}i_*\Phi) \Rightarrow j^*\mathbb{L}i_*\Phi$.
$HH_*(D) = H_*(j^*\mathbb{L}i_*\Phi)$

Application (Mirror symmetry).

X . CY – mfd $\Sigma \longrightarrow X$

A – model B – model.

mathematically, GW-inv $g=0$ Barannikov. Frobenius manifold
$g>0$ mysterious
no rigorous construction before this.

Kontsevich : formulated MS as a eq of A_∞ – categories.

Application ①

B – model : X smooth – proj. CY – variety of dim d over $\mathbb{C}$
choose holo volume form X
Write $D^b_\infty(X)$ which is A_∞ – enhancement
of derived cat of coherent sheaf.

$$A \longrightarrow K(A) \longrightarrow H_*(A)$$

lose information

Note that it is CY – A_∞ – category.

$\Rightarrow$ $J^* \mathbb{H}_i * D^b_\infty(X)$. is the B – model mirror to a TCFT constructed from Gromov witten invariant.

$$HH_i(D^b_\infty(X)) = \bigoplus_{q-p=i} H^p(X, \Omega_X^q)$$ Cohom $\Rightarrow$ tangential cohomology

(By cor), $\exists$ a map

$$H_*(M(I,J), \det^d) \longrightarrow \mathrm{Hom}(HH_*(D^b_\infty(X))^{\otimes I}, HH_*(D^b_\infty(X))^{\otimes J})$$

These operations should be the B-model. mirror to corresponding operations on the homology of s.M coming from GW-invariant.

Application 2. A: side

Conjectural

$$\mathrm{Fuk}(X) \longrightarrow \text{cld TCFT} \ (d\text{-dim}) \xrightarrow{H} HH_*(\mathrm{Fuk})$$

$\mathcal{F}$

X: Sym manifold of $2d$

Gromov-Witten Invariant $\mathcal{G} \xrightarrow{H} H_{*+d}(X)$

$HH_*(\mathrm{Fuk}) \downarrow H_{*+d}(X)$

$\mathcal{M}' \longrightarrow \overline{\mathcal{M}}$: Mor (stable algebraic curves)

$C_*(\mathcal{M}') \longrightarrow C_*(\overline{\mathcal{M}})$ 1

pull back $\Leftarrow$ chain-level theory of GW-Inv

cld TCFT $\searrow$ cld TCFT !

we want in general. Complx $C_*(X)$ $\nwarrow$ Homology level

By the grading reason, $\underline{C_{*+d}(X)}$

$\Rightarrow$ Good

By GW Invariant

Cohomological Field Theory

Conjecture 1: $\exists$ d-dim'l OC-TCFT whose D-branes are certain Lagrangs. in X. whose morphism spaces bet D-branes $L_1.L_2$. are Lagrangian Floer chain gps.

still conjectural.

$$\mathrm{Hom}_i(L_1.L_2) = CF^{-i}(L_1.L_2)$$

whose complex of cld states is the shifted singular chain complex $C_{-*+d}(X)$ of X

i.e. $\jmath^* \Phi = \mathcal{G}$.

(Cor)

By universal property. get a map $\jmath^* \amalg_{i*}(\mathrm{Fuk}(X)) \longrightarrow \Phi$

homological TCFT $\Rightarrow HH_*(\mathrm{Fuk}(X)) \longrightarrow H_{*+d}(X)$.

(Rmk) At the cohomological level,

(proof of the theorem)

Part 1 : Algebraic part.

Category tol

A.B dgscur : $A \overset{g.i}{\sim} B \Rightarrow$ A-mod $\overset{i.e}{\sim}$ B-mod.

f $H^*(f)$ is fully. faithful.

Part 2. Geometric part. f induces iso on the set of obj

(idea) what makes it difficult to prove is that

there is no way to handle our category OC_Λ^d.

If we can find some sort of generators & relations.

then we can attack this

By part 1 ET find g i model for our category

which could be described ~~and~~ by generators & rela.

Play with this model $\Rightarrow$ Everything clear.

(step 1). Combinatorial model for $M_\Lambda(\alpha.\beta)$

(step 2) " O_Λ^d. OC_Λ^d

& Give generators & relation. description.

(step 3) Prove the main thm

(Step 1) Λ: fixed. $\alpha.\beta \in ob M_\Lambda$ $C(\alpha) = \phi$

① Compactify. $M_\Lambda(\alpha.\beta) := N(\alpha.\beta)$

$\Rightarrow$ $\bar{N}(\alpha.\beta)$; orbifold w/ corners possibly w/ nodal singularity

h.e adding some exceptional curves

$N(\alpha.\beta)$

② Define $G(\alpha.\beta) \xrightarrow{h.e} \hat{N}(\alpha.\beta)$

. Irr comp ; disc or annulus of od 1 with marked pt

outgoing cld boundary component required to be smooth

- exceptional surf $\bigcirc$ ≤ 1 $\left(\begin{array}{l} \text{no boundary pt on annuli} \\ \text{cld or open} \end{array} \right)$

(Rmk) · $G(\alpha, \beta)$. right model for our category. Q? need no boundary point?

· So, we need to define a composition in an appropriate way.

(Step 2)

③ Gluing & Form a category

$C(\alpha) = C(\beta) = 0$

$$\bar{N}(\alpha, \beta) \times \bar{N}(\beta, \gamma) \longrightarrow \bar{N}(\alpha, \gamma).$$

Glue the outgoing open marked point $\rightsquigarrow$ corresponding incoming.

Exceptional surface.

$\Rightarrow$ (Lemma 6.1.4)

$\exists$ Category $\bar{N}_{open}$ obj : α of M_Λ with $C(\alpha) = 0$

$\quad\quad \cup\!|$ $\quad\quad Mor(\alpha, \beta) = \bar{N}(\alpha, \beta)$.

$\quad\quad G_{open}$

(prop 6.1.5) dgsm $C_*(\bar{N}_{open}, det^d) \overset{\&}{\simeq}$ dgsm $\mathcal{O}^d$ (idea)

$\Rightarrow$ SymOb $\mathcal{OC}^d$ - $C_*(\bar{N}_{open}, det^d)$ Mod

$\sim$ SymOb $\mathcal{OC}^d$ - $\mathcal{O}^d$ Mod

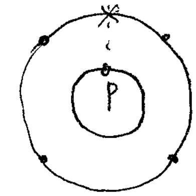
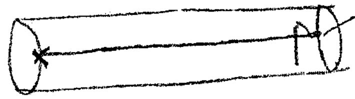
$C_*(\bar{N}, det^d)$ corresponds to $\mathcal{OC}^d$

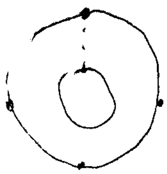
④ Not enough ! - (orbit) Cell decomposition of $G(\alpha, \beta)$

, $\Sigma \in G(\alpha, \beta)$: Cell decomposition

A - irr component of Σ

- Disc. or annulus

(1)   old outgoing boundary

(2) 

Argue that 0-cell : node. marked point
intersection point

1-cell $\partial\Sigma$ cut

2-Cell Σ

claim: Orbit-cell decomposition.

& $G(\alpha,\beta) \times G(\beta,\gamma) \longrightarrow G(\alpha,\gamma)$ cellular.

Now, define $\mathcal{D}^d(\alpha,\beta) = C_*^{cell}(G(\alpha,\beta), \det^d) \otimes \mathbb{k}$

similarly $\mathcal{D}^d_{open}$

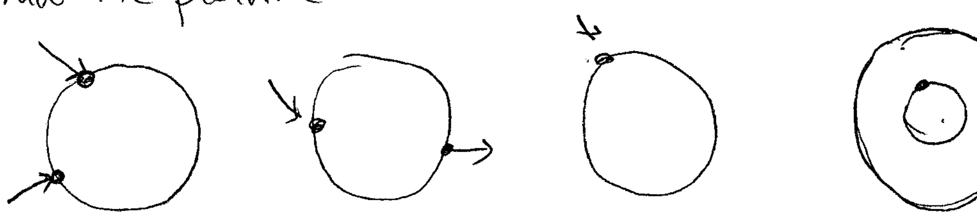
* Composition is always restricted

$\Rightarrow$ $\mathcal{D}_{open}^d \overset{q.i}{\simeq} \mathcal{O}^d$ $\Rightarrow$ $\mathcal{D}^d$ corresponds to $\mathcal{OC}^d$.

(Rmk) $\mathcal{D}_{open}^d$ is our model for $\mathcal{O}_\Lambda^d$.

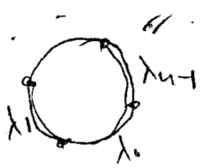
(5) Generators & relations for $\mathcal{D}_{open}^d$. boundary components are just disc.

Draw the picture.

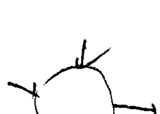


(Notation) Given an ordered set $\lambda_0 \dots \lambda_{n-1}$

$D(\lambda_0, \dots, \lambda_{n-1}) \in \mathrm{Hom}\{(\lambda_0, \dots \lambda_{n-1}\, \lambda_0), \emptyset\}$



$\mathcal{D}^{+}_{open} \subset \mathcal{D}^d_{open}$ obj same. Mor disj union of disc with precisely one outgoing marked pt

$D^+(\lambda_0, \dots, \lambda_{n-1}) \in \mathrm{Hom}\{(\lambda_0, \dots, \lambda_{n-1}), \{\lambda_0, \lambda_{n-1}\}\}$ 

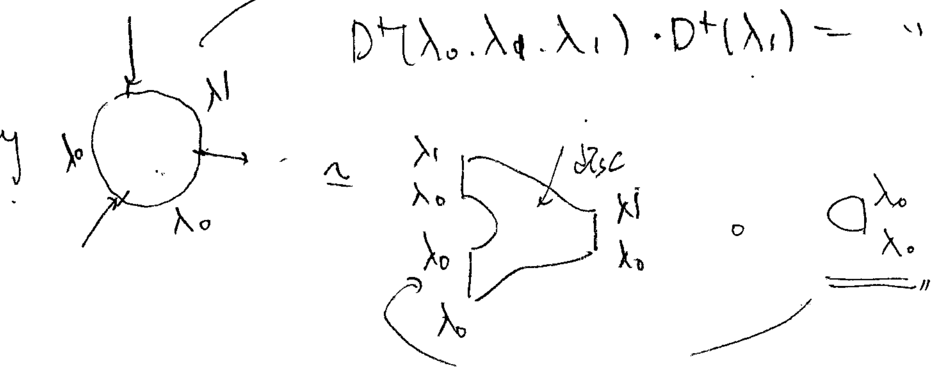
$\dot{D}^+_{open}$ is freely generated as sndg over ob D^d_{open} by discs $D^+(\lambda_0,\dots,\lambda_{n-1})$

modulo rel. $D^+(\lambda_0,\dots,\lambda_i,\lambda_i,\dots,\lambda_{n-1}) \circ D^+(\lambda_i) = 0$ $\quad n \geq 4$

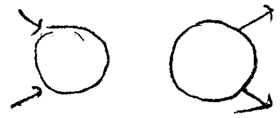
$$D^+(\lambda_0,\lambda_0,\lambda_1) \circ D^+(\lambda_0) = id\{\lambda_0,\lambda_1\}$$

$$D^+(\lambda_0,\lambda_1,\lambda_1) \circ D^+(\lambda_1) = \text{''}$$

why



• I don't know why the first relation holds

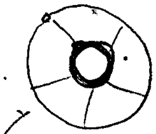
(Thm) D^d_{open} : freely gen'd by D^+_{open},  $/\sim$.

• $D(\lambda_0,\dots,\lambda_{n-1})$ obtained by $D^+(\lambda_0,\dots,\lambda_{n-1})$ 

disc w/ no boundary pt.

Annulus w/ no boundary pt included.

⑥ Generator & relation for D_c

• similar to ⑤, but more object  "annulus"

We should take it into account & get the similar result.

our main obj

$\lambda_0,\dots,\lambda_{n-1}$ D-branes $\qquad$ all incoming

$$A(\lambda_0,\dots,\lambda_{n-1}) \in D^d(\{\lambda_0,\dots,\lambda_{n-1}\}^c, (1,0))$$

(Thm) cyclic sym ob$\mathcal{OC}$ – D^d_{open} d gluing data bimodule D^d is finitely gen'd.

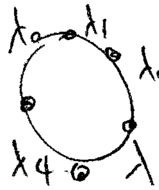
(Def) D^+ be ob$\mathcal{OC}$ – D^+_{open} bimodule with same gen rel.

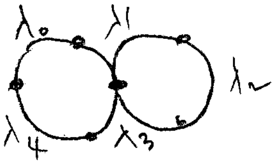
$$D^d = D^+ \otimes_{D^+_{open}} D^d_{open}$$

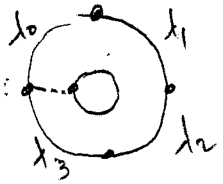
because the relation involves only the disc w/ outgoing marked pt

① differential.

Since it respects to composition. enough to describe.

d 

$= \Sigma \pm$ 

d: 

$= \Sigma \pm$.

gen'd by annulus A. and identity etc.

(Lemma)(1) The $ObOC^d - D^d_{open}$ bimodule D^d is D^d_{open}-flat.

(2) M is h-split D^d_{open} module, then

$D^d \otimes_{D^d_{open}} M$ is a h-split $ObOC^d$-mod.

$A-B$ bimodule M is A-flat if $- \otimes_A M$ is flat.

B-flat if $M \otimes_B -$ flat.

(1). generator : $A(\lambda_0, \dots, \lambda_{n-1})$. id

Give filtration on these generator

$dA(\lambda_0, \dots, \lambda_{n-1}) \in F^{n-1}$.

Say. M', M : left D^d_{open}-mod. $M \to M'$ quasi-iso.

WTS $D^d(-) \otimes_{D^d_{open}} M \to D^d(-) \otimes_{D^d_{open}} M'$ is q.i

STS ; ass graded complex. is q.i

This comes from generators & relations description of D^d

(upshot) $D^d_{open} \sim \mathcal{O}_\Lambda{}^d$

- generatored by D^+_{open} . → / relation.

- $\mathcal{D}^d \sim \mathcal{OC}_\Lambda{}^d$ as . module. $D_{open}{}^d$ action

gen. $A(\lambda_0 \cdots, \lambda_{n-1}), 1$ / ~. id discs, ...

(Step 3) - (Lemma) ① Split functor $\Phi : D^+_{open, \Lambda} \longrightarrow Comp_k$.

is the same as unital A_∞ - cat.

(ex) $D^+(\lambda_0 \ldots, \lambda_{n-1}) \to Hom(\lambda_0, \lambda_1) \otimes \cdots \otimes Hom(\lambda_{n-2}, \lambda_{n-1})$

$\to Hom(\lambda_0, \lambda_{n-1})$

I should have said about deg.

$D^+(\lambda) : 1k \longrightarrow Hom(\lambda\,\lambda)$ unit.

② " $\Phi : D^d_{open, \Lambda} \longrightarrow Comp_k$

CY. —.

Two more generators. $Hom(\lambda_0, \lambda_1) \otimes Hom(\lambda_1, \lambda_0)$

$\downarrow$

$k[d]$ & inverse.

(Aside) HMS. Hoch (co)homology. . Fukaya. category.

(Theory of. GW - Invariant).

$(\Sigma_g . J . z_1, \cdots, z_k)$.

$\mathcal{M}_{g.k}$, complex orbifold of $\dim_{\mathbb{C}} = 3g-3+k$ $\rightsquigarrow \overline{\mathcal{M}}_{g.k}$

$\mathcal{M}_{g,k}(M.J.\beta) := \{ u : \Sigma_g \longrightarrow M : J\text{-holo } [u] = \beta \} / \sim$

$\rightsquigarrow \overline{\mathcal{M}}_{g.k}(M.J.\beta) := \{$ (possibly nodal) J- hol. curves of genus g. in homology class β $\} / \sim$

$$2d = 2c_1(TM)(\beta) + (n-3)(2-2g) + 2k$$

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g.k}(M.J.\beta) & \xrightarrow{ev} & M \\ \downarrow s & & \\ \overline{\mathcal{M}}_{g.k} & & \end{array}$$

$\Rightarrow \langle \alpha_1, \cdots, \alpha_k \rangle^{\phi}_{g.\beta} :=$ # genus - g J-hol curve. in class β.
meeting cycles $c_1, \cdots, c_k$.
whose domain lies in class $S \subset \overline{\mathcal{M}}_{g.k}$

$$= \int_{\overline{\mathcal{M}}_{g.k}(M.J.\beta)} s^*\phi \times ev_1^*\alpha_1 \wedge \cdots \wedge ev_k^*\alpha_k$$

$$\langle \alpha_1, \cdots, \alpha_k \rangle^{\phi}_{g} := \sum_{\beta \in H_2(M)} \langle \alpha_1, \cdots, \alpha_k \rangle^{\phi}_{g.\beta} T^{\omega(\beta)} \in \Lambda_0$$

$(g=0)$ $H^*(M : \Lambda_0)$ admits Quantum cup-product

$$\langle \alpha * \beta . \gamma \rangle = \langle \alpha . \beta . \gamma \rangle_{g=0} \in \Lambda_0$$

Also it gives a cohomological field theory. $\left\{ \begin{array}{l} \text{marked point} \\ \longleftrightarrow \text{Boundary} \end{array} \right\}$

$I_{g,k} : H^*(\overline{M}_{g,k}) \otimes H^*(M)^{\otimes k} \longrightarrow \Lambda_0$

structure

We call such system of map. : CFT.

If we restrict it to $g=0$. understand by quantum cohomology. cup-product

* Kontsevich's recursion relation

$\sim$ One part of B-model corresponding to this is tangential cohomology.

As a CohFT $H_*(\overline{M}_{0,n})$ $\boxed{\text{coh FT}}$ $\otimes$ by. $\leftrightsquigarrow$ BK.

For higher genus $\longrightarrow$ possible $\leftrightsquigarrow$ what would be?

So, all we need to know is how to construct operadic str??

Open-strings.

A_∞ - category $\mathcal{C}$

- ob $\mathcal{C}$:
- $Mor(A, B)$: f.d complex of $\mathbb{K}$ v-sp $Hom(A, B)$

(Using homological grading convention is used)

For each seq of obj. $A_0, \cdots, A_n$, $n \geq 2$.

$$m_n : Hom(A_0, A_1) \otimes \cdots \otimes Hom(A_{n-1}, A_n) \longrightarrow Hom(A_0, A_n) \quad 2-n$$

(A_∞-relation)

$$\sum_{0 \leq i \leq j \leq n-1} \pm m_{n-j+i}(\alpha_0 \otimes \cdots \otimes \alpha_{i-1} \otimes m_{j-i+1}(\alpha_i \otimes \cdots \otimes \alpha_j) \otimes \alpha_{j+1} \cdots \otimes \alpha_{n-1}) = 0$$

CY A_∞ cat of dim d.

$\langle\ \rangle_{A.B} : \mathrm{Hom}(A.B) \otimes \mathrm{Hom}(B.A) \longrightarrow k[d]$ odd · non-deg pairing

satisfying cyclic symmetry identity.

$$\langle m_{n-1}(\alpha_0 \otimes \cdots \otimes \alpha_{n-2}), \alpha_{n-1} \rangle = (-1)^{n+1+|\alpha_0| \sum |\alpha_i|} \langle m_{n-1}(\alpha_1 \otimes \cdots \otimes \alpha_{n-2}), \alpha_0 \rangle.$$

ex) $\mathrm{Fuk}(X)$. X: sym manifold. of $2d$.

obj > L_i Lagrangian submanifold with $[\omega] \cdot \pi_1(X.L) = 0$.

rule out bad bubblings

Mor : $CF^i(L_1.L_2)$ Floer cochain.

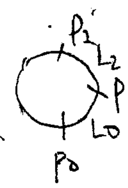
unit , $CF^i(L.L) \cong CF^i(\varphi(L)\ L)$) cohomological unital ??.

$\ \ \ \ \ \ \ \wr\!\wr$

$C^i(L) \leftarrow$ morse.

$CF^i(L_1.L_2) := \Lambda[p_i]$ p_i intersections

$$m_1 \cdot \partial p = \sum_{\substack{q \in L_1 \cap L_2 \\ \mathrm{ind}(\beta)=1}} \#(M_\beta(p.q)/_{\mathbb{R}})\, T^{\omega(\beta)} \cdot q.$$

$$m_s(p_1 \cdots p_s) := \sum_{\substack{p_0.\beta \\ \mathrm{ind}(\beta)=0}} \#\Big($$  $$\Big)\, T^{\omega(\beta)} p_0$$

Satisfying A_∞-relation.

Smart way to say this is the following

$$CC(X_0, \ldots, X_s) := \mathrm{hom}^*(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}(X_{s-1}.X_s)$$

$$CC^{std}(\mathcal{C})^s := \prod_{X_0 \cdots X_s \in \mathrm{ob}(\mathcal{C})} \mathrm{Hom}^*(\mathcal{C}(X_0 \ldots, X_s) \cdot \mathcal{C}(X_0.X_s)).$$

and $CC^*(\mathcal{C}) := \prod_{s \geq 0} CC^*(\mathcal{C})^s$

if $\mathcal{C}$ has one obj..

$CC^*(\mathcal{C}) \simeq CC^*(A)$ where. $A := \hom^*(X, X)$.

$\mu \in CC^2(A)^{\geq 1} \rightsquigarrow \underline{\mu \circ \mu = 0}$.

$\Rightarrow (\mathfrak{CC}^*(A), [\mu, -]) \Rightarrow$ Hochschild Cohom

Similarly can define $HH^*(\mathcal{C}) := H^*(\mathfrak{CC}^*(\mathcal{C}), [\mu, -])$

CWA CY-str. ok. but extremely difficult (1).

∘ About Hochschild coho / ho. Th. Algebraic Geometry.

<lecture 5. [illegible]> (11).

dg category. Perf(X).

$$\mathrm{Hom}_{\mathrm{perf}(X)}(E, F) = \Omega^{0,*}(E^\vee \otimes F)$$

$$\mathrm{Hom}_{\mathrm{perf}(X)}(E, F) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathrm{perf}(X)}(F, E) \dashrightarrow \mathbb{C}$$

of deg d. given by holomorphic volume form.

check: derived cat / D-module.