1. ___Introduction. Dijkgraaf–Witten Thy.___

* Add $\alpha = 0$ case.

$\mathrm{Cob}_\epsilon(n) \xrightarrow{F} \mathrm{Vect}$. $\quad G \in \mathrm{FinGrp}$, $\alpha \in H^{n+1}(BG, \mathbb{Z})$ $\cong H^n(BG, U(1))$

$M \in \mathrm{Hom}(\emptyset, \emptyset)$ $\quad$ (M is $n+1$-dimensional).

$$F(M) = \sum_{P \in [\mathrm{Bun}_G(M)]^{\text{iso}}} \frac{O_\alpha(P)}{\mathrm{Aut}(P)}.$$

$O_\alpha : \mathrm{Bun}_G(M) \to \mathbb{C}$

$P \in \mathrm{Bun}_G(M)$ gives $M \xrightarrow{P} BG$. (class map).

$P^*\alpha \in \cancel{H^{n+1}(BM, \mathbb{Z})}$. $H^n(M, \cancel{\mathbb{Z}}\ U(1))$.

$(P^*\alpha)[M] \in \mathbb{C}^*$ (integrating over M).

Now $M \in \mathrm{Hom}(N_1, N_2)$. $\quad (\partial M = N_1 \amalg \overline{N_2})$.

$$\begin{tikzcd} & \mathrm{Bun}_G(M) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Bun}_G(N_1) & & \mathrm{Bun}_G(N_2) \end{tikzcd} \quad \text{(Grpds)}.$$

~~$F\,\mathcal{B}(\mathrm{Bun}_G(N_2))$ = fun const fcts on $\mathrm{Bun}_G(N_1)$.~~

$f \in \cancel{F(\mathrm{Bun}_G(N_1))}$.

~~$F(M)f = \pi_{2!}\pi_1^*(f)$~~ $\cancel{ev^*}\alpha \in \cancel{H^4}$

$ev^*\alpha$. $\quad N_1 \times \mathrm{Map}(N_1, BG) \xrightarrow{ev} BG \ \alpha$.

$H^1(\cancel{\mathrm{Map}}\,N_1, BG) \to \int_{N_1} ev^*\alpha \downarrow$

$\mathrm{Map}(N_1, BG)$

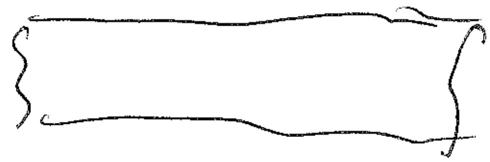
1 dl loc sys on $\mathrm{Map}(N_1, BG)$. $\ L_\alpha$

$F(N_1) = \prod (\mathcal{L}_x) \ni v$ $\left((v_x \in \mathcal{L}_x)_{x \in X}\right.$ paths take an element to the next.$\left.\right)$

<u>Defn</u>: Local system on X,

$= \varprojlim \mathcal{L}.$

N_1 M N_2



$$\begin{tikzcd} & \mathrm{Map}(M, BG) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{L}_M \quad \mathrm{Map}(N_1, BG) & & \mathrm{Map}(N, BG) \quad \mathcal{L}_N \end{tikzcd}$$

$\pi_1^* \mathcal{L}_{N_1} = \pi_2^* \mathcal{L}_{N_2} = \mathcal{L}_M$ (because N_1, N_2 cobordism)

Take a section: pull it back & push it forward.

DW thy.

$$Z(M^n) = \frac{1}{|G|} \sum_{\alpha:\pi_1 M \to G} (\bar{\alpha}^* \eta)[M].$$

$\bar{\alpha}: M \to BG$

$\eta \in H^{n+1}(BG, \mathbb{Z})$
$= H^n(BG, U(1))$
Use the integral ?.

$Z(M^{n-1}) =$

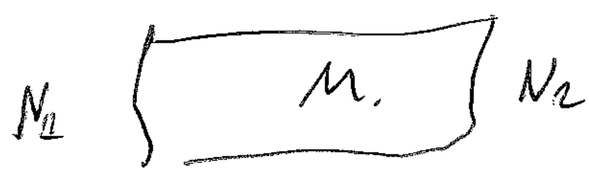
$$\begin{array}{ccc} \overset{ev^*\eta}{M \times Map(M, BG)} & \xrightarrow{ev} & BG \quad \eta \\ \downarrow & & \\ Map(M, BG). & & \end{array}$$

$\int_M (ev^*\eta)[M]$

$\uparrow$

$H^1(Map(M, BG), \mathbb{C}^*)$.

This gives a local system on $Map(M, BG)$.



$\mathcal{L}_M$ = comm pullback

$$\begin{array}{ccc} & Map(M, BG) & \\ \swarrow & & \searrow \\ Map(N_1, BG) & & Map(N_2, BG) \end{array}$$

pullback a section & pushforward.

1. pg 2.

Example 2.

$n = 2$. Then we know such maps F are characterized by 1. $F(S^1) \in$ Frob Alg α typical.

$Z(S^1) =$ loc. const $\mathbb{C}$-valued fncts on $Map(S^1, BG)$.

G bdls on S^1/iso

fix a base pt, & trivialize.
we get two trivializations,
difference is an elmt of G.
but chg trivialization ~~chg~~ is equ to conjugation.

$\Rightarrow \quad Map(S^1, BG) \simeq G/\text{conjugation}$.

$\Rightarrow \quad Z(S^1) =$ Class fncts on G.

Z_x

Topic 1.

Finite Path integral.

$$\begin{tikzcd} \mathrm{Bord}_n \arrow[dr] & & \mathrm{Fam}_n(\mathcal{C}) \arrow[dl] \\ & \mathrm{Fam}_n & \end{tikzcd}$$

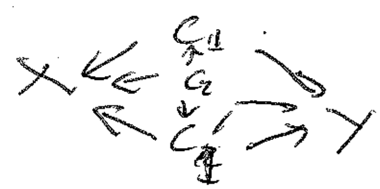
$Y \in \mapsto \mathrm{Bun}_G(Y)$.

$\mathrm{Ob}(\mathrm{Fam}_n) = \mathrm{FinGpd}$

1-Morphisms = Correspondences:

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

2-morphisms = correspondence of correspondences



3-morphisms

$\vdots$

n-category.

$\mathcal{C}$ — Sym monoidal n-category.

$\mathrm{Fam}_n \mathcal{C}$.
~~= loc system on $\mathcal{C}$~~
~~on Fam_n~~
gpds with loc syst in $\mathcal{C}$.

$$\begin{tikzcd} \mathrm{Bord}_n^{SO} \arrow[r, "I"] \arrow[dr] & \mathrm{Fam}_n(\mathcal{C}) \arrow[r, "\lim"] \arrow[d] & \mathcal{C} \\ & \mathrm{Fam}_n & \end{tikzcd}$$

(want $\underleftarrow{\lim} = \underrightarrow{\lim}$).

~~Explain what we've just done.~~

~~Example 2~~: For a correspondence

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$\chi : X \to \mathrm{Vect}_{\mathbb{C}}$, $\delta : Y \to \mathrm{Vect}_{\mathbb{C}}$. for $c \in C$

$\varphi(c) : \chi(p_1(c)) \to \delta(p_2(c))$.

$$\mathrm{Sum}_1(C,\varphi) : \mathrm{Sum}_1(X,\mathcal{X}) \to \mathrm{Sum}_1(Y,\mathcal{Y}).$$

$$\mathrm{Sum}_1(C,\varphi)(x) = \sum_{\substack{c \in \pi_0 \underbrace{\mathcal{C}_x}_{\text{fiber over } x}}} \frac{\varphi(c)}{\# \mathrm{Aut}(c)}.$$

Example $n=2$.

$\mathcal{C} = \mathcal{A}lg$: objects = Frobenius alg. maps = bimodules. 2-morphs = intertwiners.

I make

$\alpha \in H^3(BG, \mathbb{Z})$ gives twisted $\mathbb{C}^\alpha[G]$ of fields:

$$* /\!/ G \to \mathcal{A}lg.$$

$$* \longmapsto \mathbb{C}$$

$$x \in G \longmapsto K_x.$$

$$\mathrm{id} \longmapsto \mathrm{id}.$$

$F(pt_+) = \varprojlim_{*/\!/G} \mathcal{X} = A \in \mathcal{A}lg.$ s.t. $\forall B$.

(A,B)-bimod $\cong$ B bimods M s.t. $\{K_x \otimes M \cong M\}_{x\in G}$.

$$\Rightarrow A \cong \mathbb{C}^\alpha[G] = \bigoplus_{x \in G} K_x$$

Topic 2:
Exactly down.

For $n=2$. $\tau \in H^3(BG,\mathbb{Z})$.

$\mathbb{C}^\tau[G] \in \mathrm{Alg}$

{Fully dualizable object fixed under $SO(2)$}
$\cong$ {Frobenius Algebras}

Why: ~~A~~ A dual $= A^{op}$

~~as~~ $A \in (\mathbb{C}, A\otimes A^{op})$-bimod.

A fully dualizable if it is dualizable as

1) an ~~A-mod~~ $\mathbb{C}$-module $\Rightarrow$ A finite dim
2) $A\otimes A^{op}$ bimod. $\Rightarrow$ A semisimple.

~~As $A\otimes A^{op} = E(Z(S^1))$. Recall for $S^1 \times S^1 / k$~~
~~This is that $A \cong \mathrm{FactHom}(A)(D^2)$ is dualizable~~ on $\mathrm{FactHom}(S^1)$.

FACTHOM.

$Z(S^1) = A \otimes_{A\otimes A^{op}} A = A/[A,A]$; $a\otimes ba = ba$.
~~$a\otimes b$~~ $ab \cong a\otimes b \cong$ ~~$b\otimes a$~~ (by left right actions).

$\mathbb{C}$ Why does
~~$\mathbb{C}^\tau[G] \otimes_{\mathbb{C}} \mathbb{C}^\tau[G]^{op} \cong$ Class functs on G~~

$\mathbb{C}^\tau[G] \otimes_{\mathbb{C}^\tau[G]\otimes\mathbb{C}^\tau[G]^{op}} \mathbb{C}^\tau[G] \cong$ Class funcs

$\cong$

$\mathbb{C}^\tau[G] / [\mathbb{C}^\tau[G], \mathbb{C}^\tau[G]]$ (rank dim twisted class functions) for $\tau = 0$.

(Remark 4.1.27 Lurie).

~~$\mathbb{C}^\tau[G] =$~~

$\tau \in H^3(G, \mathbb{Z})$ gives a central extn

$$U(1) \to G^\tau \to G.$$

$$\begin{array}{c} K^\tau \\ \downarrow \\ G \end{array}$$

associated holomorphic line bdle

$$K^\tau := G^\tau \times_{U(1)} \mathbb{C}$$
$$\downarrow$$
$$G.$$

(by assumption $\theta_{xy} : K_x \otimes K_y \to K_{xy}$

multiplication by convolution.

$$\mathcal{L}_1 * \mathcal{L}_2(x) = \sum_{y_1 y_2 = x} \theta_{xy}(\mathcal{L}_1(y_1), \mathcal{L}_2(y_2)).$$

$$\mathbb{C}^\tau[G] \xrightarrow{\varepsilon} \mathbb{C}$$

$$\delta v \longmapsto \frac{v(e)}{|G|}.$$

$\Rightarrow$ $\mathbb{C}^\tau[G]$ is a Frobenius algebra.

Topic 2: ~~E~~ Extending down pt 2

$n=3$: $\tau \in H^4(BG, \mathbb{Z})$.

$\mathrm{Vect}^{\tau}[G]$

objects = cplx vect. bdls.

morphs = lin v.b. maps

monoidal str. $W, W' \in \mathrm{Ob}(\mathrm{Vect}^{\tau}(G))$.

$$(W \otimes W')_y = \bigoplus_{xx'=y} K^{\tau}_{x,x'} \otimes W_x \otimes W'_{x'}$$

where

$$\begin{array}{c} K^{\tau} \\ \downarrow \\ G \times G \end{array}$$

has specified prop.

$\mathrm{Vect}^{\tau}[G]$ is $SO(3)$ equiv. fully dualizable object.

~~Mon~~ Mor-3cat. $\quad$ $\mathcal{M}or^3$

objects = tensor cats.

morphs = $\mathcal{A}, \mathcal{A}'$-bimodules

2-morphs = funct. betw. bimod. cats.

3-morphs = nat. transform.

<u>Claim</u>: $\mathrm{Vect}^{\tau}[G]$ is a <u>fusion category</u>.

which (~~Sy~~ Snyder – Schommer-Pries) is a fully dualizable object in $\mathcal{M}or^3$.

Douglas –

Defn: fusion category:

- rigid (all objects left & right dualizable)
- semisimple (all objects direct sum of simples)
- linear
- monoidal category
- only finitely many isomorphism classes of simple objects. (just 1).

[[illegible]]

Result:

$$I^{\mathcal{C}}(S^1) = Z\left(F^{\mathcal{C}}(S^1)\right)$$ } Card of a category [illegible] everything else.

$= \cancel{\text{twist}}$ twisted equivariant v.b.'s $\to G$.

$(L_{x,y} \otimes W_x \to W_{yxy^{-1}})$.

$L_{x,y} = K^{\alpha}_{yxy^{-1},y} \otimes K_{y,x}$

1. Recall from last time.

$$\left(G,\ \alpha \in H^n(BG, U(1))\right) \rightsquigarrow (n, n-1)\ \text{TFT}.$$

$\partial M = \amalg N_1, N_2$

$$\begin{array}{ccc} & \mathrm{Bun}_G(M) & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G(N_1) & & \mathrm{Bun}_G(N_2) \end{array}$$

$\exists$ local system defined by α on these.
Take global sections
pullback, pushforward.

In the 2d case: fully extended

$\mathbb{C}^\tau[G] \in \mathcal{A}lg$ = Morita 2-category of algebras.

objects: Algebras.
morphs: bimodules
2-morphs: intertwiners.

<u>Next</u>: the 3d case:

$\mathrm{Vect}^\alpha[G]$ — $\alpha \in H^3(BG, \mathbb{Z})$

$\alpha \rightsquigarrow$ $K^\alpha \to G \times G$ Hermitian line bdle. with isomorphisms.

Defn 9.1 of FHLT.

$\mathrm{Vect}^\tau(G)$ category

objects = $\mathbb{C}$ vect bdls / G.
morphisms = lin. v.b. maps.
monoidal str: $W, W' \to G$

$$(W * W')_y := \bigoplus_{xx'=y} K_{xx'} \otimes W_x \otimes W'_{x'}.$$

Thy is a fusion category & is fully dualizable

in

$\mathcal{M}or^{3}_{cat}$

objects = tensor cats ($\mathbb{C}$-lin)

morphisms = $\mathcal{A}$-$\mathcal{A}'$ bimodule caty.

($\mathbb{C}$-lin cat with left ah & rgh action).

2-morphs = functors between bimodule cats.

3-morphs = natural transformations.

$\cancel{Z(S^1) =}$

$Z(I_L^1)$ is A as a left $A \otimes A^{op}$ module. $F^2(I_R)$ is

A as right module.

$\Rightarrow F^2(S^1) = A \otimes_{A \otimes A^{op}} A$ (Hochschild homology as really dependent tensor product)

trace $\theta : A \to \mathrm{Vect}$. $\begin{pmatrix} W \\ \downarrow \\ a \end{pmatrix} \mapsto W_c$. (fiber at identity)

bilin form. $W \otimes W' \mapsto \theta(W * W')$.

gives $A \cong A^{\vee}$

$\Rightarrow Z(S^1) \cong \mathrm{Hom}_{A \otimes A^{op}}(A, A) \cong \underbrace{Z(A)}_{\text{Cat pairs}}$

(X, ε_x $X \in A$
$\varepsilon_x : X * - \to - * X$
nat isomorphism)

Cat 4.10.

monoidal cat of twisted equiv v.b.s.

$\Rightarrow$ what finite Chern Simons assigns to circle.

★ This Thy = Finite Chern Simons = DW theory (Freed-Quinn).

*not fully happy w/ this yet.

FHLT Hk 1. pg 3

~~We now let~~ (X, A) ~~a~~ ptd ~~finite~~ htpy type

~~(& consider~~ $\alpha \in H^{d+1}(BG, \mathbb{Z})$).

We ~~are~~ going to construct an E_n (or E_{n-1}) algebra.

Claim: ~~(Lurie,~~ Scheimbauer, H--) In the category Alg^n, $\mathring{Alg}^n$ all objects are ~~fully~~ n-dualizable.

~~Exotic~~

FHLT pg 2

Morita cats, E_n-alg & TFT.

E^n-algebra in an $(\infty, 1)$-cat $\mathcal{C}$.
(or $(\infty, 2)$-cat).

Roughly we have a map

$$S_n \circlearrowright \mathrm{Emb}\left(\coprod_n [0,1]^n \to [0,1]^n\right) \to \mathrm{Hom}(A^{\otimes n}, A) \circlearrowleft S_n$$

& composition

$$\coprod_{i=1}^{n} A^{i_2} \to A^n \to A.$$

Example: Loop spaces.

Example: E_1-algebra = associative (A^∞) algebras

Example: $\mathrm{Rep}(G)$ — E_2-algebra.

Have a category $\mathcal{C}$.

$Alg_{E_n}(\mathcal{C})$ is an $(n+1)$-category.

$Map(Alg_{E_n}(\mathcal{C}))(B_1, B_2) = (B_1, B_2)$ - bimodules in $\mathcal{C}$

Thm Dunn Additivity: Such bimodules are E_{n-1}-algebras.

$\vdots$

$(n+1)$-morphisms = are morphisms in $\mathcal{C}$.

Also: $Alg^{0}_{E_n}(\mathcal{C})$ discard non-invertible $(n+1)$-morphisms.

Also: If $\mathcal{C}$ an $(\infty,2)$ cat of $Cat_{(\infty,1)}$, then we have an $(n+2)$-category

We have a $(\infty, n+1)$ functor:

$$Alg_{E_n}(\mathcal{C}) \to Alg_{E_{n-1}}(\mathcal{C}at).$$

$$A \mapsto Rep(A).$$

Claim: FHLT: — this is fully faithful — gives some ~~...~~ fact here.

Thm (Gwilliam, Scheimbauer, Haugseng)
All objects of $Alg^{0}_{E_n}(\mathcal{C})$ are fully dualizable.

Claim (Lurie) [Remark 4.1.27]
Require also that $A \simeq \int_{D_k} A$ dualizable as a $\int_{S^{k-1}} A$ module for $0 \leq k \leq n$ to make A dualizable in $Alg_{E_n}(\mathcal{C})$.

Proof: Recall SJ's cobordism hypothesis induction.

FHLT. pg 3. —

Factorizatn / Top Chiral Homology.

Let $A \in Alg^{\circ}_{E_n}(\mathcal{C})$.

th defns by procobordn hypnks a n-d field thy Z_A.

For M^k a bordm betwn N_1^{k-1}, N_2^{k-1}.

$$Z_A(M^k) = \int_{M^{\circ}\mathrm{int}(M^k)\times D^{n-k}} A.$$

i) This is a E_{n-k} algebr (by locality at D).

ii) This is a $Z_A(N_1^{k-1}), Z_A(N_2^{k-1})$ bimodule.

$M \simeq M^{\circ} \amalg ([0,1]\times(N_1 \sqcup N_2))$

gives embeddg

$M^{\circ}\amalg (D^1 \times \partial M) \hookrightarrow M^{\circ}$.

Top Chiral homls gs map

$$\left(\int_{M^{\circ}} A\right) \otimes \int \int_{\partial M} A) \to \int_{M} A.$$

Examples : • $\mathbb{C}^{c}[G]$. (in $Alg_{E_1}(\mathbb{C}Vect)$).

• $Vect^{\tau}[G]$ in $(Alg_{E_1}(Cat_2))$

FHLT pg4.

Defining an E_n-algebra.

generalization 1: Replace BG with X a finite homotopy type.

Basic construction.

$$X \longrightarrow \Omega_x X.$$

problem no canonical choice of base pt.

Solution.

$$R^n_{top}(X) := \varprojlim_x R^{n-1}_{top}(\Omega_x X). \qquad R^1(X) = X.$$

$$R^n(X) := R(R^n_{top}(X)).$$

$$R(X) := \cancel{\coprod_{x \in \pi_0 X} \prod \mathbb{C}[\pi_i(X,x)]}$$

disj $\coprod$ $\mathbb{C}[\pi_n]$ funcs of $\prod_{i \geq 1} \pi_i(X,x)$.

(ptwise multiplication).

$\cancel{R^n_{Cat}(X) = Rep}$

Note $R^n_{top}(X) \in E_{n-1}\text{alg}(Top)$.

& R is a functor.

so $R^n(X) \in E_n\text{-alg}(Vect)$. (it has also ptwise multiplication).

So we also have

$$R^n_{cat}(X) = Rep(R^n(X)) \in E_{n-1}\text{-alg}(Cat).$$

~~Examples. Take $X = BG$.~~

$\cancel{R(X) = \coprod \prod \mathbb{C}(\pi_i(BG))}$

Change $R^1(X)$. $\cancel{= \mathbb{C}[G]}$. $G/G = \Omega(BG)$.

FHLT pg 5.

Or: $\underset{\hat{X}}{\overset{x}{*}} \to \Omega_x X$.

gives us ~~a~~ local system of

E_1-alg(Top) over X.

We also have.

a ~~sm~~ $X \to \Omega^n_x(X)$.

$R(X) =$ $\mathbb{C}[\pi_n]$ - fcts on

$\prod(\pi_{n-1} \times \dots \times \pi_1)$.

Example:

BG. $n = 1 \to \mathbb{C}[G]$.

BG $n = 2$.

$\Omega(BG)$. fcts on π_0. So we get.

$\mathbb{C}[G]$

mult 1 := composing loops is multiplication in G.

mult 2 := mult of $\mathbb{C}$.

we then have

$Rep(\mathbb{C}[G]) \cong Vect(G)$.

*. Why?

~~we have Rep w_G = representat.~~

~~the a fcts in $\mathbb{C}[G]$~~

~~Observables & domain walls~~ & Anomalies

What do these things give at the higher levels?

Cpd ⇝ E_n-algebras
just apply it not to X but
$\mathrm{Hom}(M, X)$. (top spaces) finite homotopy types.

So then take.
M bords N_1 to N_2. N_1, N_2 odim $n-1$ d.

$$Z(M) = Z\,\mathrm{R}(\mathrm{Hom}(N_1, X)) \underset{R_n(\mathrm{Hom}(M,X))}{\otimes} \mathrm{R}(\mathrm{Hom}(N_2, X)).$$

~~Rs is an E~~
to define Z iteratively
By uniqueness of a TFT this should give the same as the cohomology definition.
& we can use the definition on the highest level!
$R_n(\emptyset) = R_0(\emptyset) = \mathbb{C}$ (unit / full object in Vect).

$$Z(M^{n-1}) = \mathbb{C} \underset{R_n(\mathrm{Hom}(M,X))}{\otimes} \mathbb{C}.$$

* Does this give the correct answer?

Yes. $F \hookrightarrow \mathrm{Hom}(M, X)$

$$\downarrow$$

$\mathrm{Hom}(\emptyset, X) \times \mathrm{Hom}(\emptyset, X)$

When F is the fibre of any of groupoids

FHLT does

~~$R_n(\mathrm{Hom}(M,X))$~~
$Z_n(\mathrm{Hom}(M,X)) = R_n(\emptyset) \underset{R_n(\mathrm{Hom}(M,X))}{\otimes} R_n(\emptyset)$
$= R_{n-1}(\mathrm{Hom}(M,X))$.
RESULT FOLLOWS.